Lecture on Additive Number Theory

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Example:

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- There are two types of problems in this subject:
- **Direct Problem:** Here we start with two sets A & B, and try to deduce information of $A + B$. Or, start with a set A and determine the structure of

$$hA := \underbrace{A + A + \ldots\ldots\ldots + A}_{h \text{ times}}$$

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- Waring Problem: $k > 0$ positive integer and $A_k = \{0^k, 1^k, 2^k, 3^k, ...\}$ = Set of all k-th powers. Then there exists a positive integer s such that $sA_k (= \underbrace{A_k + A_k + ... + A_k}_{s \text{ times}})$ contains all positive integers.

  Or,

$$sA_k = \mathbb{N}$$

.

Lower Bound on Sumset

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Proof.

$$A = \{a_1 < a_2 < a_3 ...... < a_r\};$$
$$B = \{b_1 < b_2 < b_3 ...... < b_s\}.$$

$$\implies a_1 + b_1 < a_1 + b_2 < a_1 + b_3 < .... < a_1 + b_s$$
$$< a_2 + b_s < a_3 + b_s < .... < a_r + b_s.$$
$$\implies |A + B| \geq r + s - 1 = |A| + |B| - 1.$$

Lower Bound on Sumset

- **Bound is sharp:** *example:*

$$\text{Take} \quad A = \{1, 2, 3, ...10\};$$
$$B = \{1, 2, 3, ...20\}.$$
$$\text{Then,} \quad A + B = \{2, 3, 4, ..., 30\}.$$
$$\implies |A + B| = 29 = 10 + 20 - 1 = |A| + |B| - 1.$$

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- *Exercise:* Equality holds iff both A and B are in arithmetic progression of same difference.

  Recall: Arithmetic progression :=$\{a, a + d, a + 2d, ..., a + (m - 1)d\}$

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- k-term Arithmetic progression
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conjecture 1 (Erdős)

If $A \subseteq \mathbb{N}$, and $\sum_{a \in A} \frac{1}{a}$ diverges then, A contains $k-$term arithmetic progression for any given positive integer k.

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- This conjecture is still open.

Erdős conjecture on arithmetic progressions

- A special case of this conjecture was proved by Ben Green and Terrence Tao. They proved that:

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  - we will prove the divergence of $\left(\sum_{p \text{ prime}} \frac{1}{p}\right)$ at the end of the lecture.

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- The list of work for which Terrence Tao got Fields Medel(2006) includes this one.

Weaker Statement (Erdős-Turan conjecture/ Szemerédi theorem)

- The following theorem is an weaker statement of Erdős' Conjecture. It was known as Erdős-Turan conjecture.

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Theorem 3

Let $\delta > 0$ and k be an positive integer. Then we can find an positive integer $N_0(k,\delta)$ such that, If

$$N \geq N_0(k,\delta);$$
$$\mathcal{A} \subseteq \{1,2,3,.........,N\} \textit{ with } |\mathcal{A}| \geq \delta N,$$

then $\mathcal{A}$ contains $k-$term arithmetic progression.

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- For $k = 3$, it was proved by Klaus Roth.

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- As a consequence of *Furstenberg's theorem* over $\mathbb{Z}$, we get a little stronger version of *Szemerédi theorem*.
- Gowers also gave another proof using Harmonic analysis.

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- Ultimately it may look like $S = \{0, 3, 4, 7, 12, 22, 24, 41, ...\}$

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- Check upto $N = 3^k$, for some large positive integer k.

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- Call $\mathcal{A}_0 = \{0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40, 81, 82, ...\}$

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- Call $\mathcal{A}_0 = \{0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40, 81, 82, ...\}$
- **Recall:** *Base 3 representation:* $n = [r_k r_{k-1} .... r_1 r_0]_3 \iff n = r_k 3^k + r_{k-1} 3^{k-1} + ... + 3r_1 + r_0$.
  - Denote $r_k r_{k-1} ..... r_1 r_0$ by $[n]_3$

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  - Denote $r_k r_{k-1} .....r_1 r_0$ by $[n]_3$
- *Observation:*
  - If 2 occurs as a digit in $[n]_3$, then $n \notin \mathcal{A}_0$;
  - Otherwise $n \in \mathcal{A}_0$.

Algorithm to find Large set with 3-term AP

- Call $\mathcal{A}_0 = \{0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40, 81, 82, ...\}$
- **Recall:** *Base 3 representation:* $n = [r_k r_{k-1}....r_1 r_0]_3 \Longleftrightarrow n = r_k 3^k + r_{k-1} 3^{k-1} + ... + 3r_1 + r_0$.
  - Denote $r_k r_{k-1}.....r_1 r_0$ by $[n]_3$
- *Observation:*
  - If 2 occurs as a digit in $[n]_3$, then $n \notin \mathcal{A}_0$;
  - Otherwise $n \in \mathcal{A}_0$.
- Converse of this is also true:

Theorem 4

If $\mathcal{A} \subseteq \mathbb{N} \cup 0$, with the conditions:

- *If 2 occurs in $[n]_3$, then $n \notin \mathcal{N}$;*
- *If 2 does not occur in $[n]_3$, then $n \in \mathcal{A}$.*

Then $\mathcal{A}$ has no 3-term AP.

Condition for a set not having 3-term AP

- Let $n, m, q \in \mathcal{A}$ with $n + m = 2q$.

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  with $a_i, b_i, c_i \in \{0, 1\} \quad \forall i$.

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- $\implies$ There is no carry over in the summation or doubling.

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- $\implies a_i = b_i = c_i$.

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- $\implies$ $a_i = b_i = c_i$.
- $\implies$ $n = m = q$
- $\implies$ No nontrivial 3-term AP.

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- Also $|\mathcal{A}| = 2^k$.
- $$\begin{aligned} \implies \log|\mathcal{A}| &= k \log 2 \\ &= \frac{\log N}{\log 3} \log 2 \\ &= \log N^{\frac{\log 2}{\log 3}} \\ \implies |\mathcal{A}| &= N^{\frac{\log 2}{\log 3}}. \end{aligned}$$

Generalization from base 3 to $(2d+1)$ [Behrend]

- Given N, large, choose d such that $N \sim (2d+1)^k - 1$, with $k \sim [\sqrt{\log N}]$.
  Recall: $\sim$ means equality upto multiple of a constant.

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- Let $n, m, q \in \mathcal{A}'$, with $n + m = 2q$.

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- let

$$[n]_{2d+1} = a_k a_{k-1} .... a_1 a_0;$$
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- But that does not prove $a_i = b_i = c_i$, as $a_i, b_i, c_i \in \{0, 1, 2, \ldots., d\}$, with $d \geq 2$.
- So $\mathcal{A}'$is not the right candidate. We need to modify the set $\mathcal{A}'$.

Generalization from base 3 to $(2d+1)$ [Behrend]

- Define an equivalence relation $\leftrightsquigarrow$ on $\mathcal{A}'$ by

$$n = [x_k x_{k-1} \ldots\ldots x_1 x_0]_{2d+1} \leftrightsquigarrow m = [y_k y_{k-1} \ldots\ldots y_1 y_0]_{2d+1}$$

iff,

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- $\mathcal{A} :=$ Set of elements of $\mathcal{A}'$ which belongs to the same equivalence class maximum number of element. More preciously:
  - Consider all equivalence classes of $\mathcal{A}'$.
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  - Choose one of the class which contain maximum number of element.
  - Take set of all the elements of that class as $\mathcal{A}$.
- In short $\mathcal{A}$ is a maximal sphere in $\mathcal{A}'$.

Generalization from base 3 to $(2d+1)$ [Behrend]

- So $\mathcal{A}$ a maximal set satisfying following three properties:
  - $\mathcal{A} \subset \{1, 2, 3, ..., N\}$ with $N \sim 3^k$.
  - $\forall n = [a_k a_{k-1} ..... a_1 a_0]_{2d+1} \in \mathcal{A},\ \ 0 \leq a_i \leq d\ \ \forall i$.
  - $\forall n = [a_k a_{k-1} ..... a_1 a_0]_{2d+1} \in \mathcal{A},\ \ {a_k}^2 + {a_{k-1}}^2 + ... + {a_1}^2 + {a_0}^2$ is constant.

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The set $\mathcal{A}$, defined above, has no 3-element AP.

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- **Proof:** If not then, $n, m, q \in \mathcal{A}$, with $n + m = 2q$.

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Generalization from base 3 to $(2d+1)$ [Behrend]

- $\Longrightarrow$

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\begin{aligned}
a_0 + b_0 &= 2c_0 \\
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&- - -- - - - - \\
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- Not possible unless, $n = m = q$.
- $\mathcal{A}$ does not contain any non-trivial 3-term AP.

Generalization from base 3 to $(2d+1)$ [Behrend]

- $|\mathcal{A}'| = (d+1)^k \geq (d+\frac{1}{2})^k = \frac{(2d+1)^k}{2^k} = \frac{N+1}{2^k}$

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\begin{aligned}
\implies |\mathcal{A}| &\geq \frac{|\mathcal{A}'|}{2^k d^2} \\
&\geq \frac{N+1}{4^k e^{2\log d}} \\
&\geq Ne^{-c\sqrt{\log N}}. \quad \{\because k \sim [\sqrt{\log N}]\}
\end{aligned}
$$

c is some positive constant.

Improved Size of $\mathcal{A}$

- *Note:* $Ne^{-c\sqrt{\log N}} \geq N^{\frac{\log 2}{\log 3}}$, for large N.
  - $$f(N) = \frac{Ne^{-c\sqrt{\log N}}}{N^{\frac{\log 2}{\log 3}}}$$

    To prove: $f(n) \geq 1$.

    Or, $\log(f(N)) \geq 0$.

    $$\text{Or, } \log N(1 - \frac{\log 2}{\log 3}) - c\sqrt{\log N} \geq 0.$$

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    Which is true If N is large.
- So Behrend's result gives larger set $\mathcal{A}$ with no 3-term AP.

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    - Sets of the type $\{[\frac{N}{2}] + 1, [\frac{N}{2}] + 2, ..., N\}$, for some positive integer N.

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- Fermat's Last Theorem:A_k :=all k-th powers of positive integers with $k > 2$. Then A_k is sum free.

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Sum Free Sets

- A is called sum-free if the equation $x + y = z$ has no solution with $x, y, z \in A$.
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- Number of sum-free subsets of $\{1, 2, ..., N\}$ is $\leq C2^{\frac{N}{2}}$. It was known as Cameron-Erdos conjecture and is proved by Ben Green.

Divergence of $\left(\sum_{p \text{ prime}} \frac{1}{p}\right)$

- Euler product formula:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^s)^{-1}$$

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- But convergence of the second term in the summation $\implies (\sum_{p \text{ prime}} \frac{1}{p})$ diverges.

Prime number theorem(PNT)

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Thanks!