Differential Algebraic Topology

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August 7, 2005

INTRODUCTION

In this book we present some basic concepts and results from algebraic topology. We do this in the frame of differential topology. One of the central tools of algebraic topology are the homology groups. These are abelian groups associated to topological spaces which measure certain aspects of the complexity of a space.

The idea of homology was introduced by Poincaré in 1895. The way Poincaré introduced homology in this paper is the model for our approach. Since some basics of differential topology were not far enough developed, certain difficulties with Poincaré's approach occurred and three years later lead him to a new way of looking at homology which instead of differential topology uses objects from combinatorics, in particular simplices. This approach was very successful and up to now most books on algebraic topology follow it. The idea of the original concept came up then and there but more on an advanced level (for another geometric approach to (co)homology see [**B-R-S**]). We hope it is useful to present Poincaré's original idea in an elementary textbook.

As in other papers dealing with homology along Poincaré's original lines [**Po**] the central object replacing the simplices in the common presentation are certain stratified spaces. These are topological spaces which are decomposed as a disjoint union of smooth manifolds, called strata. We will derive this decomposition from another structure. Namely we use the language of differential spaces [**Si**] by distinguishing on a topological space **S** a certain algebra of functions **C**, which will play the role of smooth functions if **S** is a smooth manifold in the ordinary sense. The properties of this algebra will provide a decomposition of **S** into smooth manifolds, the strata.

We call our objects **stratifolds**. It turns out that basic concepts from differential topology like Sard's theorem, partition of unity, transversality generalize to stratifolds and this allows a definition of homology groups based on stratifolds. It is rather easy and intuitive to derive the basic properties of homology groups in the world of stratifolds. These properties allow computation of homology groups and straight forward constructions of important homology classes like for example the fundamental class of a closed smooth oriented manifold or more generally of a compact stratifold. We also define cohomology groups but only for smooth manifolds. Again certain important cohomology classes occur very naturally in this description, in particular the characteristic classes of smooth vector bundles over smooth oriented manifolds. Another useful aspect of this approach is that one of the most fundamental results, namely Poincaré duality, is almost a triviality.

From (co)homology groups one can derive important invariants like the Euler characteristic and the signature. These invariants play a significant role in some of the most spectacular results in differential topology. As a highlight we present Milnor's exotic 7-spheres (using a result of Thom which we do not prove in this book).

We mentioned above that Poincaré left his original approach and defined homology in a combinatorial way. It is natural to ask whether the definition in this book based on stratifolds is equivalent to the common combinatorial definition of singular homology. Both constructions satisfy the basic axioms of a homology theory and this implies, that for a large class of spaces, for example all spaces which are homotopy equivalent to a CW-complex, both theories are equivalent. There is also an axiomatic characterization of cohomology for smooth manifolds which implies that our cohomology groups are equivalent to ordinary singular cohomology. All this is explained in §20. It was a surprise to the author to find out that for more general spaces than those which are homotopy equivalent to CW-complexes, our homology theory is different from ordinary singular homology. This difference occurs already for rather simple spaces like one-point compactifications of smooth manifolds!

This indicates what the main themes of this book are. Readers should be familiar with the basic notion of point set topology and of differential topology. We would like to stress that one can start reading the book if one only knows the definition of a topological space and some basic examples and methods for creating topological spaces, and concepts like Hausdorff and compact spaces. From differential topology one only needs to know the definition of smooth manifolds and some basic examples and concepts like regular values and Sard's theorem. The author has given introductory courses to algebraic topology which start with the presentation of these prerequisites from point set and differential topology and then proceed with chapter 1 of this book. Additional information like orientation of manifolds or vector bundles or later on transversality was explained when it was needed. Thus the book can serve as basis for a combined introduction to differential and algebraic topology. It also allows a quick presentation of (co)homology in a course about differential geometry.

I presented the material in this book in courses at Mainz and Heidelberg University. I would like to thank the students and the assistants in these courses for their interest and one or the other suggestion for improvements. Thanks to Anna Grinberg for not only drawing the pictures but also for careful reading of earlier versions and for several stimulating discussions. Also many thanks to Daniel Müllner and Martin Olbermann for reading the final version and suggesting several improvements. I had several fruitful discussions with Gerd Laures, Wilhelm Singhof, Stephan Stolz, and Peter Teichner about the fundamental concepts. Theodor Bröcker and Don Zagier have read a previous version of the book and suggested numerous improvements. The book was carefully refereed and I obtained from the referees valuable suggestions for improvements. I would like to thank these colleagues for their generous help. Finally I would like to thank Dorothea Heukäufer and Ursula Jagtiani for the careful typing.

Contents

CHAPTER 1

Smooth manifolds revisited

Prerequisites: We assume that the reader is familiar with some basic notation of point set topology and differentiable manifolds. Actually rather little is needed for the beginning of this book. For example, it is sufficient to know ([**Jä**], ch. 1 and 3) as background from point set topology. For the first chapters, all we need to know from differential topology is the definition of smooth ($= C^\infty$) manifolds (without boundary) and smooth ($= C^\infty$) maps (see for example ([**Hi**] chapt. I.1 and I.4) or the corresponding chapters in [**B-J**]). In later chapters, where more background is required, the reader can find this in the cited literature.

1. A word about structures

Most definitions or concepts in modern mathematics are of the following type: A mathematical object is a set together with additional information called structure. For example a group is a set G together with a map $G \times G \to G$, the multiplication, or a topological space is a set X together with certain subsets, the open subsets. Often the set is already equipped with a structure of one sort and one adds another structure, for example a vector space is an abelian group together with the second structure given by scalar multiplication, or a smooth manifold is a topological space together with a smooth atlas. Given such a structure one defines certain classes of "allowed" maps (often called morphisms) which respect this structure in a certain sense, for example group homomorphisms or continuous maps. The real numbers $\mathbb{R}$ admit many different structures, they are a group, a field, a vector space, a metric space, a topological space, a smooth manifold .... The "allowed" maps from a set with a structure, which also the reals have, to $\mathbb{R}$, are of central importance.

In this section we will **define** a structure on a topological space by specifying certain maps to the reals. This is done in such a way that the allowed maps are the maps specifying the structure. In other words we give the allowed maps (morphisms) and this way we define a structure. For example we will define a smooth manifold M by specifying the C^∞-maps to $\mathbb{R}$. This stresses the central role of the allowed maps to $\mathbb{R}$, which in many areas of mathematics, in particular in analysis, play a central role.

2. Differential spaces

We introduce the language of differential spaces [**Si**], which are topological spaces together with a distinguished set of continuous functions fulfilling certain properties. To formulate these properties the following notion is useful: If X is a topological space, we denote the continuous functions from X to $\mathbb{R}$ by $C^0(X)$.

Definition: *A subset* $\mathbf{C} \subset C^0(X)$ *is called an* **algebra** *if for* $f, g \in \mathbf{C}$ *the sum* $f+g$ *, the product* fg *and all constant functions are in* $\mathbf{C}$.

The concept of an algebra, a vector space which at the same time is a ring fulfilling the obvious axioms, is more general, but here we only need algebras which are contained in $C^0(X)$.

For example, $C^0(X)$ itself is an algebra. The set of the constant functions is also an algebra. Or if $U \subset \mathbb{R}^k$ is an open subset, we denote the set of function $f : U \longrightarrow \mathbb{R}$, where all partial derivatives of all orders exist, by $C^\infty(U)$. This is an algebra in $C^0(U)$. More generally, if M is a k-dimensional smooth manifold we consider $C^\infty(M)$, the set of smooth functions on M, which is an algebra in $C^0(M)$.

The property of a continuous function can be decided locally, i. e. a function $f : X \longrightarrow \mathbb{R}$ is continuous if and only if for all $x \in X$ there is an open neighbourhood U and $f|_U$ is continuous. The following is an equivalent—more complicated looking— formulation where we don't need to know what it means for $f|_U$ to be continuous. A function $f : X \to \mathbb{R}$ is continuous if and only if for each $x \in X$ there is an open neighbourhood U and a continuous function g such that $f|_U = g|_U$. Since this formulation makes sense for an arbitrary set of functions C, we define:

Definition: *Let* $\mathbf{C}$ *be a set of functions* $f : X \to \mathbb{R}$. *We say that* $\mathbf{C}$ *is* **locally detectable** *if a function* $f : X \longrightarrow \mathbb{R}$ *is contained in* $\mathbf{C}$ *if and only if for all* $x \in X$ *there is an open neighbourhood* U *of* x *and* $g \in \mathbf{C}$ *such that* $f|_U = g|_U$.

For those familiar with the language of sheaves it is obvious that $(X, \mathbf{C})$ is in this language equivalent to a ringed space.

As mentioned above, the set of continuous functions $C^0(X)$ is locally detectable. Similarly, if M is a smooth manifold, then $C^\infty(M)$ is locally detectable.

We can now define differential spaces.

Definition: *A* **differential space** *is a pair* $(X, \mathbf{C})$, *where* X *is a topological space and* $\mathbf{C} \subset C^0(X)$ *is an algebra of continuous functions such that*

(1) $\mathbf{C}$ *is locally detectable,*
(2) *for all* $f_1, \ldots, f_k \in \mathbf{C}$ *and* $g : \mathbb{R}^k \longrightarrow \mathbb{R}$ *a smooth function, the function* $x \mapsto g(f_1(x), \ldots, f_k(x))$ *is in* $\mathbf{C}$.

We have already discussed the use of the first condition above, and the second condition is obviously desirable in order to construct new elements of $\mathbf{C}$ by composition with smooth maps.

The considerations above show that if M is a k-dimensional smooth manifold, then $(M, C^\infty(M))$ is a differential space. This is the fundamental class of examples which will be the model for our generalization to stratifolds in the next chapter.

From a differential space $(X, \mathbf{C})$, one can often construct new differential spaces. For example, if $Y \subset X$ is a subspace, we define $\mathbf{C}(Y)$ to contain those functions $f : Y \longrightarrow \mathbb{R}$ such that for all $x \in Y$, there is a $g : X \longrightarrow \mathbb{R}$ in $\mathbf{C}$ such that $f|_V = g|_V$ for some open neighbourhood V of x in Y. The reader should check that $(Y, \mathbf{C}(Y))$ is a differential space.

There is another algebra associated to a subspace Y in X, namely the restriction of all elements in $\mathbf{C}$ to Y. Later we will consider differential spaces with additional properties which guarantee that $\mathbf{C}(Y)$ is equal to the restriction of elements in $\mathbf{C}$ to Y, if Y is a closed subspace.

For the generalization to stratifolds it is useful to note that one can define smooth manifolds in the language of differential spaces.

To prepare this, we need a way to compare differential spaces.

Definition: *Let $(X, \mathbf{C})$ and $(X', \mathbf{C}')$ be differential spaces. A homeomorphism $f : X \longrightarrow X'$ is called* **isomorphism** *if for each $g \in \mathbf{C}'$ and $h \in \mathbf{C}$, we have $gf \in \mathbf{C}$ and $hf^{-1} \in \mathbf{C}'$.*

The slogan is: Composition with f stays in $\mathbf{C}$ and with f^{-1} stays in $\mathbf{C}'$. Obviously the identity map is an isomorphism from $(X, \mathbf{C})$ to $(X, \mathbf{C})$. If $f : X \to X'$ and $f' : X' \to X''$ are isomorphisms then $f'f : X \to X''$ is an isomorphism. If f is an isomorphism then f^{-1} is an isomorphism.

For example, if X and X' are open subspaces of $\mathbb{R}^k$ equipped with the algebras of smooth functions, then an isomorphism f is the same as a diffeomorphism from X to X', a bijective map such that the map and its inverse are smooth ($= C^\infty$) maps. This equivalence is due to the fact that a map g from an open subset U of $\mathbb{R}^k$ to an open subset V of $\mathbb{R}^n$ is smooth if and only if all coordinate functions are smooth. (For a similar discussion, see the end of this chapter.)

3. Smooth manifolds revisited

We recall that if $(X, \mathbf{C})$ is a differential space and U an open subspace, the algebra $\mathbf{C}(U)$ is defined as the continuous maps $f : U \to \mathbb{R}$ such that for each $x \in U$ there is an open neighbourhood $V \subset U$ of x and $g \in \mathbf{C}$ such that $g|_V = f|_V$. We remind the reader that $(U, \mathbf{C}(U))$ is a differential space.

Definition: *A k-dimensional* **smooth manifold** *is a differential space $(M, \mathbf{C})$ where M is a Hausdorff space with a countable basis of its topology, such that for each $x \in M$*

there is an open neighbourhood $U \subseteq M$ and an open subset $V \subset \mathbb{R}^k$ and an isomorphism

$$\varphi : (V, C^\infty(V)) \to (U, \mathbf{C}(U)).$$

The slogan is: A k-dimensional smooth manifold is a differential space which is locally isomorphic to $\mathbb{R}^k$.

To justify this definition of this well known mathematical object, we have to show that it is equivalent to the definition based on a maximal smooth atlas. Starting from the definition above, we consider all isomorphisms $\varphi : (V, C^\infty(V)) \to (U, \mathbf{C}(U))$ from the definition above and note that their coordinate changes $\varphi^{-1}\varphi' : (\varphi')^{-1}(U \cap U') \to V \cap V'$ are smooth maps and so the maps $\varphi : V \to U$ give a maximal smooth atlas on M. In turn if a smooth atlas $\varphi : V \to U \subset M$ is given, then we define $\mathbf{C}$ as the continuous functions $f : M \to \mathbb{R}$ such that for all φ in the smooth atlas we have $f\varphi : V \to \mathbb{R}$ is in $C^\infty(V)$.

We want to introduce the important concept of germs of functions. Let C be a set of functions from X to $\mathbb{R}$. We define an equivalence relation on C by setting f equivalent to g if and only if there is an open neighbourhood V of x such that $f|_V = g|_V$. We call the equivalence class represented by f the **germ** of f at x and denote the equivalence class by $[f]_x$. We denote the set of germs of functions at x by $\mathbf{C}_x$. This definition of germs is different from the standard one which only considers equivalence classes of functions defined on some open neighbourhood of x. For differential spaces these sets of equivalence classes are the same, since if $f : U \to \mathbb{R}$ is defined on some open neighbourhood of x, then there is a $g \in \mathbf{C}$ such that on some smaller neighbourhood V we have $f|_V = g|_V$.

To prepare the definition of stratifolds in the next chapter, we recall the definition of the tangent space at a point $x \in M$ in terms of derivations. Let $(X, \mathbf{C})$ be a differential space. For a point $x \in X$, we consider the germ $\mathbf{C}_x$ of functions near x. If $f \in \mathbf{C}$ and $g \in \mathbf{C}$ are representatives of germs near x, then the sum $f + g$ and the product $f \cdot g$ represent well defined germs denoted $[f]_x + [g]_x \in \mathbf{C}_x$ and $[f]_x \cdot [g]_x \in \mathbf{C}_x$.

Definition: *Let $(X, \mathbf{C})$ be a differential space. A* **derivation** *at $x \in X$ is a map from the germs $\mathbf{C}_x$ of functions near x*

$$\alpha : \mathbf{C}_x \longrightarrow \mathbb{R}$$

such that

$$\alpha([f]_x + [g]_x) = \alpha([f]_x) + \alpha([g]_x),$$

$$\alpha([f]_x \cdot [g]_x) = \alpha([f]_x) \cdot g(x) + f(x) \cdot \alpha([g]_x),$$

and

$$\alpha([c]_x \cdot [f]_x) = c \cdot \alpha([f]_x)$$

for all $f, g \in \mathbf{C}$ and $[c]_x$ the germ of a constant function which maps all $y \in M$ to $c \in \mathbb{R}$.

If $U \subset \mathbb{R}^k$ is an open set and $v \in \mathbb{R}^k$, the Leibniz rule says that for $x \in U$, the map

$$\alpha_v : C_x^\infty \longrightarrow \mathbb{R}$$

$$[f]_x \longmapsto Df_x(v)$$

is a derivation. Thus the derivative in the direction of v is a derivation which justifies the name.

If α and β are derivations, then $\alpha+\beta$ mapping $[f]_x$ to $\alpha([f]_x)+\beta([f]_x)$ are derivations, and if $t \in \mathbb{R}$ then $t\alpha$ mapping $[f]_x$ to $t\alpha([f]_x)$ is a derivation. Thus the derivations at $x \in X$ form a vector space.

Definition: *Let $(X, \mathbf{C})$ be a differential space and $x \in X$. The vector space of derivations at x is called the* **tangent space** *of X at x and denoted by T_xX.*

This notation is justified by the fact that if M is a k-dimensional smooth manifold, which we interpret as a differential space $(M, C^\infty(M))$, then the definition above is one of the equivalent definitions of the tangent space ([**B-J**], p. 14). In particular, $\dim T_xX = k$.

We have already defined isomorphisms between differential spaces. We also want to introduce morphisms. If the differential spaces are smooth manifolds, then the morphisms will be the smooth maps. To generalize the definition of smooth maps to differential spaces, we reformulate the definition of smooth maps.

If M is an m-dimensional smooth manifold and U is an open subset of $\mathbb{R}^k$ then a map $f : M \longrightarrow U$ is a smooth map if and only if all components $f_i : M \longrightarrow \mathbb{R}$ are in $C^\infty(M)$ for $1 \leq i \leq k$. If we don't want to use components we can equivalently formulate that f is smooth if and only if for all $\rho \in C^\infty(U)$ we have $\rho f \in C^\infty(M)$. This is the logic behind the following definition. Let $(X, \mathbf{C})$ be a differential space and $(X', \mathbf{C}')$ another differential space. Then we define a **morphism** f from $(X, \mathbf{C})$ to $(X', \mathbf{C}')$ as a continuous map $f : X \longrightarrow X'$ such that for all $\rho \in \mathbf{C}'$ we have $\rho f \in \mathbf{C}$. We denote the set of morphisms by $\mathbf{C}(X, X')$. The following properties are obvious from the definition:

(1) $id : (X, \mathbf{C}) \longrightarrow (X, \mathbf{C})$ is a morphism,
(2) if $f : (X, \mathbf{C}) \longrightarrow (X', \mathbf{C}')$ and $g : (X', \mathbf{C}') \longrightarrow (X", \mathbf{C}")$ are morphisms, then $gf : (X, \mathbf{C}) \longrightarrow (X", \mathbf{C}")$ is a morphism,
(3) all elements of $\mathbf{C}$ are morphisms from X to $\mathbb{R}$,
(4) the isomorphisms (as defined above) are the morphisms $f : (X, \mathbf{C}) \longrightarrow (X', \mathbf{C}')$ such that there is a morphism $g : (X', \mathbf{C}') \longrightarrow (X, \mathbf{C})$ with $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_{X'}$.
(5) Let $f : (X, \mathbf{C}) \to (X', \mathbf{C}')$ be a morphism. Then we define for each $x \in X$ the **differential**

$$df_x : T_xX \to T_{f(x)}X'$$

which maps the derivation α to α' where α' assigns to $[g]_{f(x)} \in \mathbf{C}'_{x'}$ the value $\alpha([gf]_x)$.

CHAPTER 2

Stratifolds

Prerequisites: The main new ingredient is Sard's Theorem (see for example [**B-J**], chapt. 6 or [**Hi**], chapt. 3.1). It is enough to know the statement of this important result.

1. Stratifolds

We will define a stratifold as a differential space with certain properties. The main consequence of these properties is that there will be a natural decomposition of a differential space into subspaces (which in the end should be smooth manifolds). We begin with the definition of this decomposition.

Let $(\mathbf{S}, \mathbf{C})$ be a differential space. We define the subspace $\mathbf{S}^i := \{x \in \mathbf{S} | \dim T_x\mathbf{S} = i\}$.

By construction $\mathbf{S} = \mathring{\cup}\mathbf{S}^i$, i. e. $\mathbf{S}$ is the disjoint union of the subsets $\mathbf{S}^i$. In chapter 1 we introduced the differential spaces $(\mathbf{S}^i, \mathbf{C}(\mathbf{S}^i))$ given by the subspace $\mathbf{S}^i$ together with the induced algebra. Our first condition is that this differential space is a smooth manifold:

1a) We require that $(\mathbf{S}^i, \mathbf{C}(\mathbf{S}^i))$ is a smooth manifold (as defined in chapter 1).

Once this condition is fulfilled we write $C^\infty(\mathbf{S}^i)$ instead of $\mathbf{C}(\mathbf{S}^i)$. This smooth structure on $\mathbf{S}^i$ has the property that any smooth function can locally be extended to an element of $\mathbf{C}$. We want to strengthen this property by requiring that in a certain sense such an extension is unique. To formulate this we note that for points $x \in \mathbf{S}^i$, we have two sorts of germs of functions, namely $\mathbf{C}_x$, the germ of functions near x on $\mathbf{S}$, and $C^\infty(\mathbf{S}^i)_x$, the germ of smooth functions near x on $\mathbf{S}^i$, and our second condition requires that these germs are equal. More precisely, condition 1b is:

1b) Restriction defines for all $x \in \mathbf{S}^i$, a bijection

$$\mathbf{C}_x \xrightarrow{\cong} C^\infty(\mathbf{S}^i)_x$$
$$[f]_x \longrightarrow [f|_{\mathbf{S}^i}]_x.$$

Here the only new input is the injectivity, the surjectivity follows from the definition. As a consequence, the tangent space of $\mathbf{S}$ at x is isomorphic to the tangent space of $\mathbf{S}^i$ at x. In particular we conclude that

$$\dim \mathbf{S}^i = i.$$

Conditions 1a and 1b give the most important properties of a stratifold. In addition we impose some other conditions which are common in similar contexts. To formulate them we introduce the following notation.

We call $\mathbf{S}^i$ the i-**stratum** of $\mathbf{S}$. In other concepts of spaces which are decomposed as smooth manifolds, the connected components of $\mathbf{S}^i$ are called the strata but we prefer to collect the i-dimensional strata into a single stratum. We call $\cup_{i\leq r}\mathbf{S}^i =: \Sigma^r$ the $r-$ **skeleton** of $\mathbf{S}$.

Definition: *A k-dimensional* **stratifold** *is a differential space* $(\mathbf{S}, \mathbf{C})$*, where* $\mathbf{S}$ *is a locally compact (meaning each point is contained in a compact neighbourhood) Hausdorff space with countable basis, the skeleta* Σ^i *are closed subspaces. In addition we assume:*

(1) *the conditions 1a) and 1b) are fulfilled, i.e. restriction gives a smooth structure on* $\mathbf{S}^i$ *and for each* $x \in \mathbf{S}^i$ *restriction gives an isomorphism*

$$i^* : \mathbf{C}_x \xrightarrow{\cong} C^\infty(\mathbf{S}^i)_x,$$

(2) $\dim T_x\mathbf{S} \leq k$ *for all* $x \in \mathbf{S}$*, i. e. all tangent spaces have dimension* $\leq k$*,*
(3) *for each* $x \in \mathbf{S}$ *and open neighbourhood* $U \in \mathbf{S}$ *there is a function* $\rho \in \mathbf{C}$ *such that* $\rho(x) \neq 0$ *and* $\operatorname{supp}\rho \subseteq U$ *(such a function is called a* **bump function***).*

We recall that the support of a function $f : X \to \mathbb{R}$ is $\operatorname{supp} f := \overline{\{x|\, f(x) \neq 0\}}$, the closure of the points where f is non-trivial.

In our definition of a stratifold, the dimension k is always a finite number. One could easily define infinite dimensional stratifolds where the only difference is that in condition 2., we would require that $\dim T_x\mathbf{S}$ is finite for all $x \in \mathbf{S}$. Infinite dimensional stratifolds will play no role in this book.

Let me comment on these conditions. The most important condition is the first which we have already explained above. In particular, we recall that the smooth structure on $\mathbf{S}^i$ is determined by $\mathbf{C}$ which gives us a **stratification** of $\mathbf{S}$, a decomposition into smooth manifolds $\mathbf{S}^i$ of dimension i. The second condition says that the dimension of all non-empty strata is less or equal to k. We don't assume that $\mathbf{S}^k \neq \emptyset$ which, at the first glance, might look strange, but even in the definition of a k-dimensional manifold M, it is not required that $M \neq \emptyset$.

The third condition will be used later to show the existence of a partition of unity, an important tool to construct elements of $\mathbf{C}$. To do this, we will also use the topological conditions that the space is locally compact, Hausdorff, and has a countable basis. The other topological conditions on the skeleta and strata are common in similar contexts. For example, they guarantee that the **top stratum** $\mathbf{S}^k$ is open in $\mathbf{S}$, a useful and natural property. Here we note that the requirement that the skeleta are closed is equivalent to the statement that for each $j > i$ we require $\overline{\mathbf{S}^i} \cap \mathbf{S}^j = \emptyset$. This topological condition roughly says that if we "walk" in $\mathbf{S}^i$ to a limit point outside $\mathbf{S}^i$, then this point sits in $\mathbf{S}^r$

for $r < i$. These conditions are common in similar contexts such as CW-complexes.

We have chosen the letter Σ^j for the j-skeleton since Σ^{k-1} is the singular set of $\mathbf{S}$ in the sense that $\mathbf{S} - \Sigma^{k-1} = \mathbf{S}^k$ is a smooth k-dimensional manifold. Thus if $\Sigma^{k-1} = \emptyset$, then $\mathbf{S}$ is a smooth manifold.

We call our objects stratifolds since, while on the one hand they are stratified spaces, on the other hand – although stratifolds are much more general than smooth manifolds – they are in a certain sense very close to smooth manifolds. As we will see, many of the fundamental tools of differential topology are available for stratifolds. In this respect smooth manifolds and stratifolds are not very different and deserve a similar name.

Remark: *It's a nice property of smooth manifolds that once an algebra in $C^0(M)$ for a locally compact Hausdorff space M with countable basis is given, the question, whether M is a smooth manifold is a local question. The same is true for stratifolds, since the conditions 1 - 3 are again local.*

2. Local retracts

To obtain a better feeling for the central condition 1, we give an alternative description. If $(\mathbf{S}, \mathbf{C})$ is a stratifold and $x \in \mathbf{S}^i$, we will construct an open neighborhood U_x of x in $\mathbf{S}$ and a morphism $r_x : U_x \to U_x \cap \mathbf{S}^i$ such that $r_x|_{U_x \cap \mathbf{S}^i} = id_{U_x \cap \mathbf{S}^i}$. (Here we consider U_x as differential space with the induced structure on an open subset as described in chapter 1.) Such a map is called a **retract** from U_x to $V_x := U_x \cap \mathbf{S}^i$. If one has a local retract $r : U_x \to U_x \cap \mathbf{S}^i =: V_x$, we can use it to extend a smooth map $g : V_x \to \mathbb{R}$ to a map on U_x by gr. Thus composition with r gives a map

$$C^\infty(\mathbf{S}^i)_x \longrightarrow \mathbf{C}_x$$

mapping $[h]$ to $[hr]$, where we represent h by a map whose domain is contained in V_x. This gives an inverse of the isomorphism in condition 1 given by restriction.

To construct a retract we choose an open neighborhood W of x in $\mathbf{S}^i$ such that W is the domain of a chart $\varphi : W \longrightarrow \mathbb{R}^i$ (we want that $\operatorname{im} \varphi = \mathbb{R}^i$ and we achieve this by starting with an arbitrary chart, which contains one whose image is an open ball which we identify with $\mathbb{R}^i$ by an appropriate diffeomorphism). Now we consider the coordinate functions $\varphi_j : W \longrightarrow \mathbb{R}$ of φ and consider for each $x \in W$ the germ represented by φ_j. By condition 1 there is an open neighbourhood $W_{j,x}$ of x in $\mathbf{S}$ and an extension $\hat\varphi_{j,x}$ of $\varphi_j|_{W_{j,x}} \cap \mathbf{S}^i$. We denote the intersection $\cap_{j=1,\dots i} W_{j,x}$ by W_x and obtain a morphism $\hat\varphi_x : W_x \longrightarrow \mathbb{R}^i$ such that $y \mapsto (\hat\varphi_{1,x}(y), \dots, \hat\varphi_{i,x}(y))$. For $y \in W_x \cap \mathbf{S}^i$ we have $\hat\varphi_x(y) = \varphi(y)$. Next we define

$$r : W_x \longrightarrow \mathbf{S}^i$$

$$z \mapsto \varphi^{-1}\hat\varphi_x(z).$$

For $y \in W_x \cap \mathbf{S}^i$ we have $r(y) = y$. Finally we define $U_x := r^{-1}(W_x \cap \mathbf{S}^i)$ and

$$r_x := r|_{U_x} : U_x \to U_x \cap \mathbf{S}^i = W_x \cap \mathbf{S}^i =: V_x$$

is the desired retract.

We summarize these considerations.

Proposition 2.1. *(Local retracts) Let* $(\mathbf{S}, \mathbf{C})$ *be a stratifold. Then for* $x \in \mathbf{S}^i$ *there is an open neighborhood* U *of* x *in* $\mathbf{S}$ *and* V *of* x *in* $\mathbf{S}^i$ *and a morphism*

$$r : U \to V$$

such that $U \cap \mathbf{S}^i = V$ *and* $r|_V = id$. *Such a morphism is called a* **local retract** *near* x.

If $r : U \to V$ *is a local retract near* x, *then* r *induces an isomorphism*

$$C^\infty(\mathbf{S}^i)_x \to \mathbf{C}_x,$$
$$[h] \mapsto [hr]$$

the inverse of $i^* : \mathbf{C}_x \to C^\infty(\mathbf{S}^i)_x$.

The germ of local retracts near x *is unique, i.e. if* $r' : U' \to V'$ *is another local retract near* x, *then there is a* $U'' \subset U \cap U'$ *such that* $r|_U'' = r'|_U''$.

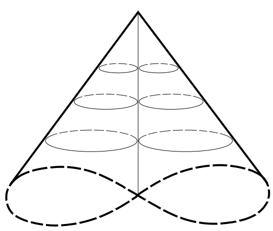
The last statement follows since $\varphi_j r|_{U_j''} = \varphi_j r'|_{U_j''}$ for an appropriate open neighbourhood U_j'', and since φ is injective, we conclude for $U'' := \cap U_j''$ the statement.

Note, that one can use the local retracts to characterize elements of $\mathbf{C}$, namely a continuous function $f : \mathbf{S} \to \mathbb{R}$ is in $\mathbf{C}$ if and only if its restriction to all strata is smooth and it commutes with appropriate local retracts. This implies that if $f : \mathbf{S} \to \mathbb{R}$ is a nowhere zero morphism then $1/f$ is in $\mathbf{C}$.

3. Examples

The first class of examples is given by the smooth k-dimensional manifolds. These are the k-dimensional stratifolds with $\mathbf{S}^i = \emptyset$ for $i < k$. It is clear that such a stratifold gives a smooth manifold and in turn a k-dimensional manifold gives a stratifold. All conditions are obvious (for the existence of a bump function see ([B-J], p. 66) or ([Hi], p. 41)).

Example 1: The most fundamental non-manifold example is the cone over a manifold. We define the open cone over a topological space Y as $Y \times [0,1)/(Y \times \{0\}) =: \overset{\circ}{C}Y$. (We call it the open cone and give the notation $\overset{\circ}{C}Y$ to distinguish it from the (closed) cone $CY := Y \times [0,1]/(Y \times \{0\})$.) We call the point $Y \times \{0\}/_{Y \times \{0\}}$ the top of the cone and abbreviate this as pt.



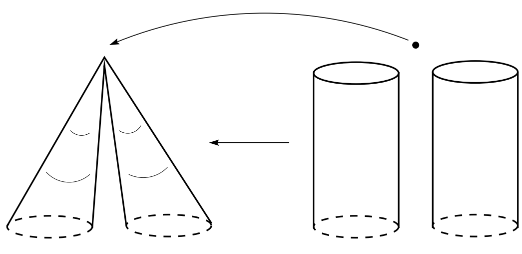
Let M be a k-dimensional compact smooth manifold. We consider the **open cone** over M and define an algebra making it a stratifold. We define the algebra $\mathbf{C} \subset C^0(\overset{\circ}{C}M)$ consisting of all functions in $C^0(\overset{\circ}{C}M)$ which are constant on some open neighbourhood U of the top of the cone and whose restriction to $M \times (0,1)$ is in $C^\infty(M \times (0,1))$. We want to show that $(\overset{\circ}{C}M, \mathbf{C})$ is a $(k+1)$-dimensional stratifold. It is clear from the definition of $\mathbf{C}$ that $\mathbf{C}$ is a locally detectable algebra, and that condition 2. in the definition of differential spaces is fulfilled.

So far we have seen that the open cone $(\overset{\circ}{C}M, \mathbf{C})$ is a differential space. We now check that the conditions of a stratifold are satisfied. Obviously, $\overset{\circ}{C}M$ is a Hausdorff space with a countable basis and, since M is compact, $\overset{\circ}{C}M$ is locally compact. The other topological properties of a stratifold are clear. We continue with the description of the stratification. For $x \neq \mathrm{pt}$, the top of the cone, since $\mathbf{C}_x$ is the set of germs of smooth functions on $M \times (0,1)$ near x, $T_x(\overset{\circ}{C}M) = T_x(M \times (0,1))$ which implies that $\dim T_x(\overset{\circ}{C}M) = k+1$. For $x = \mathrm{pt}$, the top of the cone, C_x consists of simply the germs of constant functions. Since 1 is the constant function mapping all points to 1, we see for each derivation α, we have $\alpha(1 \cdot 1) = \alpha(1) \cdot 1 + 1 \cdot \alpha(1)$ implying $\alpha(1) = 0$. But since $\alpha([c] \cdot 1) = c \cdot \alpha(1)$, we conclude that $T_{\mathrm{pt}}(\overset{\circ}{C}M) = 0$ and $\dim T_{\mathrm{pt}}(\overset{\circ}{C}M) = 0$. Thus we have two non-empty strata: $M \times (0,1)$ and the top of the cone.

The conditions 1 and 2 are obviously fulfilled. It remains to show the existence of bump functions. Near points $x \neq \mathrm{pt}$ the existence of a bump function follows from the existence of a bump function in $M \times (0,1)$ which we extend by 0 to $\overset{\circ}{C}M$. Near pt we first note that any open neighbourhood of pt contains an open neighbourhood of the form $M \times [0,\epsilon)/(M \times \{0\})$ for an appropriate $\epsilon > 0$. Then we choose a smooth function $\eta : [0,1) \to [0,\infty)$ which is 1 near 0 and 0 for $t \geq \epsilon$ (for the construction of such a function see ([**B-J**], p. 65)). With the help of η, we can now define the bump function

$$\rho([x,t]) := \eta(t)$$

which completes the proof that $(\overset{\circ}{C}M, \mathbf{C})$ is a $(k+1)$-dimensional stratifold. It has two non-empty strata: $\mathbf{S}^{k+1} = M \times (0,1)$ and $\mathbf{S}^0 = \mathrm{pt}$.



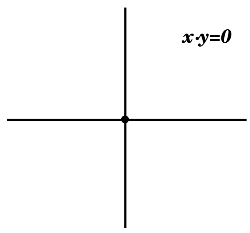
Example 2: Let M be a non-compact m-dimensional manifold. The **one-point compactification** of M is the space M^+ consisting of M and an additional point $+$. The

topology is given by defining open sets as the open sets of M together with the complements of compact subsets of M. The latter give the open neighbourhoods of $+$. The one-point compactification is a compact Hausdorff space (and so it has a countable basis). (For more information see e. g. [**Sch**].)

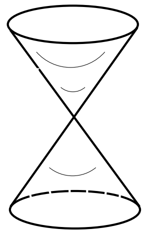
On M^+, we define the algebra $\mathbf{C}$ as the continuous functions which are constant on some open neighbourhood of $+$ and smooth on M. Then $(M^+, \mathbf{C})$ is an m-dimensional stratifold. All conditions except 3. are obvious. For the existence of a bump function near $+$ (near all other points use a bump function of M and extend it by 0 to $+$), let U be an open neighbourhood of $+$. By definition of the topology, $M - U =: A$ is a compact subset of M. Then one constructs another compact subset $B \subset M$ with $A \subset \overset{\circ}{B}$ (how?), and, starting from B instead of A, a third compact subset $C \subset M$ with $B \subset \overset{\circ}{C}$. Then B and $M - \overset{\circ}{C}$ are disjoint closed subsets of M and there is a smooth function $\rho : M \to (0, \infty)$ such that $\rho|_B = 0$ and $\rho|_{M - \overset{\circ}{C}} = 1$. We extend ρ to M^+ by mapping $+$ to 1 to obtain a bump function on M near $+$.

Thus we have given the one point compactification of a smooth non-compact m-dimensional manifold M the structure of a stratifold $\mathbf{S} = M^+$, with non-empty strata $\mathbf{S}^m = M$ and $\mathbf{S}^0 = +$.

Example 3: The most natural examples of manifolds with singularities occur in algebraic geometry as algebraic varieties, i.e. zero sets of a family of polynomials. There is a natural but not completely easy way to impose the structure of a stratifold on an algebraic variety (this proceeds in two steps, namely, one first shows that a variety is a Whitney stratified space and then one uses the retracts constructed for Whitney stratified spaces to obtain the structure of a stratifold, where the algebra consists of those functions commuting with appropriate representatives of the retracts). Here we only give a few simple examples. Consider $\mathbf{S} := \{(x, y) \in \mathbb{R}^2 | xy = 0\}$. We define $\mathbf{C}$ as the functions on $\mathbf{S}$ which are smooth outside 0 and constant in some open neighbourhood of 0. It is easy to show that $(\mathbf{S}, \mathbf{C})$ is a 1-dimensional stratifold with $\mathbf{S}^1 = \mathbf{S} - (0, 0)$ and $\mathbf{S}^0 = (0, 0)$.



Example 4: In the same spirit we consider $\mathbf{S} := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = z^2\}$. Again we define $\mathbf{C}$ as the functions on $\mathbf{S}$ which are smooth outside 0 and constant in some open neighbourhood of 0. This gives a 2-dimensional stratifold $(\mathbf{S}, \mathbf{C})$, where $\mathbf{S}^2 = \mathbf{S} - (0, 0)$ and $\mathbf{S}^0 = (0, 0)$.



Example 5: Let $(\mathbf{S}, \mathbf{C})$ be a k-dimensional stratifold and $U \subset \mathbf{S}$ an open subset. Then $(U, \mathbf{C}(U))$ is a k-dimensional stratifold. We suggest that the reader verifies this to become acquainted with stratifolds.

Example 6: Let $(\mathbf{S}, \mathbf{C})$ and $(\mathbf{S}', \mathbf{C}')$ be stratifolds of dimension k and l. Then we define a stratifold with underlying topological space $\mathbf{S} \times \mathbf{S}'$. To do this we use the local retracts (Proposition 2.1). We define $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$ as those continuous functions $f : \mathbf{S} \times \mathbf{S}' \longrightarrow \mathbb{R}$ which are smooth on all products $\mathbf{S}^i \times \mathbf{S}^j$ and such that for each $(x, y) \in \mathbf{S}^i \times (\mathbf{S}')^j$ there are local retracts $r_x : U_x \longrightarrow \mathbf{S}^i \cap U_x$ and $r_y : U_y \longrightarrow (\mathbf{S}')^j \cap U_y$ and we require that $f|_{U_x \times U_y} = f(r_x \times r_y)$. In short, we define $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$ as those continuous maps which commute with the product of appropriate local retracts onto $\mathbf{S}^i$ and $(\mathbf{S}')^j$. The detailed argument that $(\mathbf{S} \times \mathbf{S}', \mathbf{C}(\mathbf{S} \times \mathbf{S}'))$ is a $(k + l)$-dimensional stratifold is a bit lengthy and not relevant for further reading and for that reason we provide it in Appendix A. Both projections are morphisms.

In particular, if $(\mathbf{S}', \mathbf{C}')$ is a smooth m-dimensional manifold M, then we have the product stratifold $(\mathbf{S} \times M, \mathbf{C}(\mathbf{S} \times M))$.

Example 7: Combining example 6 with the method for constructing example 1, we construct the open cone over a compact stratifold $(\mathbf{S}, \mathbf{C})$. The underlying space is again $\overset{\circ}{C}\mathbf{S}$. We consider the algebra $\mathbf{C} \subset C^0(\overset{\circ}{C}\mathbf{S})$ consisting of all functions in $C^0(\overset{\circ}{C}\mathbf{S})$ which are constant on some open neighbourhood U of the top of the cone pt and whose restriction to $\mathbf{S} \times (0, 1)$ is in $\mathbf{C}(\mathbf{S} \times (0, 1))$. By arguments similar to those used, for example, on the cone over a compact manifold, one shows that $(\overset{\circ}{C}\mathbf{S}, \mathbf{C})$ is a $(k+1)$-dimensional stratifold.

Example 8: If $(\mathbf{S}, \mathbf{C})$ and $(\mathbf{S}', \mathbf{C}')$ are k-dimensional stratifolds, we define the **topological sum** whose underlying topological space is the disjoint union $\mathbf{S} + \mathbf{S}'$ (which is by definition $\mathbf{S} \times \{0\} \cup \mathbf{S}' \times \{1\}$) and whose algebra is given by those functions whose restriction to $\mathbf{S}$ is in $\mathbf{C}$ and to $\mathbf{S}'$ is in $\mathbf{C}'$. It is obvious that this is a k-dimensional stratifold.

Example 9: The following construction allows an inductive construction of stratifolds. We will not use it in this book (so the reader can skip it), but it provides a rich class of stratifolds. Let $(\mathbf{S}, \mathbf{C})$ be an n-dimensional stratifold and W a k-dimensional smooth manifold together with a collar $\mathbf{c} : \partial W \times [0, \epsilon) \to W$. We assume that $k > n$. Let $f : \partial W \to \mathbf{S}$ be a morphism, which we call attaching map. We further assume that the attaching map f is proper, which in our context is equivalent to requiring that the preimages of compact sets are compact. Then we define a new space $\mathbf{S}'$ by gluing W to

$\mathbf{S}$ via f:

$$\mathbf{S}' := W \cup_f \mathbf{S}.$$

On this space, we consider the algebra $\mathbf{C}'$ consisting of those functions $g : \mathbf{S}' \to \mathbb{R}$ whose restriction to $\mathbf{S}$ is in $\mathbf{C}$, whose restriction to $\overset{\circ}{W}$ is smooth, and such that for some $\delta < \epsilon$ we have $g\mathbf{c}(x,t) = gf(x)$ for all $x \in \partial W$ and $t < \delta$. We leave it to the reader to check that $(\mathbf{S}', \mathbf{C}')$ is a k-dimensional stratifold. If $\mathbf{S}$ consists of a single point, we obtain a stratifold whose underlying space is $W/_{\partial W}$, the space obtained by collapsing ∂W to a point. If W is compact and we apply this construction, then the result agrees with the stratifold from example 2 for $\overset{\circ}{W}$. Specializing further to $W := M \times [0,1)$, where M is a closed manifold, we obtain the stratifold from example 1, the open cone over M.

Applying this construction inductively to a finite sequence of i-dimensional smooth manifolds W_i with compact boundary equipped with collars and morphisms $f_i : \partial W_i \to \mathbf{S}^{i-1}$, where $\mathbf{S}^{i-1}$ is inductively constructed from (W_0, f_0), ... , (W_{i-1}, f_{i-1}), we obtain a rich class of stratifolds. Most stratifolds occurring in "nature" are of this type. This construction is very similar to the definition of CW-complexes. There we inductively attach cells (= closed balls), whereas here we attach arbitrary manifolds. Thus on the one hand it is more general, but on the other hand more special, since we require that the attaching maps are morphisms.

In this context it is sometimes useful to remember the data in this construction: the collars and the attaching maps. More precisely we pass from the collars to equivalence classes of collars called germs of collars, where two collars are equivalent if they agree on some small neighbourhoos of the boundary. Stratifolds constructed inductively by attaching manifolds together with the data (germs of collars and attaching maps) are called **parametrized stratifolds** or **p-stratifolds**.

From now on, we often omit the algebra C from the notation of a stratifold and write S instead of (S, C) (unless we want to make the dependence on C visible). This is in analogy to smooth manifolds where the single letter M is used instead of adding the maximal atlas or, equivalently, the algebra of smooth functions to the notation.

4. Properties of smooth maps

In analogy to maps from a smooth manifold to a smooth manifold, we call the morphisms f from a stratifold $\mathbf{S}$ to a smooth manifold **smooth maps**.

We now prove some elementary properties of smooth maps.

Proposition 2.2. *Let $\mathbf{S}$ be a stratifold and $f_i : \mathbf{S} \to \mathbb{R}$ be a family of smooth maps such that supp f_j is a locally finite family of subsets of $\mathbf{S}$. Then $\sum f_i$ is a smooth map.*

Proof: The local finiteness implies that for each $x \in \mathbf{S}$, there is a neighbourhood U of x such that $\operatorname{supp} f_i \cap U = \emptyset$ for all but finitely many i_1, ..., i_k. Then it is clear that $\sum f_i|_U = f_{i_1}|_U + ... + f_{i_k}|_U$. Since $f_{i_1} + ... + f_{i_k}$ is smooth, we conclude from the fact that

the algebra of smooth functions on $\mathbf{S}$ is locally detectable, that the map is smooth.
q.e.d.

We will now construct an important tool from differential topology, namely the existence of subordinated partitions of unity. This will make the role of the bump functions clear.

Recall that a **partition of unity** is a family of functions $\rho_i : \mathbf{S} \to \mathbb{R}_{\geq 0}$ such that their supports form a locally finite covering of $\mathbf{S}$ and $\sum \rho_i = 1$. It is called subordinated to some covering of $\mathbf{S}$, if for each i the support $\operatorname{supp} \rho_i$ is contained in one of the covering sets.

PROPOSITION 2.3. *Let* $\mathbf{S}$ *be a stratifold with an open covering. Then there is a subordinated partition of unity of smooth functions (called smooth partition of unity).*

Proof: The argument is similar to that for smooth manifolds ([**B-J**], p. 66). We choose a sequence of compact subspaces $A_i \subset \mathbf{S}$ such that $A_i \subset \overset{\circ}{A}_{i+1}$ and $\cup A_i = \mathbf{S}$. Such a sequence exists since $\mathbf{S}$ is locally compact and has a countable basis ([**Sch**], p. 81). For each $x \in A_{i+1} - \overset{\circ}{A}_i$ we choose U from our covering such that $x \in U$ and take a smooth bump function $\rho^i_x : \mathbf{S} \to \mathbb{R}_{\geq 0}$ with $\operatorname{supp} \rho^i_x \subset (\overset{\circ}{A}_{i+2} - A_{i-1}) \cap U$. Since $A_{i+1} - \overset{\circ}{A}_i$ is compact, there is a finite number of points x_ν such that $(\rho^i_{x_\nu})^{-1}(0,\infty)$ covers $A_{i+1} - \overset{\circ}{A}_i$. From Proposition 2.2 we know that $s := \sum_{i,\nu} \rho^i_{x_\nu}$ is a smooth function and $\rho^i_{x_\nu}/s$ is the desired subordinated partition of unity.
q.e.d.

As a consequence, we note that $\mathbf{S}$ is a paracompact space.

To demonstrate the use of this result, we give the following standard application.

PROPOSITION 2.4. *Let* $A \subset \mathbf{S}$ *be a closed subset of a stratifold* $\mathbf{S}$*,* U *an open neighbourhood of* A *and* $f : U \to \mathbb{R}$ *a smooth function. Then there is a smooth function* $g : \mathbf{S} \to \mathbb{R}$ *such that* $g|_A = f|_A$*.*

Proof: The subsets U and $\mathbf{S} - A$ form an open covering of $\mathbf{S}$. Consider a subordinated partition of unity $\rho_i : \mathbf{S} \to \mathbb{R}_{\geq 0}$. Then we define for $x \in U$

$$g(x) := \sum_{\operatorname{supp} \rho_i \subset U} \rho_i(x) f(x),$$

where for $x \notin U$ we define $g(x) = 0$.
q.e.d.

Another useful consequence is that if Y is a subspace of $\mathbf{S}$ then $\mathbf{C}(Y)$ is equal to the restriction of elements of $\mathbf{C}$ to Y.

Proposition 2.5. *Let Y be a closed subspace of $\mathbf{S}$. Then $\mathbf{C}(Y)$ is equal to the restriction of elements of $\mathbf{C}$ to Y.*

Proof: By definition $f : Y \to \mathbb{R}$ is in $\mathbf{C}(Y)$ if and only if for each $y \in Y$ there is $g_Y \in \mathbf{C}$ and an open neighbourhood U_y of X in $\mathbf{S}$ such that $f|_{U_y \cap Y} = g|_{U_y \cap Y}$. Since Y is closed, the subsets U_y for $y \in Y$ and $\mathbf{S} - Y$ form an open covering of $\mathbf{S}$. Let $\rho_i : \mathbf{S} \to \mathbb{R}$ be a subordinated partition of unity of smooth functions. Then for each i there is an $y(i)$ such that supp $\rho_i \subset U_{y(i)}$ or supp $\rho_i \subseteq \mathbf{S} - Y$. We consider the smooth function

$$F := \sum_{\text{supp } \rho_i \subset U_{y(i)}} \rho_i g_{y(i)}$$

For $z \in Y$ we have

$$F(z) = \sum_{\text{supp } \rho_i \subset U_{y(i)}} \rho_i(z) g_{y(i)}(z) = \sum_i \rho_i(z) f(z) = f(z).$$

Here we have used that for $z \in Y$ if supp $\rho_i \subset U_{y(i)}$, then supp $\rho_i \subset \mathbf{S} - Y$ and so $\rho_i(z) = 0$ and that if $\rho_i(z) \neq 0$, then $g_{y(i)}(z) = f(z)$.
q.e.d.

5. Consequences of Sard's Theorem

One of the most useful fundamental results in differential topology is Sard's Theorem ([**B-J**], p. 58, [**Hi**], p. 69) which implies that the regular values of a smooth map are dense (Brown's Theorem). As an immediate consequence of Sard's theorem for manifolds, we obtain a generalization of Brown's Theorem to stratifolds.

We recall that if $f : M \to N$ is a smooth map between smooth manifolds, then $x \in N$ is called a regular value of f if the differential df_y is surjective for each $y \in f^{-1}(x)$.

Definition: *Let $f : \mathbf{S} \to M$ be a smooth map from a stratifold to a smooth manifold. We say that $x \in M$ is a* **regular value** *of f, if x is a regular value of $f|_{\mathbf{S}^i}$ for all i.*

Let $f : M \to N$ be a smooth map between smooth manifolds. The image of a point $y \in M$ where the differential is not surjective is called a critical value. Sard's theorem says that the critical values have measure zero. This implies that its complement, the set of regular values is dense (Brown's theorem). Since a finite union of sets of measure zero has measure zero, we deduce the following generalization of Brown's Theorem:

Proposition 2.6. *Let $g : \mathbf{S} \to M$ be a smooth map. The set of regular values of g is dense in M.*

Regular values x of smooth maps $f : M \to N$ have the useful property that $f^{-1}(x)$, the set of solutions, is a smooth manifold of dimension $\dim M - \dim N$. An analogous result holds for a smooth map $g : \mathbf{S} \to M$, where $\mathbf{S}$ is a stratifold of dimension n and M a smooth manifold without boundary of dimension m. Consider a regular value $x \in M$. By 2.5 we can define $\mathbf{C}_{f^{-1}(x)}$ as the restriction of the smooth functions of $\mathbf{S}$ to $f^{-1}(x)$.

Proposition 2.7. *Let* $\mathbf{S}$ *be a* k*-dimensional stratifold,* M *an* m*-dimensional smooth manifold,* $g : \mathbf{S} \to M$ *be a smooth map and* $x \in M$ *a regular value. Then* $(g^{-1}(x), \mathbf{C}(g^{-1}(x)))$ *is a* $k - m$*-dimensional stratifold.*

Proof: We note that for each $y \in g^{-1}(x)$ the differential $dg_y : T_y\mathbf{S} \to T_xM$ as defined at the end of chapter 1 is surjective. This uses the property that $T_y\mathbf{S} = T_y\mathbf{S}^i$ if $y \in \mathbf{S}^i$. From this we conclude that $\dim T_y g^{-1}(x) \leq \dim T_y\mathbf{S} - m$. On the other hand, $T_y g^{-1}(x)$ contains the subspace $T_y((g|_{\mathbf{S}^i})^{-1}(x))$ and so the dimension must be equal:

$$\dim T_y g^{-1}(x) = \dim T_y\mathbf{S} - m.$$

Thus $g^{-1}(x)^{i-m} = (g|_{\mathbf{S}^i})^{-1}(x)$, the stratification is induced from the stratification of $\mathbf{S}$.

The topological conditions of a stratifold are obvious. To show condition 1, we have to prove that

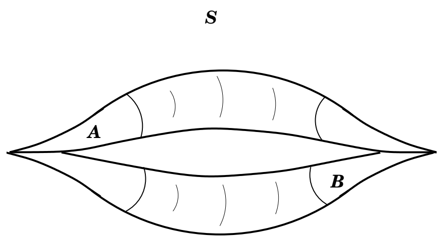
$$\mathbf{C}(g^{-1}(x))_y \to C^\infty(g^{-1}(x)^{i-m})_y$$

$$[f] \mapsto [f|_{g^{-1}(x)^{i-m}}]$$

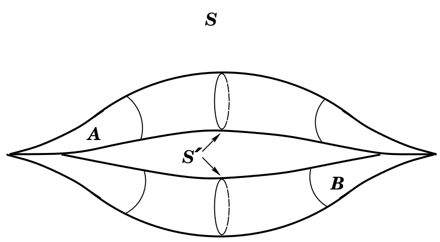
is an isomorphism. We give an inverse by applying Proposition 2.1 to choose a local retract $r : U \to V$ of $\mathbf{S}$ near y. The morphism gr is a local extension of $g|_V$ and $g|_U$ is another extension implying by condition 1, that there exists a neighbourhood U' of y such that $gr|_{U'} = g|_{U'}$. Thus $r|_{g^{-1}(x)\cap U'} : g^{-1}(x) \cap U' \to g^{-1}(x)^{i-m}$ is a morphism. Now we obtain an inverse of $\mathbf{C}(g^{-1}(x))_y \to C^\infty(g^{-1}(x)^{i-m})_y$ by mapping $[f] \in C^\infty(g^{-1}(x)^{i-m})$ to $fr|_{g^{-1}(x)\cap U'}]$. We have to show that $[fr|_{g^{-1}(x)\cap U'}]$ is in $\mathbf{C}(g^{-1}(x))_y$, i.e. is the restriction of an element of $\mathbf{C}_y$. But since $g^{-1}(x)^{i-m}$ is a smooth submanifold of $\mathbf{S}^i$, we can extend $[f]$ to a germ $[\hat{f}] \in (\mathbf{S}^i)_y$ and so $[\hat{f}r]$ is in $\mathbf{C}_y$ and $[\hat{f}r|_{g^{-1}(x)}] = [fr|_{g^{-1}(x)\cap U'}]$. Since $r|_{g^{-1}(x)\cap U'}$ is a local retract, the map $[f] \mapsto [fr|_{g^{-1}(x)\cap U'}]$ is an inverse of $\mathbf{C}(g^{-1}(x))_y \to C^\infty(g^{-1}(x)^{i-m})_y$.

This implies condition 1. The second condition is obvious and for condition 3 we note that bump functions are given by restriction of appropriate bump functions on $\mathbf{S}$.
q.e.d.

The next result will be very useful in the construction of homology. It answers the following natural question. Let $\mathbf{S}$ be a connected k-dimensional stratifold and A and B non-empty disjoint closed subsets of $\mathbf{S}$.



The question is whether there is a $(k-1)$-dimensional stratifold $\mathbf{S}'$ with underlying topological space $\mathbf{S}' \subset \mathbf{S} - (A \cup B)$. Then we say that $\mathbf{S}'$ separates A and B in $\mathbf{S}$.



The positive answer uses several of the results presented so far. We first note that there is a smooth function $\rho : \mathbf{S} \to \mathbb{R}$ which maps A to 1 and B to -1. Namely, since $\mathbf{S}$ is paracompact, it is normal ([**Sch**], p. 95) and thus there are disjoint open neighbourhoods U of A and V of B. Defining f as 1 on U and -1 on V the existence of ρ follows from Proposition 2.4.

Now we apply Proposition 2.6 to see that the regular values of ρ are dense. Thus we can choose a regular value $t \in (-1, 1)$. Proposition 2.7 implies that $\rho^{-1}(t)$ separates A and B. Thus we have proved a separation result:

PROPOSITION 2.8. *Let* $\mathbf{S}$ *be a* k*-dimensional connected stratifold and* A *and* B *disjoint closed non-empty subsets of* $\mathbf{S}$.

Then there is a non-empty $(k-1)$*-dimensional stratifold* $\mathbf{S}'$ *with* $\mathbf{S}' \subset \mathbf{S} - (A \cup B)$. *In other words,* $\mathbf{S}'$ *separates* A *and* B *in* $\mathbf{S}$.

CHAPTER 3

Stratifolds with boundary: c-stratifolds

Stratifolds are generalizations of smooth manifolds without boundary, but we also want to be able to define stratifolds with boundary. To motivate the idea of this definition, we recall that a smooth manifold W with boundary has a collar, which is a diffeomorphism $\mathbf{c} : \partial W \times [0, \epsilon) \to V$, where V is an open neighbourhood of ∂W in W, and $\mathbf{c}|_{\partial W} = \mathrm{id}_{\partial W}$. Collars are useful for many constructions such as gluing of manifolds. This makes it plausible to add a collar to the definition of a manifold with boundary as additional structure. Actually it is enough to consider the germ of collars. We call a smooth manifold together with a germ of collars a c-**manifold**. Our stratifolds with boundary will be defined as stratifolds together with a germ of collars, and so we call them c-**stratifolds**.

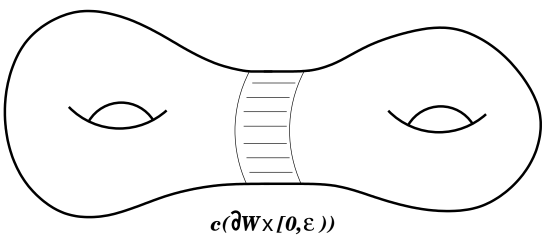
Staying with smooth manifolds for a while, we observe that we can define manifolds which are equipped with a collar as follows. We consider a topological space W together with a closed subspace ∂W. We denote $W - \partial W$ by $\overset{\circ}{W}$ and call it the interior. We assume that $\overset{\circ}{W}$ and ∂W are smooth manifolds of dimension n and $n-1$.

Definition: *Let* $(W, \partial W)$ *be a pair as above. A* **collar** *is a homeomorphism*

$$\mathbf{c} : U_\epsilon \to V,$$

where $\epsilon > 0$*,* $U_\epsilon := \partial W \times [0, \epsilon)$*, and* V *is an open neighbourhood of* ∂W *in* W *such that* $c|_{\partial W \times \{0\}}$ *is the identity map to* ∂W *and* $\mathbf{c}|_{U - (\partial W \times \{0\})}$ *is a diffeomorphism onto* $V - \partial W$*.*

The condition requiring that $\mathbf{c}(U_\epsilon)$ is open avoids the following situation:



Namely, it guarantees that the image of $\mathbf{c}$ is an "end" of W.

What is the relation to smooth manifolds equipped with a collar? If W is a smooth manifold and $\mathbf{c}$ a collar, then we obviously obtain all the ingredients of the definition above by considering W as a topological space. In turn, if $(W, \partial W, \mathbf{c})$ is given as in the definition above, we can in an obvious way extend the smooth structure of $\overset{\circ}{W}$ to a smooth manifold

W with boundary. The smooth structure on W is characterized by requiring that $\mathbf{c}$ is not only a homeomorphism but a diffeomorphism. The advantage of the definition above is that it can be given using only the language of manifolds without boundary. Thus it can be generalized to stratifolds.

Let $(\mathbf{T}, \partial\mathbf{T})$ be a pair of topological spaces. We denote $\mathbf{T} - \partial\mathbf{T}$ by $\overset{\circ}{\mathbf{T}}$ and call it the interior. We assume that $\overset{\circ}{\mathbf{T}}$ and $\partial\mathbf{T}$ are stratifolds of dimension n and $n-1$ and that $\partial\mathbf{T}$ is a closed subspace.

Definition: *Let* $(\mathbf{T}, \partial\mathbf{T})$ *be a pair as above. A* **collar** *is a homeomorphism*

$$\mathbf{c} : U_\epsilon \to V,$$

where $\epsilon > 0$*,* $U_\epsilon := \partial\mathbf{T} \times [0, \epsilon)$*, and* V *is an open neighbourhood of* $\partial\mathbf{T}$ *in* $\mathbf{T}$ *such that* $\mathbf{c}|_{\partial\mathbf{T}\times\{0\}}$ *is the identity map to* $\partial\mathbf{T}$ *and* $\mathbf{c}|_{U_\epsilon - (\partial\mathbf{T}\times\{0\})}$ *is an isomorphism of stratifolds onto* $V - \partial\mathbf{T}$*.*

Perhaps this definition needs some explanation. By examples 5 and 6 in §1 the open subset $U_\epsilon - (\partial\mathbf{T} \times \{0\})$ can be considered as a stratifold. Similarly, $V - \partial\mathbf{T}$ is an open subset of $\mathbf{T}$ and thus, by example 5 in §2, it can be considered as a stratifold.

We are only interested in a germ of collars, an equivalence class of collars where two collars $\mathbf{c} : U_\epsilon \to V$ and $\mathbf{c}' : U'_{\epsilon'} \to V'$ are called equivalent if there is a $\delta < \min\{\epsilon, \epsilon'\}$, such that $\mathbf{c}|_{U_\delta} = \mathbf{c}'|_{U_\delta}$. As usual when we consider equivalence classes, we denote the germ represented by a collar $\mathbf{c}$ by $[\mathbf{c}]$.

Now we define:

Definition: *An n-dimensional* c-**stratifold** $\mathbf{T}$ *(a collared stratifold) is a pair of topological spaces* $(\mathbf{T}, \partial\mathbf{T})$*, where* $\overset{\circ}{\mathbf{T}} = \mathbf{T} - \partial\mathbf{T}$ *is an n-dimensional stratifold and* $\partial\mathbf{T}$ *is an* $(n-1)$*-dimensional stratifold, which is a closed subspace of* $\mathbf{T}$ *together with a germ of collars* $[\mathbf{c}]$*. We call* $\partial\mathbf{T}$ *the* **boundary** *of* $\mathbf{T}$*.*

A **smooth map** *from* $\mathbf{T}$ *to a smooth manifold* M *is a continuous function* f *whose restriction to* $\overset{\circ}{\mathbf{T}}$ *and to* $\partial\mathbf{T}$ *is smooth and commutes with an appropriate representative of the germ of collars, i.e. there is an* $\epsilon > 0$ *such that* $f\mathbf{c}(x,t) = f(x)$ *for all* $x \in \partial\mathbf{T}$ *and* $t < \epsilon$*.*

We often call $\mathbf{T}$ the underlying space of the c-stratifold.

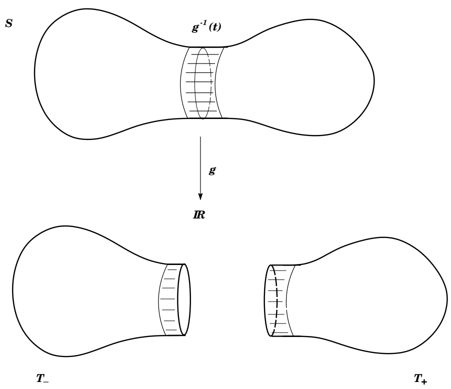
As for manifolds we allow that $\partial\mathbf{T}$ is empty. Then, of course, a c-stratifold is nothing but a stratifold (without boundary or better with an empty boundary). Thus the stratifolds are incorporated into the world of c-stratifolds as those c-stratifolds $\mathbf{T}$ with $\partial\mathbf{T} = \emptyset$.

The simplest examples of c-stratifolds are given by c-manifolds W. Here we define $\mathbf{T} = W$ and $\partial\mathbf{T} = \partial W$ and attach to $\overset{\circ}{\mathbf{T}}$ and $\partial\mathbf{T}$ the stratifold and collar structures given by the smooth manifolds. Another important class of examples is given by the product of a stratifold $\mathbf{S}$ with a c-manifold W. By this we mean the c-stratifold whose underlying topological space is $\mathbf{S} \times W$, whose interior is $\mathbf{S} \times \overset{\circ}{W}$ and the boundary is $\mathbf{S} \times \partial W$, and whose germ of collars is represented by $\mathrm{id}_{\mathbf{S}} \times \mathbf{c}$, where $[\mathbf{c}]$ is the germ of collars of W. We abbreviate this c-stratifold by $\mathbf{S} \times W$. In particular, we obtain the half open cylinder $\mathbf{S} \times [0, 1)$ or the **cylinder** $\mathbf{S} \times [0, 1]$. A third simple class of c-stratifolds is obtained by the product of a c-stratifold $\mathbf{T}$ with a smooth manifold M. The underlying topological space of this stratifold is given by $\mathbf{T} \times M$ with interior $\overset{\circ}{\mathbf{T}} \times M$ and boundary $\partial\mathbf{T} \times M$ and germ of collars $[\mathbf{c} \times \mathrm{id}_M]$, where $[\mathbf{c}]$ is the germ of collars of $\mathbf{T}$.

The next example is the (closed) cone $C(\mathbf{S})$ over a stratifold $\mathbf{S}$. The underlying topological space is the (closed) cone $\mathbf{T} := \mathbf{S} \times [0, 1]/_{\mathbf{S}\times\{0\}}$ whose interior is $\mathbf{S} \times [0, 1)/_{\mathbf{S}\times\{0\}}$ and whose boundary is $\mathbf{S} \times 1$. The collar is given by the map $\mathbf{S} \times [0, 1/2) \to C(\mathbf{S})$ mapping (x, t) to $(x, 1 - t)$.

In contrast to manifolds with boundary, where the boundary can be recognized from the underlying topological space, this is not the case with **c**-stratifolds. For example we can consider a **c**-manifold W as a stratifold without boundary with algebra $\mathbf{C}$ given by the functions which are smooth on the boundary and interior and commute with the retract given by a representative of the germ of collars. Here the strata are the boundary and the interior of W. On the other hand it is—as mentioned above—a **c**-stratifold with boundary ∂W. In both cases the smooth functions agree.

The following construction of cutting along a codimension 1 stratifold will be useful later on. Suppose in the situation of Proposition 2.7, where $g : \mathbf{S} \to \mathbb{R}$ is a smooth map to the reals with regular value t, that there is an open neighbourhood U of $g^{-1}(t)$ and an isomorphism from $g^{-1}(t) \times (t - \epsilon, t + \epsilon)$ to U for some $\epsilon > 0$, whose restriction to $g^{-1}(t) \times \{0\}$ is the identity map to $g^{-1}(t)$. Such an isomorphism is often called a **bicollar**. Then we consider the spaces $\mathbf{T}_+ := g^{-1}[t, \infty)$ and $\mathbf{T}_- := g^{-1}(-\infty, t]$. We define their boundary as $\partial\mathbf{T}_+ := g^{-1}(t)$ and $\partial\mathbf{T}_- := g^{-1}(t)$. Since $\overset{\circ}{\mathbf{T}}_+$ and $\overset{\circ}{\mathbf{T}}_-$ are open subsets of $\mathbf{S}$ they are stratifolds. The restriction of the isomorphism to $g^{-1}(t) \times [t, t + \epsilon)$ is a collar of $\mathbf{T}_+$ and the restriction of the isomorphism to $g^{-1}(t) \times (t - \epsilon, t]$ is a collar of $\mathbf{T}_-$. Thus we obtain two c-stratifolds $\mathbf{T}_+$ and $\mathbf{T}_-$. We say that $\mathbf{T}_+$ and $\mathbf{T}_-$ are obtained from $\mathbf{S}$ by **cutting along a codimension** 1 **stratifold**, namely along $g^{-1}(t)$.



Now we construct the reverse process and introduce **gluing of stratifolds along the common boundary**. Let $\mathbf{T}$ and $\mathbf{T}'$ be c-stratifolds with same boundary, $\partial\mathbf{T} = \partial\mathbf{T}'$. By passing to the minimum of ϵ and ϵ' we can assume that the domains of the collars are equal: $\mathbf{c} : \partial\mathbf{T} \times [0,\epsilon) \to V \subset \mathbf{T}$ and $\mathbf{c}' : \partial\mathbf{T}' \times [0,\epsilon) \to V \subset \mathbf{T}'$. Then we consider the topological space $\mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}'$ obtained from the disjoint union of $\mathbf{T}$ and $\mathbf{T}'$ by identifying the boundary. We have a bicollar (in the world of topological spaces), a homeomorphism $\varphi : \partial\mathbf{T} \times (-\epsilon,\epsilon) \to V \cup V'$ by mapping $(x,t) \in \partial\mathbf{T} \times (-\epsilon, 0]$ to $\mathbf{c}(x,-t)$ and $(x,t) \in \partial\mathbf{T} \times [0,\epsilon)$ to $\mathbf{c}'(x,t)$.

With respect to this underlying topological space, we define the algebra $\mathbf{C}(\mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}')$ to consist of those continuous maps $f : \mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}' \to \mathbb{R}$, such that the restrictions to $\mathring{\mathbf{T}}$ and $\mathring{\mathbf{T}}'$ are in $\mathbf{C}$ and $\mathbf{C}'$, respectively, and where the composition $f\varphi : \partial\mathbf{T} \times (-\epsilon,\epsilon) \to \mathbb{R}$ is in $\mathbf{C}(\partial\mathbf{T} \times (-\epsilon,\epsilon))$. It is easy to see that $\mathbf{C}(\mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}')$ is a locally detectable algebra. Since condition (2) in the definition of differential spaces is obviously fulfilled, we have a differential space. Clearly, $\mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}'$ is a locally compact Hausdorff space with countable basis. The conditions 1. - 3. in the definition of a stratifold are local conditions. Since they hold for $\mathring{\mathbf{T}}$, $\mathring{\mathbf{T}}'$, and $\partial\mathbf{T} \times (-\epsilon,\epsilon)$ and φ is an isomorphism, they hold for $\mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}'$. Thus $(\mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}', \mathbf{C}(\mathbf{T} \cup_{\partial\mathbf{T}=\partial\mathbf{T}'} \mathbf{T}'))$ is a stratifold.

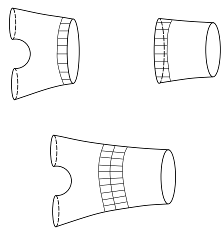
One can generalize the context of the above construction by assuming only the existence of an isomorphism g: $\partial\mathbf{T} \to \partial\mathbf{T}'$ rather than $\partial\mathbf{T} = \partial\mathbf{T}'$. Then we glue the spaces via g to obtain a space $\mathbf{T} \cup_g \mathbf{T}'$. If we replace in the definition of the algebra the homeomorphism φ by $\varphi : \partial\mathbf{T} \times (-\epsilon,\epsilon) \to V \cup V'$ mapping $(x,t) \in \partial\mathbf{T} \times (-\epsilon,0]$ to $\mathbf{c}(x,-t)$ and $(x,t) \in \partial\mathbf{T} \times [0,\epsilon)$ to $\mathbf{c}'(g(x),t)$, then we obtain a locally detectable algebra $\mathbf{C}(\mathbf{T} \cup_g \mathbf{T}')$. The same arguments as above used for $g = \mathrm{id}$ imply that $(\mathbf{T} \cup_g \mathbf{T}', \mathbf{C}(\mathbf{T} \cup_g \mathbf{T}'))$ is a stratifold. We summarize this as:

PROPOSITION 3.1. *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be k-dimensional c-stratifolds and* $g : \partial\mathbf{T} \to \partial\mathbf{T}'$ *be an isomorphism. Then*

$$(\mathbf{T} \cup_g \mathbf{T}', \mathbf{C}(\mathbf{T} \cup_g \mathbf{T}'))$$

is a k-dimensional stratifold.

Of course, if g is an isomorphism between some components of the boundary of $\mathbf{T}$ and some components of the boundary of $\mathbf{T}'$, we can glue as above via g to obtain a c-stratifold, whose boundary is the union of the complements of these boundary components.



Finally we note that if $f : \mathbf{T} \to \mathbb{R}$ is a smooth function and s is a regular value of $f|_{\overset{\circ}{\mathbf{T}}}$ and $f|_{\partial \mathbf{T}}$, then $f^{-1}(s)$ is a $\mathbf{c}$-stratifold with collar given by restriction.

CHAPTER 4

$\mathbb{Z}/2$-homology

Prerequisites: We use the classification of 1-dimensional compact manifolds [**Mi 2**], Appendix.

1. Motivation of homology

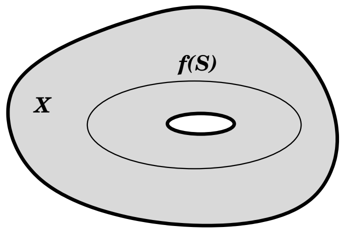
We begin by motivating the concept of a homology theory. We will construct in this book several homology theories which are all in the same spirit in the sense that they all attempt to heuristically measure the complexity of a space by analyzing the holes in X. The understanding of holes typically is: Let Y be a topological space and L a non-empty subspace. Then we say that $X := Y - L$ has the hole L. We call such a hole an **extrinsic hole** since we need to know the bigger space Y to say that X has a hole. We also want to say what it means that X has a hole without knowing Y. Such a hole we would call an **intrinsic hole**. The idea is rather simple: We try to detect holes by fishing them with a net. We throw (= map) the net into X and try to shrink the net to a point. If this is not possible, we have "caught" a hole. For example, if we consider $X = \mathbb{R}^n - 0$, then we would say that X is obtained from $\mathbb{R}^n$ by introducing the hole 0. We can detect the hole by mapping the "net" S^{n-1} to X via the inclusion. Since we cannot shrink S^{n-1} in X continuously to a point, we have "fished" the hole without using that X sits in $\mathbb{R}^n$.

This is a very flexible concept since we are free in choosing the shape of our net. In this chapter our nets will be certain compact stratifolds. Later we will consider other classes of stratifolds. Further flexibility will come from the fact that we can use stratifolds of different dimensions for detecting "holes of these dimensions". Finally, we are free in making precise what we mean by shrinking a net to a point. Here we will use a very rough criterion: We say that a net given by a map from a stratifold $\mathbf{S}$ to X can be "shrunk" to a point if there is a compact c-stratifold $\mathbf{T}$ with $\partial \mathbf{T} = \mathbf{S}$ and one can extend the map from $\mathbf{S}$ to $\mathbf{T}$, in other words, instead of shrinking the hole, we "fill" it with a compact stratifold $\mathbf{T}$.

To explain this idea further, we start again from the situation where the space X is obtained from a space Y by deleting a set L. Depending on the choice of L, this may be a very strange space. Since we are more interested in nice spaces, let us assume that L is the interior of a compact c-stratifold $\mathbf{T} \subset Y$, i.e. $L = \overset{\circ}{\mathbf{T}}$. Then we can consider the inclusion of $\partial \mathbf{T}$ into $X = Y - L$ as our net. We say that this inclusion detects the hole obtained by deleting $\overset{\circ}{\mathbf{T}}$ if we cannot extend the inclusion from $\partial \mathbf{T}$ to X to a map from $\mathbf{T}$ into X.

We now weaken our knowledge of X by assuming that it is obtained from Y by deleting the interior of some compact c-stratifold, but we do not know which one. We only know the boundary $\mathbf{S}$ of the deleted c-stratifold. Then the only way to test if we have a hole with boundary—the boundary of the deleted stratifold—is to consider **all** compact c-stratifolds $\mathbf{T}$ having the same boundary $\mathbf{S}$ and to try to extend the inclusion of the boundary to a continuous map from $\mathbf{T}$ to X. If this is impossible for all $\mathbf{T}$, then we say that X has a hole.

We have found a formulation which makes sense for arbitrary spaces X. It has a hole with the boundary shape of a compact stratifold $\mathbf{S}$ without boundary if there is an embedding of $\mathbf{S}$ into X which cannot be extended to any compact c-stratifold with boundary $\mathbf{S}$. Furthermore, instead of fishing holes only by embeddings, we consider arbitrary continuous maps from compact stratifolds $\mathbf{S}$ to X. We say that such a map fishes a hole if we cannot extend it to a continuous map of any compact c-stratifold $\mathbf{T}$ with boundary $\mathbf{S}$. Finally, we collect all these continuous maps from all compact stratifolds $\mathbf{S}$ of a fixed dimension m to X modulo those extending to a compact c-stratifold with boundary $\mathbf{S}$ into a set, and find an obvious group structure on it to obtain our first homology group denoted $\mathbf{H}_m(X;\mathbb{Z}/2)$.



The idea for introducing homology this way is essentially contained in Poincaré's original paper from 1895 [**Po**]. Instead of using the concept of stratifolds, he uses objects called "variétés". The definition of these objects is not very clear in this paper, which leads to serious difficulties. As a consequence, he suggested another combinatorial approach which turned out to be successful, and is the basis of the standard approach to homology. The original idea of Poincaré was taken up by Thom [**Th 1**] around 1950 and later on by Conner and Floyd [**C-F**] who introduced a homology theory in the spirit of Poincaré's original approach using smooth manifolds. The construction of this homology theory is very easy but computations are much harder than for ordinary homology. In this book, we use their ideas (with some technical modification) to realize Poincaré's original idea in a textbook.

2. $\mathbb{Z}/2$-oriented stratifolds

We begin with the construction of our first homology theory by following the motivation above. The elements of our first homology groups for a topological space X will be equivalence classes of certain pairs $(\mathbf{S}, g)$ of m-dimensional stratifolds $\mathbf{S}$ together with a continuous map $g : \mathbf{S} \to X$, with respect to an equivalence relation called a bordism.

But before we define bordisms, we must introduce the concept of an isomorphism between pairs $(\mathbf{S}, g)$, and $(\mathbf{S}', g')$.

Definition: *Let X be a topological space and $g : \mathbf{S} \to X$ and $g' : \mathbf{S}' \to X$ be continuous maps, where $\mathbf{S}$ and $\mathbf{S}'$ are m-dimensional stratifolds. An* **isomorphism** *from $(\mathbf{S}, g)$ to $(\mathbf{S}', g')$ is an isomorphism of stratifolds $f : \mathbf{S} \to \mathbf{S}'$ such that*

$$g = g'f.$$

If such an isomorphism exists, we call $(\mathbf{S}, g)$ and $(\mathbf{S}', g')$ isomorphic.

For a space X, the collection of pairs $(\mathbf{S}, g)$, where $\mathbf{S}$ is an m-dimensional stratifold and $g : \mathbf{S} \to X$ a continuous map, does not form a set. To see this, start with a fixed pair $(\mathbf{S}, g)$ and consider the pairs $(\mathbf{S} \times \{i\}, g)$, where i is an arbitrary index. For example, we could take i to be any set. Thus, there are at least as many pairs as sets and the class of all sets is not a set. But the isomorphism classes form a set.

PROPOSITION 4.1. *The isomorphism classes of pairs $(\mathbf{S}, g)$ form a set.*

The proof of this Proposition does not help with the understanding of homology. Thus we have postponed it to the end of Appendix A (as we have done with other proofs which are more technical and whose understanding is not needed for reading the rest of the book).

We introduce the relation which leads to homology groups. Given two pairs $(\mathbf{S}_1, g_1)$ and $(\mathbf{S}_2, g_2)$, we introduce the sum

$$(\mathbf{S}_1, g_1) + (\mathbf{S}_2, g_2) := (\mathbf{S}_1 + \mathbf{S}_2, g_1 + g_2),$$

where $g_1 + g_2 : \mathbf{S}_1 + \mathbf{S}_2 \to X$ is the disjoint sum of the maps g_1 and g_2. If $\mathbf{T}$ is a c-stratifold and $f : \mathbf{T} \to X$ a map, we abbreviate

$$\partial(\mathbf{T}, f) := (\partial\mathbf{T}, f|_{\partial\mathbf{T}}).$$

We will now characterize "certain" stratifolds from which we would like to construct our homology. There are two conditions we impose: $\mathbb{Z}/2$-orientability and regularity.

Definition: *We call an n-dimensional c-stratifold $\mathbf{T}$ with boundary $\mathbf{S} = \partial\mathbf{T}$ (we allow the possibility that $\partial\mathbf{T}$ is empty)* $\mathbb{Z}/2$**-oriented** *if $(\overset{\circ}{\mathbf{T}})^{n-1} = \emptyset$, i. e. if the stratum of codimension 1 is empty.*

We note that if $(\overset{\circ}{\mathbf{T}})^{n-1} = \emptyset$, then $\mathbf{S}^{n-2} = \emptyset$. The reason is that via c we have an embedding of $U = \mathbf{S}^{n-2} \times (0, \epsilon)$ into $(\overset{\circ}{\mathbf{T}})^{n-1}$ and so $U = \emptyset$ if $(\overset{\circ}{\mathbf{T}})^{n-1} = \emptyset$. But if $U = \emptyset$ then also $\mathbf{S}^{n-2} = \emptyset$. Thus the boundary of a $\mathbb{Z}/2$-oriented stratifold is itself $\mathbb{Z}/2$-oriented.

Remark: *It is not clear at this moment what the notion $\mathbb{Z}/2$-oriented has to do with our intuitive imagination of orientation (knowing what is "left" and "right"). For a connected closed smooth manifold, we know what "oriented" means* [**B-J**]. *If M is a closed* *manifold, this concept can be translated to a homological condition using integral homology. It is equivalent to the existence of the so called fundamental class. Our definition of $\mathbb{Z}/2$-oriented stratifolds guarantees that a closed smooth manifold always has a $\mathbb{Z}/2$-fundamental class as we shall explain later.*

3. Regular stratifolds

We distinguish another class of stratifolds by imposing a further local condition.

Definition: *A stratifold* $\mathbf{S}$ *is called a* **regular stratifold** *if for each* $x \in \mathbf{S}^i$ *there is an open neighborhood* U *of* x *in* $\mathbf{S}$*, a stratifold* $\mathbf{F}$ *with* $\mathbf{F}^0$ *a single point pt, an open subset* V *of* $\mathbf{S}^i$*, and an isomorphism*

$$\varphi : V \times \mathbf{F} \to U,$$

whose restriction to $V \times pt$ *is the identity.*

To obtain a feeling for this condition, we look at some examples. We note that a smooth manifold is a regular stratifold. If $\mathbf{S}$ is a regular stratifold and M a smooth manifold, then $\mathbf{S} \times M$ is a regular stratifold. Namely for $(x, y) \in \mathbf{S} \times M$, we consider an isomorphism $\varphi : V \times \mathbf{F} \to U$ near x for $\mathbf{S}$ as above and then

$$\varphi \times \mathrm{id} : (V \times \mathbf{F}) \times M \to U \times M$$

is an isomorphism near (x, y). Thus $\mathbf{S} \times M$ is a regular stratifold. A similar consideration shows that the product $\mathbf{S} \times \mathbf{S}'$ of two regular stratifolds $\mathbf{S}$ and $\mathbf{S}'$ is regular.

Another example of a regular stratifold is the open cone over a compact smooth manifold M. More generally, if $\mathbf{S}$ is a regular stratifold, then the open cone $\overset{\circ}{C}\mathbf{S}$ is a regular stratifold. Namely, since the open subset $\mathbf{S} \times (0, 1)$ is by the considerations above a regular stratifold, it remains to check the condition for the 0-stratum, but this is clear (we can take $U = \mathbf{F} = \overset{\circ}{C}\mathbf{S}$ and $V = \mathrm{pt}$).

It is obvious that the topological sum of two regular stratifolds is regular.

Thus the constructions of stratifolds using regular stratifolds from the examples in chapter 2 lead to regular stratifolds.

It is also obvious that gluing of regular stratifolds as explained in Proposition 3.1 leads to regular stratifolds. The reason is that points in the gluing look locally like points in either $\overset{\circ}{\mathbf{T}}$, $\overset{\circ}{\mathbf{T}}{}'$ or $\partial T \times (-\epsilon, \epsilon)$. Since these stratifolds are regular and regularity is a local condition, the statement follows.

Finally, if $\mathbf{S}$ is a regular stratifold and $f : \mathbf{S} \to \mathbb{R}$ is a smooth map with regular value t, then $f^{-1}(t)$ is a regular stratifold. To see this, it is enough to consider the local situation near x in $\mathbf{S}^i$ and use φ to consider the case, where the stratifold is $V \times \mathbf{F}$ for some i-dimensional manifold V and $\mathbf{F}^0$ is a point pt. Now we consider the maps $(f|_{V\times \mathrm{pt}})\, p_1$, where p_1 is the projection to V, and f and note that they agree on $V \times \mathrm{pt}$, which is the i-stratum of $V \times \mathbf{F}$. By condition 1b.) of a stratifold there is some open neighbourhood W of pt in $\mathbf{F}$ such that the maps agree on $V \times W$. Thus $f|_{V\times W} = (f|_{V\times \mathrm{pt}})\, p$, where p is the projection from $V \times W$ to V. Since t is a regular value of $f|_{V\times \mathrm{pt}}$, we see that $f^{-1}(t) \cap (V \times W) = f|^{-1}_{V\times \mathrm{pt}}(t) \times W$ showing that the conditions of a regular stratifold are fulfilled. Since we will apply this result in the next chapter, we summarize this as

Proposition 4.2. *Let $\mathbf{S}$ be a regular stratifold, $f : \mathbf{S} \to \mathbb{R}$ a smooth function and t a regular value. Then $f^{-1}(t)$ is a regular stratifold.*

The main reason for introducing regular stratifolds in our context is the following result. A regular point of a smooth map is a point x in $\mathbf{S}$ such that the derivative at x is non-zero.

Proposition 4.3. *Let $\mathbf{S}$ be a regular stratifold. Then the regular points of a smooth map $f : \mathbf{S} \to \mathbb{R}$ are an open subset of $\mathbf{S}$. If in addition $\mathbf{S}$ is compact, the set of regular values is open.*

Proof: To see the first statement consider a regular point $x \in \mathbf{S}^i$. Since $\mathbf{S}^i$ is a smooth manifold and the regular points of a smooth map on a smooth map are open (use the continuity of the determinant to see this), there is an open neighbourhood U of x in $\mathbf{S}^i$ consisting of regular points. Since $\mathbf{S}$ is regular, there is an open neighbourhood U_x of x in $\mathbf{S}$ isomorphic to $V \times \mathbf{F}$, where $V \subset U$ is an open neighbourhood of x in $\mathbf{S}^i$, such that f corresponds on $V \times \mathbf{F}$ to a map which commutes with the projection from $V \times \mathbf{F}$ to V (this uses the fact that a smooth map has locally a unique germ of extensions to an open neighbourhood). But for a map which commutes with this projection a point $(x, y) \in V \times \mathbf{F}$ is a regular point if and only if x is a regular point of $f|_V$. Since V is contained in U and U consists of regular points, U_x consists of regular points only finishing the proof of the first statement.

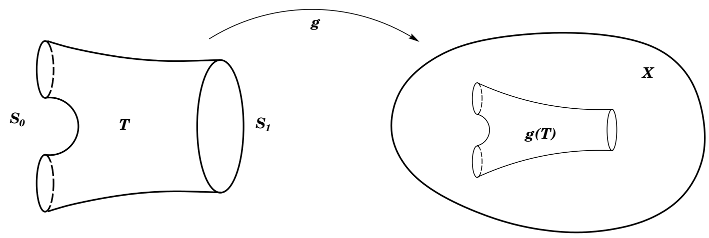
If the regular points are an open set then the singular points, which are the complement, are a closed set. If $\mathbf{S}$ is compact, the singular points are compact, and so the image under f is closed implying that the regular values are open.
q.e.d.

A c-stratifold $\mathbf{T}$ is called **regular** if $\overset{\circ}{\mathbf{T}}$ and $\partial\mathbf{T}$ are regular.

4. $\mathbb{Z}/2$-homology

We call a c-stratifold $\mathbf{T}$ **compact** if the underlying space $\mathbf{T}$ is compact. Since $\partial\mathbf{T}$ is a closed subset of $\mathbf{T}$, the boundary of a compact regular stratifold is compact.

Definition: *Two pairs $(\mathbf{S}_0, g_0)$ and $(\mathbf{S}_1, g_1)$, where $\mathbf{S}_i$ are compact m-dimensional $\mathbb{Z}/2$-oriented regular stratifolds and $g_i : \mathbf{S}_i \to X$ are continuous maps, are called* **bordant** *if* *there is a compact $m+1$-dimensional $\mathbb{Z}/2$-oriented regular c-stratifold* $\mathbf{T}$*, and a continuous map* $g : \mathbf{T} \to X$ *such that* $(\partial \mathbf{T}, g) = (\mathbf{S}_0, g_0) + (\mathbf{S}_1, g_1)$*. The pair* $(\mathbf{T}, g)$ *is called a* **bordism** *between* $(\mathbf{S}_0, g_0)$ *and* $(\mathbf{S}_1, g_1)$*.*



We will later see why we imposed the condition that the stratifolds are $\mathbb{Z}/2$-oriented and regular (the latter condition can be replaced by other conditions as long as Proposition 4.3 holds). If we would not require that the c-stratifolds are compact, we would obtain a single bordism class, since otherwise we could use $(\mathbf{S} \times [0, \infty), gp)$ as a bordism between $(\mathbf{S}, g)$ and the empty stratifold.

Proposition 4.4. *"Bordant" is an equivalence relation and the topological sum*

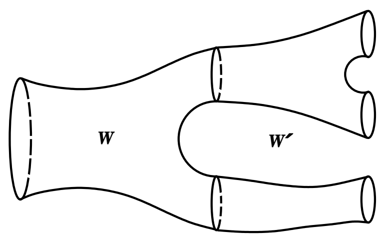
$$(\mathbf{S}_0, g_0) + (\mathbf{S}_1, g_1) := (\mathbf{S}_0 + \mathbf{S}_1, g_0 + g_1)$$

induces the structure of an abelian group on the set of equivalence classes. This group is denoted by $\mathbf{H}_m(X; \mathbb{Z}/2)$*, the m-th* **singular homology group with** $\mathbb{Z}/2$**-coefficients** *or shortly* $\mathbb{Z}/2$**-homology***. As usual, we denote the equivalence class represented by* $(\mathbf{S}, g)$ *by* $[\mathbf{S}, g]$*.*

Proof: $(\mathbf{S}, g)$ is bordant to $(\mathbf{S}, g)$ via the bordism $(\mathbf{S} \times [0,1], h)$, where $h(x,t) = g(x)$. We call this bordism the cylinder over $(\mathbf{S}, g)$. We observe that if $\mathbf{S}$ is $\mathbb{Z}/2$-oriented and regular, then $\mathbf{S} \times [0,1]$ is $\mathbb{Z}/2$-oriented and regular. Thus the relation is reflexive.

The relation is obviously symmetric.

To show transitivity we consider a bordism $(\mathbf{T}, g)$ between $(\mathbf{S}_0, g_0)$ and $(\mathbf{S}_1, g_1)$ and $(\mathbf{T}', g')$ a bordism between $(\mathbf{S}_1, g_1)$ and $(\mathbf{S}_2, g_2)$, where $\mathbf{T}$, $\mathbf{T}'$ and all $\mathbf{S}_i$ are regular $\mathbb{Z}/2$-oriented stratifolds. We glue $\mathbf{T}$ and $\mathbf{T}'$ along $\mathbf{S}_1$ as explained in Proposition 3.1. The result is regular and $\mathbb{Z}/2$-oriented. The boundary of $\mathbf{T} \cup_{\mathbf{S}_1} \mathbf{T}'$ is $\mathbf{S}_0 + \mathbf{S}_2$. Since g and g' agree on $\mathbf{S}_1$, they induce a map $g \cup g' : \mathbf{T} \cup_{\mathbf{S}_1} \mathbf{T}' \to X$, whose restriction to $\mathbf{S}_0$ is g_0 and to $\mathbf{S}_2$ is g_2. Thus $(\mathbf{S}_0, g_0)$ and $(\mathbf{S}_2, g_2)$ are bordant, and the relation is transitive.



Next, we check that the equivalence classes form an abelian group with respect to the topological sum. We first note that if $(\mathbf{S}_1, g_1)$ and $(\mathbf{S}_2, g_2)$ are isomorphic, then they are bordant. A bordism is given by gluing the cylinders $(\mathbf{S}_1 \times [0,1], h)$ and $(\mathbf{S}_2 \times [0,1], h)$ via the isomorphism considered as a map from $(\mathbf{S}_1 \times \{1\})$ to $(\mathbf{S}_2 \times \{0\})$ (as explained after Proposition 3.1). Since the isomorphism classes of pairs $(\mathbf{S}, g)$ are a set and isomorphic pairs are bordant, the bordism classes are a quotient set of the isomorphism classes, and thus are a set.

All axioms of an abelian group on $\mathbf{H}_m(X; \mathbb{Z}/2)$ for the composition given by the topological sum are rather obvious. The topological sum is associative and commutative. An element $(\mathbf{S}, g)$ represents the zero element if an only if there is a bordism $(\mathbf{T}, h)$ with $\partial(\mathbf{T}, h) = (\mathbf{S}, g)$. The inverse of $[\mathbf{S}, g]$ is given by $[\mathbf{S}, g]$ again, since $[\mathbf{S}, g] + [\mathbf{S}, g]$ is the boundary of $(\mathbf{S} \times [0, 1], h)$, the cylinder over $(\mathbf{S}, g)$.
q.e.d.

Remark: *By the last argument, each element $[\mathbf{S}, g]$ in $\mathbf{H}_m(X; \mathbb{Z}/2)$ is 2-torsion, i.e. $2[\mathbf{S}, g] = 0$. In other words, $\mathbf{H}_m(X; \mathbb{Z}/2)$ is a vector space over the field $\mathbb{Z}/2$.*

Here we abbreviate the quotient group $\mathbb{Z}/2\mathbb{Z}$, which is a field, as $\mathbb{Z}/2$. Later we will define $\mathbf{H}_m(X; \mathbb{Q})$, which will be a $\mathbb{Q}$-vector space. This indicates the role of $\mathbb{Z}/2$ in the notation of homology groups.

To obtain a feeling for homology groups, we compute $\mathbf{H}_0(\mathrm{pt}; \mathbb{Z}/2)$, the 0-th homology group of a point. A 0-dimensional stratifold is the same as a 0-dimensional manifold, and a 1-dimensional c-stratifold that is $\mathbb{Z}/2$-oriented, is the same as a 1-dimensional manifold with a germ of collars since the codimension-1 stratum is empty and there is only one possible non-empty stratum. We recall from ([**Mi 2**], Appendix) that a compact 1-dimensional manifold W with boundary has an even number of boundary points. Thus the number of points modulo 2 of a closed 0-dimensional manifold is a bordism invariant. On the other hand, an even number of points is the boundary of a disjoint union of intervals. We conclude:

THEOREM 4.5. *$\mathbf{H}_0(pt; \mathbb{Z}/2) \cong \mathbb{Z}/2$, the isomorphism is given by the number of points modulo 2. The non-trivial element is $[pt, id]$.*

There is a generalization of Theorem 4.5; one can determine $\mathbf{H}_0(X; \mathbb{Z}/2)$ for an arbitrary space X. To develop this, we remind the reader of the following definition:

Definition: *A topological space X is called* **path connected** *if any two points x and y in X can be connected by a path, i.e. there is a continuous map $\alpha : [a, b] \to X$ with $\alpha(a) = x$ and $\alpha(b) = y$.*

The relation that two points are equivalent if they can be joined by a path is an equivalence relation. The equivalence classes are called the **path components** of X. A path connected space is connected (why?) but the converse is in general not true (why?) although it is, for example, true for manifolds (why?).

The number of path components is an interesting invariant of a topological space. It can be computed via homology. Recall that since all elements of $\mathbf{H}_m(X;\mathbb{Z}/2)$ are 2-torsion, we consider $\mathbf{H}_m(X;\mathbb{Z}/2)$ as a $\mathbb{Z}/2$-vector space.

THEOREM 4.6. *The number of path components of a topological space X is equal to* $\dim_{\mathbb{Z}/2} \mathbf{H}_0(X;\mathbb{Z}/2)$. *A basis of* $\mathbf{H}_0(X,\mathbb{Z}/2)$ *(as $\mathbb{Z}/2$ vector space) is given by the homology classes $[pt, g_i]$, where g_i maps the point to an arbitrary point of the i-th path component of X.*

Proof: We recall that $\mathbb{Z}/2$-oriented c-stratifolds of dimension ≤ 1 are the same as manifolds with a germ of collars. Choose for each path component X_i a point x_i in X_i. Then we consider the bordism class $\alpha_i := [\mathrm{pt}, x_i]$, where the latter means the 0-dimensional manifold pt together with the map mapping this point to x_i. We claim that the bordism classes α_i form a basis of $\mathbf{H}_0(X;\mathbb{Z}/2)$. This follows from the definition of path components and bordism classes once we know that for points x and y in X, we have $[\mathrm{pt}, x] = [\mathrm{pt}, y]$ if and only if there is a path joining x and y. Obviously if x and y can be joined by a path, then $[\mathrm{pt}, x] = [\mathrm{pt}, y]$. Conversely, if there is a bordism between $[\mathrm{pt}, x]$ and $[\mathrm{pt}, y]$, we consider the path components of this bordism that have a non-empty boundary. We know that each path component is homeomorphic to $[0, 1]$ ([**Mi 2**], Appendix). Since the boundary consists of two points, there can be only one path component with non-empty boundary. Then this bordism is a path joining x and y.
q.e.d.

As one can see from the proof, this result is more or less a tautology. Nevertheless, it turns out that the interpretation of the number of path components as the dimension of $\mathbf{H}_0(X;\mathbb{Z}/2)$ is very useful. We will develop methods for the computation of $\mathbf{H}_0(X;\mathbb{Z}/2)$ which involve also higher homology groups $\mathbf{H}_k(X;\mathbb{Z}/2)$ for $k > 0$ and apply them, for example, to prove a sort of Jordan separation theorem in §6.

One of the main reasons for introducing singular homology groups is that one can use them to distinguish spaces. To compare the singular homology of different spaces we define **induced maps**.

Definition: *For a continuous map $f : X \to Y$, define $f_* : \mathbf{H}_m(X;\mathbb{Z}/2) \to \mathbf{H}_m(Y;\mathbb{Z}/2)$ by $f_*([\mathbf{S}, g]) := [\mathbf{S}, fg]$.*

By construction, this is a group homomorphism. The following property is an immediate consequence of the definition.

PROPOSITION 4.7. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then*

$$(gf)_* = g_* f_*$$

and

$$id_* = id.$$

One says that $\mathbf{H}_m(X;\mathbb{Z}/2)$ together with the induced maps f_* is a **functor** (which means that the two properties of Proposition 4.7 are fulfilled). The functor properties imply that if $f : X \to Y$ is a homeomorphism, then $f_* : \mathbf{H}_m(X;\mathbb{Z}/2) \to \mathbf{H}_m(Y;\mathbb{Z}/2)$ is an isomorphism. The reason is that $(f^{-1})_*$ is an inverse since $(f^{-1})_* f_* = (f^{-1}f)_* = \mathrm{id}_* = \mathrm{id}$ and similarly $f_*(f^{-1})_* = \mathrm{id}$.

We earlier motivated the idea of homology by fishing a hole using a continuous map $g : \mathbf{S} \to X$. It is plausible that a deformation of g detects the same hole. These deformations play an important role in homology. The precise definition of a deformation is the notion of **homotopy**.

Definition: *Two continuous maps f and f' between topological spaces X and Y are called* **homotopic** *if there is a continuous map $h : X \times I \to Y$ such that $h|_{X\times\{0\}} = f$ and $h|_{X\times\{1\}} = f'$. Such a map h is called a* **homotopy** *from f to f'.*

One should think of a homotopy as a continuous family of maps $h_t : X \to Y$, $x \mapsto h(x,t)$ joining f and f'. Homotopy is an equivalence relation between maps which we often denote by $\simeq$. Namely, $f \simeq f$ with homotopy $h(x,t) = f(x)$. If $f \simeq f'$ via h this implies $f' \simeq f$ via $h'(x,t) := h(x, 1-t)$. If $f \simeq f'$ via h and $f' \simeq f''$ via h' then $f \simeq f''$ via

$$h''(x,t) := \begin{cases} h(x,2t) \text{ for } 0 \le t \le 1/2 \\ h'(x,2t-1) \text{ for } 1/2 \le t \le 1 \end{cases}$$

The reader should check that this map is continuous.

The set of all continuous maps between given topological spaces is huge and hard to analyze. Often one is only interested in those properties of a map which are unchanged under deformations. This is the reason for introducing the homotopy relation.

As suggested above, $\mathbb{Z}/2$-homology cannot distinguish homotopic objects. This is made precise in the next result which is one of the fundamental properties of homology and is given the name **homotopy axiom**:

PROPOSITION 4.8. *Let f and f' be homotopic maps from X to Y. Then*

$$f_* = f'_* : \mathbf{H}_m(X;\mathbb{Z}/2) \to \mathbf{H}_m(Y;\mathbb{Z}/2).$$

Proof: Let $h : X \times I \to Y$ be a homotopy between maps f and f' from X to Y. Consider $[\mathbf{S}, g] \in \mathbf{H}_m(X)$. Then the cylinder $(\mathbf{S} \times [0,1], h \circ (g \times \mathrm{id}))$ is a bordism between $(\mathbf{S}, fg)$ and $(\mathbf{S}, f'g)$, and thus $f_*[\mathbf{S}, g] = f'_*[\mathbf{S}, g]$.
q.e.d.

We mentioned above that homeomorphisms induce isomorphisms between $\mathbf{H}_m(X;\mathbb{Z}/2)$ and $\mathbf{H}_m(Y;\mathbb{Z}/2)$. This can be generalized by introducing **homotopy equivalences**. We say that a continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ such that gf and fg are homotopic to Id on X and Y. Such a map g is called a **homotopy inverse** of f. Roughly, a homotopy equivalence is a deformation from one space to another. For example, the inclusion $i : S^m \to \mathbb{R}^{m+1} - \{0\}$ is a homotopy equivalence with homotopy inverse given by $g : x \mapsto x/||x||$. We have $gi = id$ and $h(x,t) = tx + (1-t)\frac{x}{||x||}$ is a homotopy between ig and Id. As explained above, a homotopy equivalence induces an isomorphism in singular bordism:

PROPOSITION 4.9. *A homotopy equivalence $f : X \to Y$ induces isomorphisms $f_* : \mathbf{H}_k(X;\mathbb{Z}/2) \to \mathbf{H}_k(Y;\mathbb{Z}/2)$ for all k.*

The reason is that if g is a homotopy inverse of f then g_* is an inverse of f_*.

A space is called **contractible** if it is homotopy equivalent to a point. For example, $\mathbb{R}^n$ is contractible. Thus for contractible spaces one has an isomorphism between $\mathbf{H}_n(X;\mathbb{Z}/2)$ and $\mathbf{H}_n(\mathrm{pt};\mathbb{Z}/2)$. This gives additional motivation to determine the higher homology groups of a point. The answer is very simple:

THEOREM 4.10. *For $n > 0$ we have*

$$\mathbf{H}_n(pt;\mathbb{Z}/2) = 0.$$

Proof: Since there is only the constant map to the space consisting of a single point we can omit the maps in our bordism classes if the space X is a point. Thus we have to show that each $\mathbb{Z}/2$-oriented compact regular stratifold $\mathbf{S}$ of dimension > 0 is the boundary of a $\mathbb{Z}/2$-oriented compact regular c-stratifold $\mathbf{T}$. There is an obvious candidate, the closed cone $C\mathbf{S}$ defined in §2. This is obviously $\mathbb{Z}/2$-oriented since $\mathbf{S}$ is $\mathbb{Z}/2$-oriented and the dimension of $\mathbf{S}$ is > 0. (If $\dim \mathbf{S} = 0$, then the 0-dimensional stratum, the top of the cone, is not empty in a 1-dimensional stratifold, and so the cone is not $\mathbb{Z}/2$-oriented!) We have seen already that the cone is regular if $\mathbf{S}$ is regular.
q.e.d.

This is a good place to see the effect of restricting to $\mathbb{Z}/2$-oriented stratifolds. If we considered arbitrary stratifolds, then even in dimension 0 the homology group of a point would be trivial. But if all homology groups of a point are zero, then— at least for nice spaces—their homology groups would also be zero. This follows from the Mayer-Vietoris sequence which we will introduce in the next chapter. Similarly, the homology groups would be uninteresting if we would not require that the stratifolds are compact since, for instance, the half open cylinder $\mathbf{S} \times [0, 1)$ could be taken to show that $[\mathbf{S}]$ is zero in the homology of a point.

CHAPTER 5

The Mayer-Vietoris sequence and homology groups of spheres

1. The Mayer-Vietoris sequence

While on the one hand the definition of $\mathbf{H}_n(X;\mathbb{Z}/2)$ is elementary and intuitive, on the other hand it is hard to imagine how one can compute these groups. We will prove in this chapter a rather effective method which—in combination with the homotopy axiom (4.8)—often will allow us to reduce the computation to $\mathbf{H}_m(\mathrm{pt};\mathbb{Z}/2)$. We will discuss interesting applications of these computations in the next chapter.

The method for reducing $\mathbf{H}_n(X;\mathbb{Z}/2)$ to $\mathbf{H}_m(\mathrm{pt};\mathbb{Z}/2)$ is based on Propositions 4.7, 4.8 and 4.9, and the following long exact sequence. To formulate the method, we have to introduce the notation of **exact sequences**. A sequence of homomorphisms between abelian groups

$$\cdots \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \dots$$

is called **exact** if for each n, holds: $\ker f_{n-1} = \operatorname{im} f_n$.

For example,

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

is exact where the map $\mathbb{Z} \to \mathbb{Z}/2$ is the reduction mod 2. The zeros on the left and right side mean in combination with exactness that the map $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ is injective and $\mathbb{Z} \to \mathbb{Z}/2$ is surjective, which is clearly the case. The exactness in the middle means that the kernel of the reduction mod 2 is the image of the multiplication by 2, which is also clear.

To get a feeling for exact sequences, we observe that if we have an exact sequence

$$A \xrightarrow{0} B \xrightarrow{f} C \xrightarrow{0} D,$$

then f is injective (following from the 0-map on the left side) and surjective (following from the 0-map on the right side). Thus, in this situation, f is an isomorphism. Of course, if there is only a 0 on the left, then f is only injective, and if there is only a 0 on the right, then f is only surjective.

Another elementary but useful consequence of exactness concerns sequences of abelian groups where each group is a finite dimensional vector space over a field K and the maps are linear maps. If

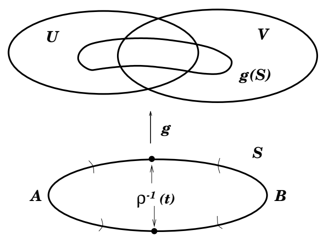
$$0 \to A_n \to A_{n-1} \cdots \to A_1 \to A_0 \to 0$$

is an exact sequence of finite dimensional K-vector spaces and linear maps, then:

$$\sum_{i=0,\dots,n} (-1)^i \dim A_i = 0,$$

the alternating sum of the dimensions is 0. The reader is recommended to prove this elementary exercise in linear algebra.

To formulate the method, we consider the following situation. Let U and V be open subsets of a space X. We want to relate the homology groups of U, V, $U \cap V$ and $U \cup V$. To do so, we need maps between the homology groups of these spaces. There are some obvious maps induced by the different inclusions. In addition, we need a less obvious map, the so-called boundary operator $d : \mathbf{H}_m(U \cup V; \mathbb{Z}/2) \to \mathbf{H}_{m-1}(U \cap V; \mathbb{Z}/2)$. We begin with its description. Consider an element $[\mathbf{S}, g] \in \mathbf{H}_m(U \cup V; \mathbb{Z}/2)$. We note that $A := g^{-1}(X - V)$ and $B := g^{-1}(X - U)$ are disjoint closed subsets of $\mathbf{S}$.



By arguments similar to the proof of Proposition 2.8, there is a separating stratifold $\mathbf{S}' \subset \mathbf{S} - (A \cup B)$ of dimension $m-1$ (the picture above explains the idea of the proof of 2.8, where $\mathbf{S}' = \rho^{-1}(t)$ for a smooth function $\rho : \mathbf{S} \to \mathbb{R}$ with $\rho(A) = 1$ and $\rho(B) = -1$ and t a regular value) and we define

$$d([\mathbf{S}, g]) := [\mathbf{S}', g|_{\mathbf{S}'}].$$

We will show in Appendix B (the proof is purely technical and plays no essential role in understanding homology) that this construction gives a well defined map

$$d : \mathbf{H}_m(U \cup V; \mathbb{Z}/2) \to \mathbf{H}_{m-1}(U \cap V; \mathbb{Z}/2).$$

If we apply this construction to a topological sum, it leads to the topological sum of the corresponding pairs and so this map is a homomorphism.

Proposition 5.1. *The construction above assigning to* $(\mathbf{S}, g)$ *the pair* $(\mathbf{S}', g|_{\mathbf{S}'})$ *gives a well defined homomorphism*

$$d : \mathbf{H}_m(U \cup V; \mathbb{Z}/2) \to \mathbf{H}_{m-1}(U \cap V; \mathbb{Z}/2).$$

This map is called the **boundary operator**.

Now we can give the fundamental tool for relating homology groups of a space X to those of a point:

THEOREM 5.2. *For open subsets U and V of X the following sequence (***Mayer-Vietoris sequence***) is exact:*

$$\ldots \mathbf{H}_n(U \cap V;\mathbb{Z}/2) \to \mathbf{H}_n(U;\mathbb{Z}/2) \oplus \mathbf{H}_n(V;\mathbb{Z}/2) \to \mathbf{H}_n(U \cup V;\mathbb{Z}/2) \xrightarrow{d} \mathbf{H}_{n-1}(U \cap V;\mathbb{Z}/2) \to \mathbf{H}_{n-1}(U;\mathbb{Z}/2) \oplus \mathbf{H}_{n-1}(V;\mathbb{Z}/2) \to$$

It commutes with induced maps.

Here the map $\mathbf{H}_n(U \cap V;\mathbb{Z}/2) \to \mathbf{H}_n(U;\mathbb{Z}/2) \oplus \mathbf{H}_n(V;\mathbb{Z}/2)$ is $\alpha \mapsto ((i_U)_*(\alpha),(i_V)_*(\alpha))$, the map $\mathbf{H}_n(U;\mathbb{Z}/2) \oplus \mathbf{H}_n(V;\mathbb{Z}/2) \to \mathbf{H}_n(U \cup V;\mathbb{Z}/2)$ is $(\alpha,\beta) \mapsto (j_U)_*(\alpha) - (j_V)_*(\beta)$.

We give some explanation. The maps i_U and i_V are the inclusions from $U \cap V$ to U and V, the maps j_U and j_V are the inclusions from U and V to $U \cup V$. The sequence extends arbitrarily far to the left and ends as

$$\cdots \to \mathbf{H}_0(U;\mathbb{Z}/2) \oplus \mathbf{H}_0(V;\mathbb{Z}/2) \to \mathbf{H}_0(U \cup V;\mathbb{Z}/2) \to 0$$

on the right side. Finally, the last condition in the definition means that if we have a space X' with open subspaces U' and V' and a continuous map $f : X \to X'$ with $f(U) \subset U'$ and $f(V) \subset V'$, then the diagram

$$\begin{array}{ccccccccc}
\ldots \to & \mathbf{H}_n(U \cap V;\mathbb{Z}/2) & \longrightarrow & \mathbf{H}_n(U;\mathbb{Z}/2) \oplus \mathbf{H}_n(V;\mathbb{Z}/2) & \longrightarrow & \mathbf{H}_n(U \cup V;\mathbb{Z}/2) & \longrightarrow & \mathbf{H}_{n-1}(U \cap V;\mathbb{Z}/2) \to \ldots \\
 & \downarrow (f|_{U \cap V})_* & & \downarrow (f|_U)_* \oplus (f|_V)_* & & \downarrow (f|_{U \cup V})_* & & \downarrow (f|_{U \cap V})_* \\
\cdots \to & \mathbf{H}_n(U' \cap V';\mathbb{Z}/2) & \longrightarrow & \mathbf{H}_n(U';\mathbb{Z}/2) \oplus \mathbf{H}_n(V';\mathbb{Z}/2) & \longrightarrow & \mathbf{H}_n(U' \cup V';\mathbb{Z}/2) & \longrightarrow & \mathbf{H}_{n-1}(U \cap V;\mathbb{Z}/2) \to \ldots
\end{array}$$

commutes. That is to say that the two compositions of maps going from the upper left corner to the lower right corner in any rectangle agree.

The reader might wonder why we have taken the difference map $(j_U)_*(\alpha) - (j_V)_*(\beta)$ instead of the sum $(j_U)_*(\alpha) + (j_V)_*(\beta)$, which is equivalent in our situation since for all homology classes $\alpha \in \mathbf{H}_m(X;\mathbb{Z}/2)$ we have $\alpha = -\alpha$. The reason is that a similar sequence exists for other homology groups (the Mayer-Vietoris sequence is actually one of the basic axioms for a homology theory as will be explained later) where the elements do not have order 2, and thus one has to take the difference map to obtain an exact sequence. We actually will give the proof in such a way that it will extend verbally to the other main homology groups in this book, integral homology, so that we don't have to repeat the argument.

The idea of the proof of Theorem 5.2 is very intuitive but there are some technical points which make it a bit lengthy. We give now a short proof explaining the fundamental steps. The understanding of this short proof is very helpful for getting in general a feeling for homology theories. In Appendix B we add the details which are unnecessary to study for a first reading of the book.

Short proof of Theorem 5.2: We will show the exactness of the Mayer-Vietoris sequence step by step. We first recall that a sequence

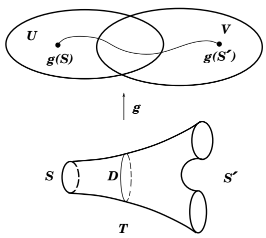
$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if $gf = 0$ (i.e. $\operatorname{im} f \subset \ker\, g$) and $\ker\, g \subset \operatorname{im} f$.

The first case is:

$$\mathbf{H}_n(U \cap V; \mathbb{Z}/2) \to \mathbf{H}_n(U; \mathbb{Z}/2) \oplus \mathbf{H}_n(V; \mathbb{Z}/2) \to \mathbf{H}_n(U \cup V; \mathbb{Z}/2)$$

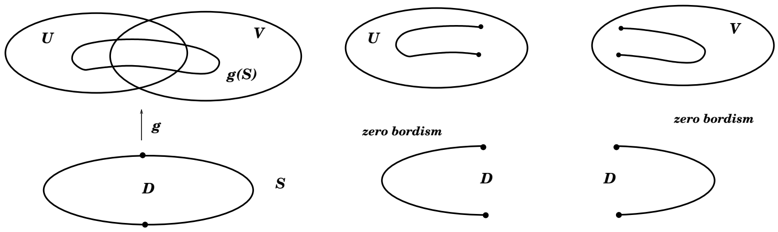
Obviously the composition of the two maps is zero. To show the other inclusion, we consider $[\mathbf{S}, g] \in \mathbf{H}_n(U; \mathbb{Z}/2)$ and $[\mathbf{S}', g'] \in \mathbf{H}_n(V; \mathbb{Z}/2)$ that map to zero in $\mathbf{H}_n(U \cup V; \mathbb{Z}/2)$. Let $(\mathbf{T}, h)$ be a zero bordism of $[\mathbf{S}, j_U g] - [\mathbf{S}', j_V g']$. Then we separate $\mathbf{T}$ using Proposition 2.8 along a compact regular stratifold $\mathbf{D}$ with $h(\mathbf{D}) \subset U \cap V$. We will show in the detailed proof that we actually can choose $\mathbf{T}$ such that there is an open neighbourhood U of $\mathbf{D}$ in $\mathbf{T}$ and an isomorphism of $\mathbf{D} \times (-\epsilon, \epsilon)$ to U, which on $\mathbf{D} \times \{0\}$ is the identity map. In other words: there exists a bicollar (this is where we apply the property that homology classes consist of regular stratifolds). Then—as explained in §4—we can cut along $\mathbf{D}$ to obtain a bordism $(\mathbf{T}_-, h|_{\mathbf{T}_-})$ between $(\mathbf{S}, g)$ and $(\mathbf{D}, h|_{\mathbf{D}})$ as well as a bordism $(\mathbf{T}_+, g|_{\mathbf{T}_+})$ between $(\mathbf{D}, h|_{\mathbf{D}})$ and $(\mathbf{S}', g')$. Thus $[\mathbf{D}, h|_{\mathbf{D}}] \in \mathbf{H}_n(U \cap V; \mathbb{Z}/2)$ maps to $([\mathbf{S}, g], [\mathbf{S}', g']) \in \mathbf{H}_n(U; \mathbb{Z}/2) \oplus \mathbf{H}_n(V; \mathbb{Z}/2)$.



Next we consider the exactness of

$$\mathbf{H}_n(U \cup V; \mathbb{Z}/2) \xrightarrow{d} \mathbf{H}_{n-1}(U \cap V; \mathbb{Z}/2) \to \mathbf{H}_{n-1}(U; \mathbb{Z}/2) \oplus \mathbf{H}_{n-1}(V; \mathbb{Z}/2).$$

The composition of the two maps is zero. For this we show in the detailed proof that as above we can choose a representative for the homology class in $U \cup V$ such that we can cut along the separating manifold defining the boundary operator. The argument is demonstrated in the following picture.



The other inclusion is again self-explanatory demonstrated by the same pictures read in reverse order, where instead of cutting we glue.

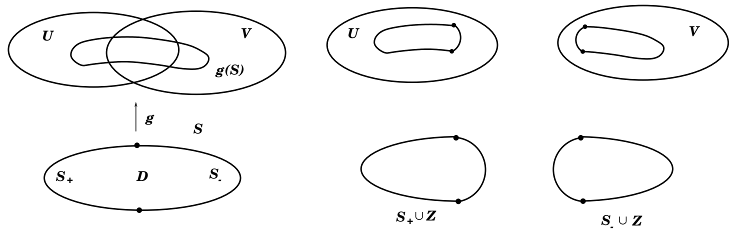
Finally, we prove exactness of

$$\mathbf{H}_n(U;\mathbb{Z}/2)\oplus\mathbf{H}_n(V;\mathbb{Z}/2)\to\mathbf{H}_n(U\cup V;\mathbb{Z}/2)\xrightarrow{d}\mathbf{H}_{n-1}(U\cap V;\mathbb{Z}/2).$$

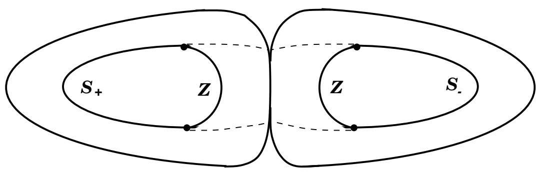
If $[\mathbf{S},g]\in\mathbf{H}_n(U;\mathbb{Z}/2)$ we show $d(j_U)_*[\mathbf{S},g]=0$. This is obvious by the construction of the boundary operator since we can choose ρ and the regular value t such that the separating regular stratifold $\mathbf{D}$ is empty. By the same argument $d(j_V)_*$ is the trivial map.

To show the other inclusion we consider $[\mathbf{S},g]\in\mathbf{H}_n(U\cup V;\mathbb{Z}/2)$ with $d([\mathbf{S},g])=0$. We will show in Appendix B that we can choose $(\mathbf{S},g)$ in such a way that the regular stratifold $\mathbf{S}$ is obtained from two regular c-stratifolds $\mathbf{S}_+$ and $\mathbf{S}_-$ with same boundary $\mathbf{D}$ by gluing them along $\mathbf{D}$. Furthermore we have $g(\mathbf{S}_+)\subset U$ and $g(\mathbf{S}_-)\subset V$.

If $d([\mathbf{S},g])=0$, there is a compact regular c-stratifold $\mathbf{Z}$ with $\partial\mathbf{Z}=\mathbf{D}$ and an extension of $g|_{\mathbf{D}}$ to $r:\mathbf{Z}\to U\cap V$. We glue $\mathbf{S}_+$ and $\mathbf{S}_-$ to $\mathbf{Z}$ to obtain $\mathbf{S}_+\cup_{\mathbf{D}}\mathbf{Z}$ and $\mathbf{S}_-\cup_{\mathbf{D}}\mathbf{Z}$, and map the first to U via $g|_{\mathbf{S}_+}\cup r$ and the second to V via $g|_{\mathbf{S}_-}\cup r$. This gives an element of $\mathbf{H}_n(U;\mathbb{Z}/2)\oplus\mathbf{H}_n(V;\mathbb{Z}/2)$.



We are finished if in $U\cup V$ the difference of these two bordism classes is equal to $[\mathbf{S},g]$. For this we take $(\mathbf{S}_+\cup_{\mathbf{D}}\mathbf{Z})\times[0,1]$ and $(\mathbf{S}_-\cup_{\mathbf{D}}\mathbf{Z})\times[1,2]$ and paste it together along $\mathbf{Z}\times 1$.



We will show in Appendix A that this can be given the structure of a regular c-stratifold with boundary $\mathbf{S}_+\cup_{\mathbf{D}}\mathbf{Z}+\mathbf{S}_-\cup_{\mathbf{D}}\mathbf{Z}+\mathbf{S}_+\cup_{\mathbf{D}}\mathbf{S}_-$. Since $\mathbf{S}_+\cup_{\mathbf{D}}\mathbf{S}_-=\mathbf{S}$ and our maps extend to a map from this regular c-stratifold to $U\cup V$, we have a bordism between $[\mathbf{S}_+\cup_{\mathbf{D}}\mathbf{Z},g|_{\mathbf{S}_+}\cup r]+[\mathbf{S}_-\cup_{\mathbf{D}}\mathbf{Z},g|_{M_-}\cup r]$ and $[\mathbf{S},g]$.
q. e. d.

As an application we compute the homology groups of a topological sum. Let X and Y be topological spaces and $X+Y$ the topological sum (the disjoint union). Then X and Y are open subspaces of $X+Y$ and we denote them by U and V. Since the intersection $U\cap V$ is the empty set and the homology groups of the empty set are 0 (this is a place where it is necessary to allow the empty set as k-dimensional stratifold whose corresponding class is of course 0) the Mayer-Vietoris sequence gives short exact sequences:

$$0 \to \mathbf{H}_k(X;\mathbb{Z}/2) \oplus \mathbf{H}_k(Y;\mathbb{Z}/2) \to \mathbf{H}_k(X+Y;\mathbb{Z}/2) \to 0,$$

where the zeroes on the left and right side correspond to $\mathbf{H}_n(\emptyset;\mathbb{Z}/2) = 0$ and $\mathbf{H}_{n-1}(\emptyset;\mathbb{Z}/2) = 0$, respectively. The map in the middle is $(j_X)_* - (j_Y)_*$. As explained above, the exactness implies that this map is an isomorphism:

$$(j_X)_* - (j_Y)_* : \mathbf{H}_n(X;\mathbb{Z}/2) \oplus \mathbf{H}_n(Y;\mathbb{Z}/2) \to \mathbf{H}_n(X+Y;\mathbb{Z}/2)$$

is an isomorphism. Of course, this also implies that the sum $(j_X)_* + (j_Y)_*$ is an isomorphism.

2. Reduced homology groups and homology of spheres

For computations it is often easier to split the homology groups into the homology groups of a point and the "rest", which will be called reduced homology. Let $p : X \to pt$ be the constant map to the space consisting of a single point. The n-th **reduced homology group** is $\tilde{\mathbf{H}}_n(X;\mathbb{Z}/2) := \ker\,(p_* : \mathbf{H}_n(X;\mathbb{Z}/2) \to \mathbf{H}_n(pt;\mathbb{Z}/2))$. A continuous map $f : X \to Y$ induces a homomorphism on the reduced homology groups $\tilde{\mathbf{H}}_n(X;\mathbb{Z}/2)$ to $\tilde{\mathbf{H}}_n(Y;\mathbb{Z}/2)$ by restriction to the kernels and we denote it again by $f_* : \tilde{\mathbf{H}}_n(X;\mathbb{Z}/2) \to \tilde{\mathbf{H}}_n(Y;\mathbb{Z}/2)$. If X is non-empty, there is a simple relation between the homology and the reduced homology: $\mathbf{H}_n(X;\mathbb{Z}/2)$ is isomorphic to $\tilde{\mathbf{H}}_n(X;\mathbb{Z}/2) \oplus \mathbf{H}_n(pt;\mathbb{Z}/2)$, where the isomorphism maps a homology class $a \in \mathbf{H}_n(X;\mathbb{Z}/2)$ to $(a - i_*p_*(a), p_*a)$, where i is the inclusion from pt to an arbitrary point in X:

$$\mathbf{H}_n(X;\mathbb{Z}/2) \cong \tilde{\mathbf{H}}_n(X;\mathbb{Z}/2) \oplus \mathbf{H}_n(pt;\mathbb{Z}/2),$$

which for $n > 0$ means that the reduced homology is the same as the unreduced homology, but for $n = 0$ it differs by a summand $\mathbb{Z}/2$.

Since it is often useful to work with reduced homology, it would be nice to know if there is also a Mayer-Vietoris sequence for reduced homology. This is the case. We prepare the argument by developing a useful algebraic result. Consider a commutative diagram (meaning that the composition of any two maps starting from the same group and ending in the same group are equal) of abelian groups and homomorphisms

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 \end{array}$$

where the horizontal sequences are exact and the map h_1 is surjective. Then we consider the sequence

$$\ker\, h_1 \xrightarrow{f_1|} \ker\, h_2 \xrightarrow{f_2|} \ker\, h_3 \xrightarrow{f_3|} \ker\, h_4$$

where the maps $f_i|$ are $f_i|_{\ker\, h_i}$. The statement is that the sequence

$$\ker\, h_2 \xrightarrow{f_2|} \ker\, h_3 \xrightarrow{f_3|} \ker\, h_4$$

is again exact. This is proved by a general method called **diagram chasing**, which we introduce in proving this statement. We chase in the commutative diagram given by A_i and B_j above. The first step is to show that $\operatorname{im} f_2| \subset \ker f_3|$ or equivalently $(f_3|)(f_2|) = 0$. This follows since $f_3 f_2 = 0$. To show that $\ker f_3| \subset \operatorname{im} f_2|$, we start the chasing by considering $x \in \ker h_3$ with $f_3(x) = 0$. By exactness of the sequence given by the A_i, there is $y \in A_2$ with $f_2(y) = x$. Since $h_3(x) = 0$ and $h_3 f_2(y) = g_2 h_2(y)$, we have $g_2 h_2(y) = 0$ and thus by the exactness of the lower sequence and the surjectivity of h_1, there is $z \in A_1$ with $g_1 h_1(z) = h_2(y)$. Since $g_1 h_1(z) = h_2 f_1(z)$, we conclude $h_2(y - f_1(z)) = 0$ or $y - f_1(z) \in \ker h_2$. Since $f_2 f_1(z) = 0$, we are done since we have found $y - f_1(z) \in \ker h_2$ with $f_2(y - f_1(z)) = f_2(y) = x$.

With this algebraic information, we can compare the Mayer-Vietoris sequences for $X = U \cup V$ with that of the space pt given by $U' := pt =: V'$:

$$
\begin{array}{ccccccc}
\to \mathbf{H}_n(U\cap V;\mathbb{Z}/2) & \to & \mathbf{H}_n(U;\mathbb{Z}/2)\oplus\mathbf{H}_n(V;\mathbb{Z}/2) & \to & \mathbf{H}_n(X;\mathbb{Z}/2) & \to & \mathbf{H}_{n-1}(U\cap V;\mathbb{Z}/2)\to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to \mathbf{H}_n(U'\cap V';\mathbb{Z}/2) & \to & \mathbf{H}_n(U';\mathbb{Z}/2)\oplus\mathbf{H}_n(V';\mathbb{Z}/2) & \to & \mathbf{H}_n(U'\cup V';\mathbb{Z}/2) & \to & \mathbf{H}_{n-1}(U'\cap V';\mathbb{Z}/2)\to
\end{array}
$$

Since $U' \cap V' = U' = V' = U' \cup V' = pt$, all vertical maps are surjective if $U \cap V$ is non-empty (and thus U and V as well), and therefore by the argument above, the reduced Mayer-Vietoris sequence

$$\cdots \to \tilde{\mathbf{H}}_n(U\cap V;\mathbb{Z}/2) \to \tilde{\mathbf{H}}_n(U;\mathbb{Z}/2)\oplus\tilde{\mathbf{H}}_n(V;\mathbb{Z}/2) \to \tilde{\mathbf{H}}_n(X;\mathbb{Z}/2) \to \tilde{\mathbf{H}}_{n-1}(U\cap V;\mathbb{Z}/2) \to \cdots$$

is exact if $U \cap V$ is non-empty.

Now we use the homotopy axiom (Proposition 4.8) and the reduced Mayer-Vietoris sequence to express the homology of the sphere $S^m := \{x \in \mathbb{R}^{m+1} \mid ||x|| = 1\}$ in terms of the homology of a point. For this we decompose S^m into the complement of the north pole $N = (0, ..., 0, 1)$ and the south pole $S = (0, ..., 0, -1)$, and define $S^m_+ : S^m - \{S\}$ and $S^m_- := S^m - \{N\}$. The inclusion $S^{m-1} \to S^m_+ \cap S^m_-$ mapping $y \longmapsto (y, 0)$ is a homotopy equivalence with homotopy inverse $r : (x_1, \ldots, x_{m+1}) \longmapsto (x_1, \ldots, x_m)/_{||(x_1,\ldots,x_m)||}$ (why?). Both S^m_+ and S^m_- are homotopy equivalent to a point, or equivalently the identity map on these spaces is homotopic to the constant map (why?). Since $S^m_+ \cup S^m_-$ is S^m, the reduced Mayer-Vietoris sequence gives an exact sequence

$$\cdots \to \tilde{\mathbf{H}}_n(S^m_+\cap S^m_-;\mathbb{Z}/2) \to \tilde{\mathbf{H}}_n(S^m_+;\mathbb{Z}/2)\oplus\tilde{\mathbf{H}}_n(S^m_-;\mathbb{Z}/2) \to \tilde{\mathbf{H}}_n(S^m;\mathbb{Z}/2) \xrightarrow{d} \tilde{\mathbf{H}}_{n-1}(S^m_+\cap S^m_-;\mathbb{Z}/2) \to \ldots$$

or if we use the isomorphisms induced by the homotopy equivalences above.

$$\cdots \to \tilde{\mathbf{H}}_n(S^{m-1};\mathbb{Z}/2) \to \tilde{\mathbf{H}}_n(pt;\mathbb{Z}/2)\oplus\tilde{\mathbf{H}}_n(pt;\mathbb{Z}/2) \to \tilde{\mathbf{H}}_n(S^m;\mathbb{Z}/2) \xrightarrow{d} \tilde{\mathbf{H}}_{n-1}(S^{m-1};\mathbb{Z}/2) \to \ldots.$$

Since $\tilde{\mathbf{H}}_k(pt;\mathbb{Z}/2) = 0$, we obtain an isomorphism

$$d : \tilde{\mathbf{H}}_n(S^m;\mathbb{Z}/2) \xrightarrow{\cong} \tilde{\mathbf{H}}_{n-1}(S^{m-1};\mathbb{Z}/2)$$

and from this inductively:

$$\tilde{\mathbf{H}}_n(S^m;\mathbb{Z}/2) \xrightarrow{\cong} \tilde{\mathbf{H}}_{n-m}(S^0;\mathbb{Z}/2).$$

S^0 consists of two points $\{+1\}$ and $\{-1\}$ which are open subsets and by the formula above for the homology of a topological sum we have $\tilde{\mathbf{H}}_n(S^0;\mathbb{Z}/2) \cong \mathbf{H}_n(pt;\mathbb{Z}/2)$. We summarize:

THEOREM 5.3. *$\tilde{\mathbf{H}}_n(S^m;\mathbb{Z}/2) \cong \mathbf{H}_{n-m}(pt;\mathbb{Z}/2)$ or $\mathbf{H}_n(S^m;\mathbb{Z}/2) \cong \mathbf{H}_n(pt;\mathbb{Z}/2) \oplus \mathbf{H}_{n-m}(pt;\mathbb{Z}/2)$. In particular, for $m > 0$ we have for $k = 0$ or $k = m$*

$$\mathbf{H}_k(S^m;\mathbb{Z}/2) = \mathbb{Z}/2$$

and

$$\mathbf{H}_k(S^m;\mathbb{Z}/2) = 0$$

otherwise.

It is natural to ask for an explicit representative of the non-trivial element in $\mathbf{H}_m(S^m;\mathbb{Z}/2)$. For this we introduce the fundamental class of a compact $\mathbb{Z}/2$-oriented regular stratifold. Let $\mathbf{S}$ be a n-dimensional $\mathbb{Z}/2$-oriented compact regular stratifold. We define its **fundamental class** as $[\mathbf{S}]_{\mathbb{Z}/2} := [\mathbf{S}, \mathrm{id}] \in \mathbf{H}_n(\mathbf{S},\mathbb{Z}/2)$. As the name indicates this class is important. We will see that it is always non-trivial. In particular, we obtain for each compact smooth manifold the fundamental class $[M]_{\mathbb{Z}/2} = [M, \mathrm{id}]$ where M is the stratifold associated to M. In the case of the spheres the non-vanishing is clear since by the inductive computation one sees that the non-trivial element of $\mathbf{H}_m(S^m;\mathbb{Z}/2)$ is given by the fundamental class $[S^m]_{\mathbb{Z}/2}$.

As an immediate consequence of Theorem 5.3 the spheres S^n and S^m are not homotopy equivalent for $m \neq n$, for otherwise their homology groups would all be isomorphic. In particular, for $n \neq m$ the spheres are not homeomorphic. In the next chapter, we will show for arbitrary manifolds that the dimension is a homeomorphism invariant.

CHAPTER 6

Brouwer's fixed point theorem, separation and invariance of dimension

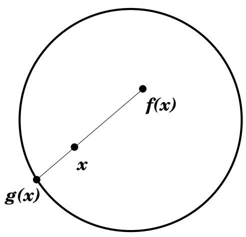
Prerequisites: The only new ingredient used in this chapter is the definition of topological manifolds which can be found either in the first pages of [**B-J**] or [**Hi**].

1. Brouwer's fixed point theorem

Let $D^n := \{x \in \mathbb{R}^n | \ ||x|| \leq 1\}$ be the **unit ball** and $B^n := \{x \in \mathbb{R}^n | \ ||x|| < 1\}$ be the **open unit ball**.

Theorem 7.1 : (Brouwer) *A continuous map $f : D^n \to D^n$ has a fixed point, i.e. there is a point $x \in D^n$ with $f(x) = x$.*

Proof: The case $n = 0$ is clear and so we assume that $n > 0$. If there is a continuous map $f : D^n \to D^n$ without fixed points, define $g : D^n \to S^{n-1}$ by mapping $x \in D^n$ to the intersection of the ray from $f(x)$ to x with S^{n-1} (give a formula for this map and see that it is continuous).



Then $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, the identity on S^{n-1}.

Now consider $\mathrm{id} = \mathrm{id}_* = (g \circ i)_* = g_* \circ i_* : \mathbf{H}_{n-1}(S^{n-1}; \mathbb{Z}/2) \xrightarrow{i_*} \mathbf{H}_{n-1}(D^n; \mathbb{Z}/2) \xrightarrow{g_*} \mathbf{H}_{n-1}(S^{n-1}; \mathbb{Z}/2)$, where $i : S^{n-1} \to D^n$ is the inclusion. By Theorem 5.3 we have $\mathbf{H}_{n-1}(S^{n-1}; \mathbb{Z}/2) = \mathbb{Z}/2$ for $n > 1$.

Thus the identity on $\mathbf{H}_{n-1}(S^{n-1}; \mathbb{Z}/2)$ is non-trivial. On the other hand, since D^n is homotopy equivalent to a point, $\mathbf{H}_{n-1}(D^n; \mathbb{Z}/2) \cong \mathbf{H}_{n-1}(\mathrm{pt}; \mathbb{Z}/2) = \{0\}$ if $n - 1 > 0$, implying a contradiction for $n > 1$. For $n = 1$ we have $\mathbf{H}_0(S^0, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\mathbf{H}_0(D^1, \mathbb{Z}/2) = \mathbb{Z}/2$ giving again a contradiction.
q.e.d.

2. A separation theorem

As an application of the relation between the number of path components of a space X and the dimension of $\mathbf{H}_0(X;\mathbb{Z}/2)$, we prove a theorem which generalizes a special case of the Jordan curve theorem. A topological manifold M is called closed if it is compact and has no boundary.

THEOREM 6.1. *Let M be a closed path connected topological manifold and $f : M \times (-\epsilon, \epsilon) \to U \subset \mathbb{R}^n$ be a homeomorphism on an open subset U of $\mathbb{R}^n$. Then $\mathbb{R}^n - f(M)$ has two path components.*

In other words, a nicely embedded closed topological manifold M of dimension $n-1$ in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into two connected components. Here "nice" means that the embedding can be extended to an embedding of $M \times (-\epsilon, \epsilon)$. If M is a smooth submanifold, then it is automatically nice [**B-J**].

Proof: Denote $\mathbb{R}^n - f(M)$ by V. Since $U \cup V = \mathbb{R}^n$ and $\mathbf{H}_1(\mathbb{R}^n;\mathbb{Z}/2) \cong \mathbf{H}_1(\mathrm{pt};\mathbb{Z}/2) = 0$ ($\mathbb{R}^n$ is contractible), the Mayer-Vietoris sequence implies

$$0 \to \mathbf{H}_0(U \cap V;\mathbb{Z}/2) \to \mathbf{H}_0(U;\mathbb{Z}/2) \oplus \mathbf{H}_0(V;\mathbb{Z}/2) \to \mathbf{H}_0(\mathbb{R}^n;\mathbb{Z}/2) \to 0$$

Now, $U \cap V$ is homeomorphic to $M \times (-\epsilon, \epsilon) - M \times \{0\}$ and thus has two path components implying that $\mathbf{H}_0$ is 2-dimensional by Theorem 4.6, and U is homeomorphic to $M \times (-\epsilon, \epsilon)$ which is path connected implying that the dimension is 1. Since the alternating sum of the dimensions is 0 we conclude $\dim_{\mathbb{Z}/2} \mathbf{H}_0(V;\mathbb{Z}/2) = 2$, which by Theorem 4.6 implies the statement of Theorem 6.1.
q.e.d.

As earlier announced, this result, although equivalent to a statement about $\mathbf{H}_0(\mathbb{R}^n - f(M);\mathbb{Z}/2)$ uses higher homology groups, namely the vanishing of $\mathbf{H}_1(\mathbb{R}^n;\mathbb{Z}/2)$.

3. Invariance of dimension

Next we want to prove the invariance of the dimension of a topological manifold under a homeomorphism. Here we only need a weak definition of an m-dimensional topological manifold M, namely that M is locally homeomorphic to an open subset of $\mathbb{R}^m$. For this we define the local homology of a space. To define the local homology of a topological space X at a point $x \in X$, we consider the space $X \cup_{X-x} C(X-x)$, the union of X and the cone over $X - x$, where $CX = X \times [0,1]/_{X \times \{0\}}$ and we identify $X \times \{1\}$ in CX with X. Equivalently, we may define it as $C(X) - (x \times (0,1))$. We define the **local homology** of X at x as $\mathbf{H}_k(X \cup C(X-x);\mathbb{Z}/2)$. We will use the local homology of a topological manifold to characterize its dimension. For this, we need the following consideration.

LEMMA 6.2. *Let M be a non-empty m-dimensional topological manifold. Then for each $x \in M$ we have*

$$\tilde{\mathbf{H}}_k(M \cup C(M-x);\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & k = m \\ 0 & \mathit{else} \end{cases}$$

Proof: Since M is non-empty, there is an $x \in M$ and so we choose a homeomorphism φ from the open ball B^m to an open neighborhood of x. We apply the Mayer-Vietoris sequence and decompose $M \cup C(M-x)$ into $U := C(M-x)$ and $V := \varphi(B^m - \{0\}) \times (\frac{1}{2}, 1] \cup \{x\}$. The projection of V to $\varphi(B^m)$ is a homotopy equivalence and so V is contractible. Also U is contractible, since it is a cone. $U \cap V$ is homotopy equivalent (again via the projection) to $\varphi(B^m - \{0\})$ and so $U \cap V$ is homotopy equivalent to S^{m-1}. The reduced Mayer-Vietoris sequence is

$$\ldots \tilde{\mathbf{H}}_k(U; \mathbb{Z}/2) \oplus \tilde{\mathbf{H}}_k(V; \mathbb{Z}/2) \to \tilde{\mathbf{H}}_k(M \cup C(M-x); \mathbb{Z}/2) \to \tilde{\mathbf{H}}_{k-1}(U \cap V; \mathbb{Z}/2) \to \ldots.$$

Since $\tilde{\mathbf{H}}_k(U; \mathbb{Z}/2)$ and $\tilde{\mathbf{H}}_k(V; \mathbb{Z}/2)$ are zero, and $\tilde{\mathbf{H}}_{k-1}(U \cap V; \mathbb{Z}/2) \cong \tilde{\mathbf{H}}_{k-1}(S^{m-1}; \mathbb{Z}/2)$ we have an isomorphism

$$\tilde{\mathbf{H}}_k(M \cup C(M-x); \mathbb{Z}/2) \cong \tilde{\mathbf{H}}_{k-1}(S^{m-1}; \mathbb{Z}/2)$$

and the statement follows from 5.3.
q.e.d.

Now we are in the situation to characterize the dimension of a non-empty topological manifold M in terms of homology. Namely by 6.2 we know that $dim(M) = m$ if and only if $\tilde{\mathbf{H}}_m(M \cup C(M-x); \mathbb{Z}/2) \neq 0$, where x is an arbitrary point in M. If $f : M \to N$ is a homeomorphism, then f can be extended to a homeomorphism $g : M \cup C(M-x) \to N \cup C(N - g(x))$ and so the corresponding local homology groups are isomorphic. Thus

$$\dim M = \dim N.$$

We summarize

THEOREM 6.3. *Let $f : M \to N$ be a homeomorphism between non-empty manifolds. Then*

$$\dim\ M = \dim\ N.$$

Remark: *Let $Y \subset X$ be a subspace, then one gives the reduced homology of $X \cup C(Y)$ often a name: it is called* **relative homology**

$$\mathbf{H}_k(X, Y; \mathbb{Z}/2) := \tilde{\mathbf{H}}_k(X \cup C(Y); \mathbb{Z}/2).$$

CHAPTER 7

$\mathbb{Z}/2$-homology of some important spaces and the Euler characteristic

1. The fundamental class

It is very useful given a space X to have some explicit non-trivial homology classes. The most important example is the fundamental class of a compact m-dimensional $\mathbb{Z}/2$-oriented regular stratifold $\mathbf{S}$ which we introduced as $[\mathbf{S}]_{\mathbb{Z}/2} := [\mathbf{S}, \mathrm{id}] \in \mathbf{H}_m(\mathbf{S}; \mathbb{Z}/2)$. We have shown that for a sphere the fundamental class is non-trivial. In the following result, we generalize this.

Proposition 7.1. *Let $\mathbf{S}$ be a compact m-dimensional $\mathbb{Z}/2$-oriented regular stratifold with $\mathbf{S}^m \neq \emptyset$. Then the fundamental class $[\mathbf{S}]_{\mathbb{Z}/2} \in \mathbf{H}_m(\mathbf{S}; \mathbb{Z}/2)$ is non-trivial.*

Proof: The 0-dimensional case is clear and so we assume that $m > 0$. We reduce the statement to the spheres. For this we consider a smooth embedding $\psi : B^m \hookrightarrow \mathbf{S}^m$, where B^m is the open unit ball, and we decompose $\mathbf{S}$ as $\psi(B^m) =: U$ and $\mathbf{S} - \psi(0) =: V$. Then $U \cap V = \psi(B^m - 0)$. We want to determine $d([\mathbf{S}]_{Z/2})$, where d is the boundary operator in the Mayer-Vietoris sequence corresponding to the covering of $\mathbf{S}$ by U and V. We choose a smooth function $\eta : [0,1] \to [0,1]$, which is 0 near 0, 1 near 1 and $\eta(t) = t$ near $1/2$, and then define $\rho : \mathbf{S} \to [0,1]$ by mapping $\psi(x)$ to $\eta(||x||)$ and $\mathbf{S} - im\,\psi$ to 1. Then $1/2$ is a regular value of ρ and by definition of the boundary operator we have

$$d([\mathbf{S}]_{Z/2}) = [\rho^{-1}(1/2), i],$$

where $i : \rho^{-1}(1/2) \to U \cap V$ is the inclusion. Thus we are finished if $[\rho^{-1}(1/2), i] \neq 0$. Since $\psi|_{\frac{1}{2}S^{m-1}}$ is a diffeomorphism from $\frac{1}{2}S^{m-1} = \{x \in \mathbb{R}^m \mid ||x|| = 1/2\}$ to $\psi(\frac{1}{2}S^{m-1}) = \rho^{-1}(1/2)$, we have

$$[\rho^{-1}(1/2), i] = \psi_*[\frac{1}{2}S^{m-1}, Id] = \psi_*[\frac{1}{2}S^{m-1}].$$

The inclusion $\rho^{-1}(1/2) \to U \cap V$ is a homotopy equivalence and thus we are finished since $[S^{m-1}] \neq 0$.
q.e.d.

2. $\mathbb{Z}/2$-homology of projective spaces

The most important geometric spaces are the classical Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, the home of **affine geometry**. It was an important breakthrough in the history of mathematics when **projective geometry** was invented. The basic idea is to add certain points at infinity to $\mathbb{R}^n$ and $\mathbb{C}^n$. The effect of this change is not so easy to describe. One important difference is that projective spaces are compact. Another is that the intersection of two hyperplanes (projective subspaces of codimension 1) is always non-empty. Many interesting spaces, in particular the projective algebraic varieties, are contained in projective spaces so that they are the "home" of algebraic geometry. In topology they play an important role for classifying line bundles and so are the heart of characteristic classes.

Many important questions can be formulated and solved using the homology (and cohomology) of projective spaces. Before we compute the homology groups, we have to define projective spaces. They are the set of all lines through 0 in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$. The lines which are not contained in $\mathbb{R}^n \times 0$ or $\mathbb{C}^n \times 0$ are in a $1-1$ correspondence with $\mathbb{R}^n$ or $\mathbb{C}^n$, where the bijection maps a point x in $\mathbb{R}^n$ or $\mathbb{C}^n$ to the line given by $(x, 1)$. Thus $\mathbb{R}^n$ resp. $\mathbb{C}^n$ are contained in $\mathbb{RP}^n$ resp. $\mathbb{CP}^n$. The lines which are contained in $\mathbb{R}^n \times 0$ or $\mathbb{C}^n \times 0$ are called points at infinity. They are parametrized by $\mathbb{RP}^{n-1}$ resp. $\mathbb{CP}^{n-1}$. Thus we obtain a decomposition of $\mathbb{RP}^n$ as $\mathbb{R}^n \cup \mathbb{RP}^{n-1}$ and $\mathbb{CP}^n$ as $\mathbb{C}^n \cup \mathbb{CP}^{n-1}$.

To see that the projective spaces are compact, we give a slightly different definition by representing a line by a vector of norm 1.

We begin with the **complex projective space** $\mathbb{CP}^m$. This is defined as a quotient space of $S^{2m+1} = \{x = (x_0, ..., x_m) \in \mathbb{C}^{m+1} | \; ||x|| = 1\}$, where $||\;||$ is the norm on $\mathbb{C}^{m+1}$, by the equivalence relation $\sim$ where $x \sim y$ if and only if there is a complex number λ such that $\lambda x = y$. In other words two points in S^{2m+1} are equivalent if they span the same line. $\mathbb{CP}^m$ is a topological manifold of dimension $2m$ and one introduces in a natural way a smooth structure [**Hi**], p. 14. Actually, here the coordinate changes are not only smooth maps but holomorphic maps, and thus $\mathbb{CP}^m$ is what one calls a complex manifold, but we don't need this structure and consider it as a smooth manifold.

To compute its homology, we decompose it into open subspaces

$$U := \{[x] \in \mathbb{CP}^m | \; x_m \neq 0\}$$

and

$$V := \{[x] \in \mathbb{CP}^m | \; |x_m| < 1\}.$$

The reader should check the following properties: U is homotopy equivalent to a point (a homotopy between the identity on U and a constant map is given by $h([x], t) := [tx_0, \dots, tx_{m-1}, x_m]$), and the inclusion from $\mathbb{CP}^{m-1}$ to V is a homotopy equivalence. A homotopy between the identity on V and a map from V to $\mathbb{CP}^{m-1}$ is given by $h([x], t) := [x_0, \dots, x_{m-1}, tx_m]$). Furthermore the intersection $U \cap V$ is homotopy equivalent to S^{2m-1}. The reason is that we actually have a homeomorphism from U to the open unit ball by mapping $[(x_0, \dots x_m)]$ to $(x_0/x_m, \dots x_{m-1}/x_m)$ and under this homeomorphism $U \cap V$ is mapped to the complement of 0, which is homotopy equivalent to S^{2m-1}.

Thus the homotopy axiom together with the Mayer-Vietoris sequence for $\mathbb{Z}/2$-homology gives an exact sequence:

$$\to \tilde{\mathbf{H}}_k(S^{2m-1}; \mathbb{Z}/2) \to \tilde{\mathbf{H}}_k(\mathbb{CP}^{m-1}; \mathbb{Z}/2) \to \tilde{\mathbf{H}}_k(\mathbb{CP}^m; \mathbb{Z}/2) \to \tilde{\mathbf{H}}_{k-1}(S^{2m-1}; \mathbb{Z}/2) \to$$

Since $\tilde{\mathbf{H}}_r(S^{2m-1}; \mathbb{Z}/2) = 0$ for $r \neq 2m-1$, we conclude inductively:

THEOREM 7.2. $\mathbf{H}_k(\mathbb{CP}^m; \mathbb{Z}/2) \cong \mathbb{Z}/2$ *for k even and $k \leq 2m$, and is* 0 *otherwise. The nontrivial homology class in $\mathbf{H}_{2n}(\mathbb{CP}^m; \mathbb{Z}/2)$ for $n \leq m$ is given by $[\mathbb{CP}^n, i]$, where i is the inclusion from $\mathbb{CP}^n$ to $\mathbb{CP}^m$.*

The last statement follows from Proposition 7.1.

To compute the homology of the **real projective space** $\mathbb{RP}^m := S^m/x \sim -x$, which is a closed smooth m-dimensional manifold ([**Hi**] p. 13), we use the same approach as for the complex projective spaces. We decompose $\mathbb{RP}^m$ as $U := \{[x] \in \mathbb{RP}^m |\ x_{m+1} \neq 0\}$ and $V := \{[x] \in \mathbb{RP}^m |\ |x_{m+1}| < 1\}$. A similar argument as above shows: U is homotopy equivalent to a point, and the inclusion from $\mathbb{RP}^{m-1}$ to V is a homotopy equivalence. Furthermore the intersection $U \cap V$ is homotopy equivalent to S^{m-1}.

The decomposition $\mathbb{RP}^m = U \cup V$ gives an exact sequence:

$$\tilde{\mathbf{H}}_k(S^{m-1}; \mathbb{Z}/2) \to \tilde{\mathbf{H}}_k(\mathbb{RP}^{m-1}; \mathbb{Z}/2) \xrightarrow{i_*} \tilde{\mathbf{H}}_k(\mathbb{RP}^m; \mathbb{Z}/2)$$
$$\to \tilde{\mathbf{H}}_{k-1}(S^{m-1}; \mathbb{Z}/2) \to \tilde{\mathbf{H}}_{k-1}(\mathbb{RP}^{m-1}; \mathbb{Z}/2)$$

This implies that for k different from m or $m-1$ the inclusion is an isomorphism $i_* : \tilde{\mathbf{H}}_k(\mathbb{RP}^{m-1}; \mathbb{Z}/2) \to \tilde{\mathbf{H}}_k(\mathbb{RP}^m; \mathbb{Z}/2)$ and that this map is injective for $k = m$, and surjective for $k = m-1$. Since by Proposition 7.1 $\mathbf{H}_m(\mathbb{RP}^m; \mathbb{Z}/2) \neq 0$, we conclude inductively:

THEOREM 7.3. $\mathbf{H}_k(\mathbb{RP}^m; \mathbb{Z}/2) \cong \mathbb{Z}/2$ *for $k \leq m$, and* 0 *otherwise. The nontrivial element in $\mathbf{H}_k(\mathbb{RP}^m; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $k \leq m$ is given by $[\mathbb{RP}^k, i]_{\mathbb{Z}/2}$ where i is the inclusion from $\mathbb{RP}^k$ to $\mathbb{RP}^m$.*

3. Betti numbers and the Euler characteristic

The Betti numbers are important invariants for topological spaces and for some topological spaces X one can use them to define the Euler characteristic.

Definition: *Let X be a topological space. The k-th* $\mathbb{Z}/2$**-Betti** *number is $b_k(X; \mathbb{Z}/2) := \dim_{\mathbb{Z}/2} \mathbf{H}_k(X; \mathbb{Z}/2)$.*

A topological space X is called $\mathbb{Z}/2$**-homologically finite**, *if for all but for finitely many k, the homology groups $\mathbf{H}_k(X; \mathbb{Z}/2)$ are zero, and finite dimensional in the remaining cases.*

For a $\mathbb{Z}/2$-homologically finite space X, we define the **Euler characteristic** *as* $e(X) := \sum_i (-1)^i b_i(X;\mathbb{Z}/2)$.

We will prove at the end of this chapter that all compact smooth manifolds are $\mathbb{Z}/2$-homologically finite. Thus their Euler characteristic can be defined.

The computations in the previous section imply:

i) Suppose $m > 0$. Then $b_k(S^m;\mathbb{Z}/2) = 1$ for $k = 0$ or $k = m$ and 0 otherwise. Thus $e(S^m) = 2$ for m even and $e(S^m) = 0$ for m odd.

ii) $b_k(\mathbb{CP}^m;\mathbb{Z}/2) = 1$ for k even and $0 \le k \le 2m$ and $b_k(\mathbb{CP}^m;\mathbb{Z}/2) = 0$ else. Thus $e(\mathbb{CP}^m) = m + 1$.

iii) $b_k(\mathbb{RP}^m;\mathbb{Z}/2) = 1$ for $0 \le k \le m$ and $b_k(\mathbb{RP}^m;\mathbb{Z}/2) = 0$ otherwise. Thus $e(\mathbb{RP}^m) = 1$ for m even and $e(\mathbb{RP}^m) = 0$ for m odd.

The relevance of the Euler characteristic cannot immediately be seen from its definition. To indicate its importance we list the following fundamental properties without proof.

i) The Euler characteristic is an obstruction for the existence of nowhere vanishing vector fields on a closed smooth manifold, i.e. if such a vector field exists, then the Euler characteristic vanishes. We will show that S^m has a nowhere vanishing vector field if and only if m is odd.

ii) The Euler characteristic has to be even if a closed smooth manifold is the boundary of a compact smooth manifold. An example of a closed smooth manifold with odd Euler characteristic is given by one of the examples above, the Euler characteristic of $\mathbb{RP}^{2k}$ is 1. Thus $\mathbb{RP}^{2k}$ is not the boundary of a compact smooth manifold.

iii) For a finite polyhedron, the Euler characteristic can be computed from its combinatorial data: it is the alternating sum of the number of k-dimensional faces.

The following property is very useful for computing the Euler characteristic without knowing the homology.

THEOREM 7.4. *Let U and V be $\mathbb{Z}/2$-homologically finite open subspaces of a topological space X, and suppose also that $U \cap V$ is $\mathbb{Z}/2$-homologically finite. Then $U \cup V$ is $\mathbb{Z}/2$-homologically finite and*

$$e(U \cup V) = e(U) + e(V) - e(U \cap V).$$

Proof: The result follows from the Mayer-Vietoris sequence. On the one hand, exactness of the sequence implies that $U \cup V$ is $\mathbb{Z}/2$-homologically finite. The formula is a consequence of the fact we explained earlier: Let $0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \to A_0 \to 0$ be an exact sequence of finite dimensional K-vector spaces, where K is some field. Then

$$\sum_{i=0}^{n}(-1)^i \dim A_i = 0.$$

Applying this formula to the exact Mayer-Vietoris sequence, we obtain

$$e(U \cup V) = e(U) + e(V) - e(U \cap V)$$

q.e.d.

We finish this chapter by proving the previously claimed result that compact manifolds are $\mathbb{Z}/2$-homologically finite.

THEOREM 7.5. *A compact smooth c-manifold is $\mathbb{Z}/2$-homologically finite.*

Proof: It is enough to prove this for closed manifolds. The bounded case W can be reduced to this case by considering the double $W \cup_{\partial W} W$, which is a closed manifold. Thus, $W \cup_{\partial W} W$ and ∂W have finitely generated homology groups, and we decompose $W \cup_{\partial W} W$ as $U \cup V$, when U is the union of one copy of W together with the bicollar used to glue and V is the union of the other copy of W together with the bicollar. U and V are both homotopy equivalent to W and $U \cap V$ is homotopy equivalent to ∂W and so a similar argument as in the proof of Theorem 7.4 shows that if $U \cup V$ and $U \cap V$ are homologically finite, then U and V are homologically finite. The Mayer-Vietoris sequence together with the collar neighbourhood theorem implies that W is as well.

To prove the theorem for a closed manifold M, we embed M into $\mathbb{R}^N$ for some N ([**Hi**] thm I.3.4) and consider a tubular neighbourhood U ([**Hi**] thm IV.5.2). Let $r : U \to M$ be the retract corresponding to the projection of the normal bundle to M. Now we choose for each point $x \in M$ an open cube in U containing x. Since M is compact, we can cover M by finitely many cubes C_i:

$$M \subset \cup C_i \subset U.$$

The union of finitely many open cubes is $\mathbb{Z}/2$-homologically finite. This follows inductively. It is clear for a single cube. We suppose that the union of $k-1$ cubes is $\mathbb{Z}/2$-homologically finite. If we add another cube then the intersection of the new cube with the union of the $k-1$ cubes is a union of at most $k-1$ cubes since the intersection of two cubes is again a cube or empty. Thus Theorem 7.4 implies that the union of k cubes is $\mathbb{Z}/2$-homologically finite.

Since $r|_{\cup C_i}$ is a retract, we conclude that the homology groups of $\cup C_i$ are mapped surjectively onto the homology groups of M, which finishes the argument.
q.e.d.

CHAPTER 8

Integral homology and the mapping degree

Prerequisites: The only new ingredient used in this chapter is the definition of orientation of smooth manifolds, which can be found in [**B-J**] or [**Hi**].

1. Integral homology groups

In this chapter, we will introduce integral homology. This is the most powerful tool in topology, fundamental in studying all sorts of classification problems. The definition is completely analogous to that of $\mathbb{Z}/2$-homology, the only difference being that we require the top-dimensional stratum to be oriented.

Definition: *An* **oriented m-dimensional c-stratifold** *is an m-dimensional c-stratifold $\mathbf{T}$ with $\overset{\circ}{\mathbf{T}}{}^{m-1} = \emptyset$ and an orientation on $\overset{\circ}{\mathbf{T}}{}^{m}$.*

An orientation on $\mathbf{T}$ induces an orientation of $\partial\mathbf{T}$ which is fixed by requiring that the collar of $\mathbf{T}$ preserves the product orientation on $(\partial\mathbf{T})^{m-1} \times (0, \epsilon)$. If we change the orientation of $\overset{\circ}{\mathbf{T}}{}^{m}$, we call the corresponding oriented stratifold $-\mathbf{T}$.

In complete analogy with the case of smooth manifolds, we define bordism groups of compact oriented m-dimensional regular stratifolds denoted $\mathbf{H}_m(X)$:

$$\mathbf{H}_m(X) := \{(\mathbf{S}, g)\}/\text{bord},$$

where $\mathbf{S}$ is an m-dimensional compact oriented regular stratifold and $g : \mathbf{S} \to X$ is a continuous map. The relation "bord" means that two such pairs $(\mathbf{S}, g)$ and $(\mathbf{S}', g')$ are equivalent if there is a compact oriented regular c-stratifold $\mathbf{T}$ with boundary $\mathbf{S} + (-\mathbf{S}')$ and $g + g'$ extends to a map $G : \mathbf{T} \to X$. The role of the negative orientation on $\mathbf{S}'$ is the following. To show that the relation is transitive, we proceed as for $\mathbb{Z}/2$-homology and glue a bordism $\mathbf{T}$ between $\mathbf{S}$ and $\mathbf{S}'$ and a bordism $\mathbf{T}'$ between $\mathbf{S}'$ and $\mathbf{S}''$ along $\mathbf{S}'$. We have to guarantee that the orientations on the top stratum of $\mathbf{T}$ and of $\mathbf{T}'$ fit together to give an orientation of the top stratum of $\mathbf{T} \cup_{\mathbf{S}'} \mathbf{T}'$. This is the case if the orientations on $\mathbf{S}'$ induced from $\mathbf{T}$ and $\mathbf{T}'$ are opposite.

With this clarification the proof that the relation is an equivalence relation is the same as for $\mathbb{Z}/2$-homology (Proposition 4.4). It is useful to note that $-[\mathbf{S}, f] = [-\mathbf{S}, f]$, the inverse of $(\mathbf{S}, f)$ is given by changing the orientation of $\mathbf{S}$.

Defining the induced map as for $\mathbb{Z}/2$-homology by composition we obtain a functor. This functor is again a homology theory which means that homotopic maps induce the same map and that there is a Mayer-Vietoris sequence commuting with induced maps (for the definition of a homology theory see also the next chapter). The construction of the boundary operator in the Mayer-Vietoris sequence which we gave for $\mathbb{Z}/2$-homology extends once we convince ourselves that the constructions used there (like cutting and gluing) transform $\mathbb{Z}$-oriented regular stratifolds into $\mathbb{Z}$-oriented regular stratifolds. But these facts are obvious once we have fixed an orientation on the preimage of a regular value s of a smooth map $f : M \to \mathbb{R}$ on an oriented manifold M. We orient such a preimage by requiring that the orientation of it together with a vector v in the normal bundle to $f^{-1}(s)$ is an orientation of M, if the image of v under the differential of f is positive.

THEOREM 8.1. *The functor $\mathbf{H}_m(X)$ is a homology theory. This functor is called* **integral homology**.

To determine the integral homology groups of a point, we first note that for $m > 0$ the cone over an oriented regular stratifold $\mathbf{S}$ is an oriented regular stratifold with boundary $\mathbf{S}$. Thus for $m > 0$ we have $\mathbf{H}_m(pt) = 0$. To determine $\mathbf{H}_0(pt)$, we remind the reader that an orientation of a 0-dimensional manifold assigns to each point x a number $\epsilon(x) \in \pm 1$, and that the boundary of an oriented interval $[a, b]$ has an induced orientation such that $\epsilon(a) = -\epsilon(b)$ [**B-J**]. Thus, if a compact 0-dimensional manifold M is the boundary of a compact oriented 1-dimensional manifold, then $\sum_{x \in M} \epsilon(x) = 0$. In turn, if $\sum_{x \in M} \epsilon(x) = 0$, then we can group the points in M in pairs with opposite orientation and take as zero bordism for these pairs an interval. Since oriented regular stratifolds of dimension 0 and 1 are the same as oriented manifolds, we conclude:

THEOREM 8.2. *The map*

$$\mathbf{H}_0(pt) \to \mathbb{Z}$$

mapping $[M]$ to $\sum_{x \in M} \epsilon(x)$ is an isomorphism. Furthermore for $m \neq 0$ we have

$$\mathbf{H}_m(pt) = 0$$

Since an oriented regular stratifold is automatically $\mathbb{Z}/2$-oriented we have a forgetful homomorphism

$$\mathbf{H}_k(X) \to \mathbf{H}_k(X; \mathbb{Z}/2)$$

We will discuss this homomorphism at the end of this chapter.

As for $\mathbb{Z}/2$-homology, we say that a space X is **homologically finite** if for all but finitely many k, the homology groups $\mathbf{H}_k(X)$ are zero and the remaining homology groups are finitely generated. The same argument as for $\mathbb{Z}/2$-homology implies that compact smooth manifolds are homologically finite. We define the **Betti numbers** $b_k(X)$ as the rank of $\mathbf{H}_k(X)$. This is an important invariant of spaces. We recall from algebra that the rank of an abelian group G is equal to the dimension of the $\mathbb{Q}$-vector space $G \otimes \mathbb{Q}$ (for some basic information about tensor products, see Appendix C). It is useful here to remind the reader of the **fundamental theorem for finitely generated abelian groups** G, which says that G is isomorphic to $\mathbb{Z}^r \oplus \mathrm{tor}(G)$, where $\mathrm{tor}(G) = \{g \in G \,|\, ng = 0 \text{ for some natural number } n \neq 0\}$ is the torsion subgroup of G. Since $\mathrm{tor}(G) \otimes \mathbb{Q} = 0$, the number r is equal to the rank of G. The torsion subgroup T is itself isomorphic to a sum of finite cyclic groups: $\mathrm{tor}(G) \cong \oplus_i \mathbb{Z}/n_i$. If X is homologically finite, then $b_k(X)$ is zero for all but finitely many k and finite otherwise.

Using the Mayer-Vietoris sequence, one computes the integral homology of the sphere S^m for $m > 0$ as for $\mathbb{Z}/2$-homology. The result is:

$$\mathbf{H}_k(S^m) \cong \mathbb{Z}$$

for $k = 0$ or $k = m$ and

$$\mathbf{H}_k(S^m) = 0$$

for k different from 0 and m. A generator of $\mathbf{H}_m(S^m)$ is given by the homology class $[S^m, \mathrm{id}]$. Here we orient S^m as the boundary of D^{m+1}, which we equip with the orientation induced from the standard orientation of $\mathbb{R}^{m+1}$. (Note that this orientation is characterized by the property that a basis of $T_x S^m$ is oriented, if the juxtaposition of it with an inward pointing normal vector is the orientation of $\mathbb{R}^{m+1}$.)

As a first important application of integral homology we define the degree of a map from a compact oriented m-dimensional regular stratifold to a connected oriented smooth manifold M. We start with the definition of the fundamental class.

Definition: *Let* $\mathbf{S}$ *be a compact oriented* m*-dimensional regular stratifold. The* **fundamental class** *of* $\mathbf{S}$ *is* $[\mathbf{S}, id] \in \mathbf{H}_m(\mathbf{S})$. *We abbreviate it as* $[\mathbf{S}] := [\mathbf{S}, id]$.

If we change the orientation of $\mathbf{S}$ passing to $-\mathbf{S}$, then the fundamental class changes orientation as well: $[-\mathbf{S}] = -[\mathbf{S}]$. Under the homomorphism $\mathbf{H}_m(\mathbf{S}) \to \mathbf{H}_m(\mathbf{S}; \mathbb{Z}/2)$, the fundamental class maps to the $\mathbb{Z}/2$-fundamental class: $[\mathbf{S}] \mapsto [\mathbf{S}]_{\mathbb{Z}/2}$. This implies that the fundamental class is non-trivial. But one actually knows more:

THEOREM 8.3. *Let* $\mathbf{S}$ *be a compact oriented* m*-dimensional regular stratifold. Then* $k[\mathbf{S}] \in \mathbf{H}_m(\mathbf{S})$ *is non-trivial for all* $k \in \mathbb{Z} - \{0\}$ *(we say that* $[\mathbf{S}]$ *has infinite order) and* $[\mathbf{S}]$ *is primitive, i.e. not divisible by* $r > 1$.

Proof: The proof is similar to the proof of Proposition 7.1. The case $m = 0$ is trivial. For $m > 0$ we construct—as for $\mathbb{Z}/2$-homology—with the help of the Mayer-Vietoris sequence a homomorphism from $\mathbf{H}_m(\mathbf{S}) = \tilde{\mathbf{H}}_m(\mathbf{S}) \to \tilde{\mathbf{H}}_{m-1}(S^{m-1})$ mapping $[\mathbf{S}]$ to $[S^{m-1}]$. Here we orient S^{m-1} as the boundary of D^m which we embed orientation preserving into the top stratum of $\mathbf{S}$. Then the statement follows by induction.
q.e.d.

2. The degree

Now we define the **degree** and begin by defining it only for maps from compact oriented m-dimensional regular stratifolds $\mathbf{S}$ to S^m. We recall that we have $\mathbf{H}_m(S^m) \cong \mathbb{Z}$ generated by $[S^m]$ for all $m > 0$.

Definition: *Let* $\mathbf{S}$ *be a compact oriented* m*-dimensional regular stratifold,* $m > 0$*, and* $f : \mathbf{S} \to S^m$ *be a continuous map. Then we define*

$$\deg f := k \in \mathbb{Z}$$

where $[\mathbf{S}, f] = k[S^m]$.

In other words, $f_*([\mathbf{S}]) = \deg(f)[S^m]$. By construction, homotopic maps have the same degree. For $h : S^m \to S^m$, we see that $h_* : \mathbf{H}_m(S^m) \to \mathbf{H}_m(S^m)$ is multiplication by $\deg h$. As a consequence, we conclude that the degree of the composition of two maps $f, g : S^n \to S^n$ is the product of the degrees:

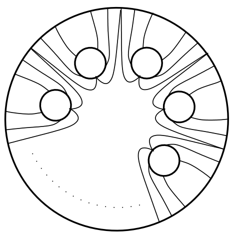
$$deg(fg) = deg(f)deg(g).$$

One can actually generalize the definition of the degree to maps from $\mathbf{S}$ to a connected oriented m-dimensional smooth manifold M: Namely one chooses an orientation preserving embedding of a disc D^m into M and considers the map $p : M \to S^m = D^m/S^{m-1}$ which is on $\overset{\circ}{D}{}^m$ the identity and maps the rest to the point represented by S^{m-1}. Then we define the degree of $f : \mathbf{S} \to M$ as

$$\deg(f) := \deg pf$$

Since any two orientation preserving embeddings of D^m into M are isotopic [**B-J**], the definition of the degree is independent on the choice of this embedding.

To get a feeling for the degree, we compute it for the map $z^m : S^1 \to S^1$, where we consider S^1 as a subspace of $\mathbb{C}$ and map z to z^m. The degree of z^m is k, where $[S^1, z^m] = k \cdot [S^1] \in \mathbf{H}_1(S^1)$. We will show that $k = m$. We have to construct a bordism between $[S^1, z^m]$ and $m \cdot [S^1]$. The following picture explains how this can be done.



Here we remove $|m|$ open balls from D^2 sitting concentrically equally distributed in the disk. This is a bordism between S^1 and $\underbrace{S^1 + \cdots + S^1}_{|m|}$. To construct a map from this bordism to S^1, we map the curved lines joining the small circles with the large circle to the image of endpoint in the large circle under the map z^m. We extend this to a map on the whole bordism by mapping the rest constantly to $1 \in S^1$. If m is positive, this induces the identity map $\mathrm{id} : S^1 \to S^1$ on each small circle. Thus we conclude $[S^1, z^m] = m \cdot [S^1, \mathrm{id}] = m \cdot [S^1]$. If m is negative, the induced map on each circle is $z^{-1} = \overline{z} = (z_1, -z_2)$. Thus we obtain that for $m < 0$ the degree of z^m is $-m \cdot \deg z^{-1}$. The degree of z^{-1} is -1. To see this we prove that $[S^1, z] = -[S^1, z^{-1}]$. A bordism between these two objects is given by $W := (S^1 \times [0, 1/2]) \cup_{z^{-1}} ((-S^1) \times [1/2, 1])$ and the map which is given by $z \cup z^{-1}$. The point here is that z^{-1} reverses the orientation (why?) and thus is an orientation preserving diffeomorphism between S^1 and $(-S^1)$ giving $\partial W = S^1 + S^1$, where both S^1's have the same orientation.

We summarize

Proposition 8.4. *The degree of $z^m : S^1 \to S^1$ is m.*

From this one can deduce the **fundamental theorem of algebra**.

Theorem 8.5. *Each complex polynomial $f : \mathbb{C} \to \mathbb{C}$ of positive degree has a zero.*

Proof: We can assume that $f(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$. If $a_0 = 0$ then $z = 0$ is a zero and so we assume $a_0 \neq 0$. We assume that f has no zero and consider the map $S^1 \to S^1$, $z \mapsto f(z)/|f(z)|$. This map is homotopic to $a_0/|a_0|$ under the homotopy $f(tz)/|f(tz)|$. On the other hand, it is also homotopic to z^n under the following homotopy. For $t \neq 0$ we take $f(t^{-1}z)/|f(t^{-1}z)|$. As t tends to 0, this map tends to z^n. We obtain a contradiction since the degree of $a_0/|a_0|$ is zero while the degree of z^n is n by Proposition 8.4.
q.e.d.

Since $z^{-1} = \overline{z} = (z_1, -z_2)$, we conclude that the degree of this reflection is -1. Using the inductive computation of $\mathbf{H}_m(S^m)$, we conclude that the degree of the reflection map $S^m \to S^m$ mapping $(z_1, z_2, \cdots, z_{m+1}) \longmapsto (z_1, -z_2, z_3, \cdots, z_{m+1})$ is also -1. Since all reflection maps $s_i : S^m \to S^m$ mapping $(z_1, \cdots, z_{m+1})$ to $(z_1, \cdots, z_{i-1}, -z_i, z_{i+1}, \cdots, z_{m+1})$ are conjugate to s_2, we conclude that for each i the degree of s_i is -1. Since $-\mathrm{id} = s_1 \circ \cdots \circ s_{m+1}$, we conclude

Proposition 8.6. *For $m > 0$ the degree of $-id : S^m \to S^m$ is $(-1)^{m+1}$.*

As a consequence, for m even the identity is not homotopic to $-\mathrm{id}$. This consequence answers an important question, namely which spheres admit a nowhere vanishing continuous vector field. Recall that the tangent bundle of S^m is $TS^m = \{(x, w) \in S^m \times \mathbb{R}^{m+1} | w \perp x\}$. For those who are not familiar with tangent bundles, we suggest to take the right side as a definition, but to convince themselves that for each x the vectors w with $w \perp x$ fit with our intuitive imagination of the tangent space of S^m at x.

A continuous **vector field** on a smooth manifold M is a continuous map $v : M \to TM$ such that $pv = \mathrm{id}$, where p is the projection of the tangent bundle. In the case of the sphere a nowhere vanishing continuous vector field is the same as a map $v : S^m \to \mathbb{R}^{m+1} - \{0\}$ with $v(x) \perp x$ for all $x \in S^m$. Replacing $v(x)$ by $v(x)/_{||v(x)||}$, we can assume that $v(x) \in S^m$ for all $x \in S^m$. But then $H : S^m \times I \to S^m$ mapping $(x,t) \mapsto (cos(\pi \cdot t))x + (sin(\pi \cdot t)) \cdot v(x)$ is a homotopy between id and $-$id giving a contradiction, if m is even. Thus we have proved

Theorem 8.7. *There is no nowhere vanishing continuous vector field on S^{2k}.*

For S^2 this result runs under the name of the **hedgehog theorem** and says that it is impossible to comb the spines of a hedgehog continuously.

On S^{2k+1} there is a nowhere vanishing vector field, for example

$$v(x_1, x_2, \cdots, x_{2k+1}, x_{2k+2}) := (-x_2, x_1, -x_4, x_3, \cdots, -x_{2k+2}, x_{2k+1}).$$

or in complex coordinates

$$v(z_1, \cdots, z_{k+1}) := (iz_1, \cdots, iz_{k+1}).$$

Thus we have shown:

There exists a nowhere vanishing vector field on S^m if and only if m is odd.

Remark: *This is a special case of a much more general theorem: There is a nowhere vanishing vector field on a compact m-dimensional smooth manifold M if and only if the Euler characteristic $e(M)$ vanishes. Note that this is consistent with our previous calculation that the Euler characteristic of S^m is 0, if m is odd, and 2, if m is even.*

3. Integral homology groups of projective spaces

We want to compute the integral homology of our favorite spaces. We recall that for $m > 0$ we have

$$\mathbf{H}_k(S^m) \cong \mathbb{Z}$$

for $k = 0, m$ and zero otherwise.

The complex projective spaces are inductively treated as for $\mathbb{Z}/2$-homology. Using the decomposition of $\mathbb{CP}^m$ into U and V as in §7, we conclude from the Mayer-Vietoris sequence:

Theorem 8.8. *$\mathbf{H}_k(\mathbb{CP}^m) \cong \mathbb{Z}$ for k even and $0 \leq k \leq 2m$ and 0 otherwise. The non-trivial homology class in $\mathbf{H}_{2n}(\mathbb{CP}^m)$ for $n \leq m$ is given by $[\mathbb{CP}^n, i]$, where i is the inclusion from $\mathbb{CP}^n$ to $\mathbb{CP}^m$.*

Finally we compute the integral homology of $\mathbb{RP}^m$.

Theorem 8.9. *$\mathbf{H}_k(\mathbb{RP}^m) \cong \mathbb{Z}$ for $k = 0$ and $k = m$, if m is odd. $\mathbf{H}_k(\mathbb{RP}^m) \cong \mathbb{Z}/2$ for k odd and $k < m$. The other homology groups are zero. Generators of the non-trivial homology groups for k odd are represented by $[\mathbb{RP}^k, i]$, where i is the inclusion.*

Proof: Again we use from §7 the decomposition of $\mathbb{RP}^m$ into U and V with U homotopy equivalent to a point, V homotopy equivalent to $\mathbb{RP}^{m-1}$, and $U \cap V$ homotopy equivalent to S^{m-1}. Then we conclude from the Mayer-Vietoris sequence by induction that for $k < m-1$ we have isomorphisms

$$i_* : \mathbf{H}_k(\mathbb{RP}^{m-1}) \cong \mathbf{H}_k(\mathbb{RP}^m).$$

To finish the induction we consider the exact Mayer-Vietoris sequence

$$0 \to \tilde{\mathbf{H}}_m(\mathbb{RP}^m) \to \tilde{\mathbf{H}}_{m-1}(S^{m-1}) \to \tilde{\mathbf{H}}_{m-1}(\mathbb{RP}^{m-1}) \to \tilde{\mathbf{H}}_{m-1}(\mathbb{RP}^m) \to 0$$

If m is odd, we conclude by induction that $\mathbf{H}_m(\mathbb{RP}^m) \cong \mathbb{Z}$ and from Theorem 8.3 that $[\mathbb{RP}^m] = [\mathbb{RP}^m, \mathrm{id}]$ is a generator. Here we use the fact that $\mathbb{RP}^m$ is orientable and we orient it in such a way that $dp_x : T_xS^m \to T_x\mathbb{RP}^m$ is orientation preserving. Since by induction $\mathbf{H}_{m-1}(\mathbb{RP}^{m-1}) = 0$, we have $\mathbf{H}_{m-1}(\mathbb{RP}^m) = 0$.

If m is even, we first note that $2i_*([\mathbb{RP}^{m-1}]) = 0$. The reason is that the reflection $r([x_1, ..., x_m]) := [-x_1, x_2, ..., x_m]$ is an orientation reversing diffeomorphism of $\mathbb{RP}^{m-1}$. Thus $[\mathbb{RP}^{m-1}] = [-\mathbb{RP}^{m-1}, r] = -r_*([\mathbb{RP}^{m-1}])$. Now consider the homotopy $h([x], t) := [cos(\pi t)x_1, x_2, ..., x_m, sin(\pi t)x_1]$ between i and ir. Thus $i_*([\mathbb{RP}^{m-1}]) = -i_*([\mathbb{RP}^{m-1}])$.

Next we note that $i_*([\mathbb{RP}^{m-1}]) \neq 0$, since it represents a non-trivial element in $\mathbb{Z}/2$-homology by Theorem 7.3, i.e. it is not even the boundary of a non-oriented regular stratifold with a map to $\mathbb{RP}^m$. Then the statement follows from the exact Mayer-Vietoris sequence above: The group $\mathbf{H}_{m-1}(\mathbb{RP}^m)$ is cyclic of order 2 generated by $i_*([\mathbb{RP}^{m-1}])$. From the non-triviality of $\mathbf{H}_{m-1}(\mathbb{RP}^m)$, we conclude that the map $\mathbf{H}_{m-1}(S^{m-1}) \to \mathbf{H}_{m-1}(\mathbb{RP}^{m-1})$ is non-trivial. Since both groups are isomorphic to $\mathbb{Z}$, this implies that $\mathbf{H}_m(\mathbb{RP}^m) = 0$.
q.e.d.

4. A comparison between integral and $\mathbb{Z}/2$-homology

An oriented stratifold $\mathbf{S}$ is automatically $\mathbb{Z}/2$-oriented. Thus we have a homomorphism

$$r : \mathbf{H}_n(X) \longrightarrow \mathbf{H}_n(X; \mathbb{Z}/2)$$

for each topological space X and each n. One often calls it the **reduction mod** 2. This homomorphism commutes with induced maps and the boundary operator, i. e. if $f : X \longrightarrow Y$ is a continuous map, then

$$f_* r = r f_* : \mathbf{H}_n(X) \longrightarrow \mathbf{H}_n(Y; \mathbb{Z}/2),$$

and if $X = U \cup V$, then

$$rd = dr : \mathbf{H}_n(U \cup V) \longrightarrow \mathbf{H}_n(U \cap V; \mathbb{Z}/2),$$

where d is the boundary operator in the Mayer-Vietoris sequence. A map r (for each space X and each n) fulfilling these two properties is called a **natural transformation** from the functor integral homology to the functor $\mathbb{Z}/2$-homology. Below and in the next chapter, we will consider other natural transformations.

If we want to use r to compare integral homology with $\mathbb{Z}/2$–homology, we need information about the kernel and cokernel of r. The answer is given in terms of an exact sequence.

THEOREM 8.10. *(Bockstein sequence:) There is a natural transformation*

$$d : \mathbf{H}_n(X;\mathbb{Z}/2) \longrightarrow \mathbf{H}_{n-1}(X)$$

and, if X is a smooth manifold, or a finite CW-complex (as defined in the next chapter), then the following sequence is exact:

$$\ldots \mathbf{H}_n(X) \xrightarrow{\cdot 2} \mathbf{H}_n(X) \xrightarrow{r} \mathbf{H}_n(X;\mathbb{Z}/2) \xrightarrow{d} \mathbf{H}_{n-1}(X) \longrightarrow \ldots$$

Since we will not apply the Bockstein sequence in this book, we will not give a proof. At the end of this book, we will explain the relation between our definition of homology and the classical definition using singular chains. The groups are naturally isomorphic if X is a smooth manifold or a finite CW-complex (we will define finite CW-complexes in the next chapter). We will prove this in §20. If one uses the classical approach, the proof of the Bockstein sequence is simple and it actually is a special case of a more general result. Besides reflecting different geometric aspects, the two definitions of homology groups both have specific strengths and weaknesses. For example, the description of the fundamental class of a closed smooth (oriented) manifold is simpler in our approach whereas the Bockstein sequence is more complicated.

The Bockstein sequence gives an answer to a natural question. Let X be a topological space such that all Betti numbers $b_k(X)$ are finite and only finitely many are non-zero. Then one can consider the alternating sum

$$\sum_k (-1)^k b_k(X).$$

The question is what is the relation between this expression and the Euler characteristic

$$e(x) = \sum_k (-1)^k b_k(X;\mathbb{Z}/2).$$

THEOREM 8.11. *Let X be a smooth manifold or a finite CW-complex. Then $b_k(X)$ is finite and non-trivial only for finitely many k, and*

$$e(X) = \sum_k (-1)^k b_k(X).$$

Proof: We decompose $\mathbf{H}_k(X;\mathbb{Z}) \cong \mathbb{Z}^{r(k)} \oplus \mathbb{Z}/2a_1 \oplus \ldots \mathbb{Z}/2a_{s(k)} \oplus T$, where T consists of odd torsion elements. Then the kernel of multiplication with 2 is $(\mathbb{Z}/2)^{s(k)}$ and the cokernel is

$$(\mathbb{Z}/2)^{s(k)} \oplus (\mathbb{Z}/2)^{r(k)}.$$

Thus we have a short exact sequence

$$0 \to (\mathbb{Z}/2)^{s(k)} \oplus (\mathbb{Z}/2)^{r(k)} \to \mathbf{H}_k(X;\mathbb{Z}/2) \to (\mathbb{Z}/2)^{s(k-1)} \to 0$$

implying that $\dim \mathbf{H}_k(X;\mathbb{Z}/2) = s(k) + s(k-1) + r(k)$ and from this we conclude the theorem by a cancellation argument.
q.e.d.

CHAPTER 9

A comparison theorem for homology theories and CW-complexes

1. The axioms of a homology theory

We have constructed already two homology theories. We now give a general definition of a homology theory.

Definition: *A* **generalized homology theory** *assigns to each topological space X a sequence of abelian groups $h_n(X)$ for $n \in \mathbb{Z}$, and to each continuous map $f : X \to Y$ a homomorphism $f_* : h_n(X) \to h_n(Y)$. One requires that the groups are trivial if $X = \emptyset$, or $n < 0$ and X arbitrary, and that the following properties hold:*

i) $id_* = id, (gf)_* = g_* f_*$, i.e. h is a **functor**,

ii) if f is homotopic to g, then $f_* = g_*$, i.e. h is **homotopy invariant**,

*iii) for open subsets U and V of X there is a long exact sequence (***Mayer-Vietoris sequence***)*

$$\cdots \to h_n(U \cap V) \to h_n(U) \oplus h_n(V) \to h_n(U \cup V) \xrightarrow{d} h_{n-1}(U \cap V) \longrightarrow h_{n-1}(U) \oplus h_{n-1}(V) \to \dots$$

commuting with induced maps (the Mayer-Vietoris sequence is natural). Here the map $h_n(U \cap V) \to h_n(U) \oplus h_n(V)$ is $\alpha \mapsto ((i_U)_*(\alpha), (i_V)_*(\alpha))$, the map $h_n(U) \oplus h_n(V) \to h_n(U \cup V)$ is $(\alpha, \beta) \mapsto (j_U)_*(\alpha) - (j_V)_*(\beta)$ and the map d is a group homomorphism called the **boundary operator**. *Note that d is an essential part of the homology theory.*

We required that $h_n(X) = 0$ for $n < 0$. Such a theory is often called **connective homology theory**. As before we say that h_n is a **functor** from the category of topological spaces and continuous maps to the category of abelian groups and group homomorphisms. As for singular homology, the map d is an essential datum of a homology theory. The maps i_U and i_V are the inclusions from $U \cap V$ to U and V, the maps j_U and j_V are the inclusions from U and V to $U \cup V$. The sequence extends arbitrarily far to the left and ends as

$$\cdots \to h_0(U) \oplus h_0(V) \to h_0(U \cup V) \to 0$$

on the right.

2. Comparison of homology theories

We want to show that under appropriate conditions two homology theories are—in a certain sense—equivalent. We begin with the definition of a natural transformation between two homology theories A and B.

Definition: *Let A and B be homology theories. A* **natural transformation** τ *assigns to each space X a homomorphism $\tau : A(X) \to B(X)$ such that for each $f : X \to Y$ the diagram*

$$\begin{array}{ccc} A(X) & \xrightarrow{\tau} & B(X) \\ \downarrow f_* & & \downarrow f_* \\ A(Y) & \xrightarrow{\tau} & B(Y) \end{array}$$

commutes.

We furthermore require that the diagram

$$\begin{array}{ccc} A_n(U \cup V) & \xrightarrow{\tau} & B_n(U \cup V) \\ \downarrow d_A & & \downarrow d_B \\ A_{n-1}(U \cap V) & \xrightarrow{\tau} & B_{n-1}(U \cap V) \end{array}$$

commutes.

A natural transformation is called a **natural equivalence** *if for each X the homomorphism $\tau : A(X) \to B(X)$ is an isomorphism.*

In the following chapters, we will sometimes consider two homology theories and a natural transformation between them, and we may want to check whether this is a natural equivalence — at least for a suitable class of spaces X. It turns out that for the spaces under consideration this can very easily be decided: one only has to check that $\tau : A_n(pt) \to B_n(pt)$ is an isomorphism for all n.

To characterize such a class of suitable spaces, we introduce the notion of homology with compact support. A space is called weakly compact if each open covering has a finite subcovering. If the space is Hausdorff, then this is equivalent to being compact.

Definition: *A homology theory h is a* **homology with compact support** *if for each homology class $x \in h_n(X)$ there is a weakly compact subspace $K \subset X$ and $\beta \in h_n(K)$ such that $x = j_*(\beta)$, where $j : K \to X$ is the inclusion, and if for each weakly compact $K \subset X$ and $x \in h_n(K)$ mapping to 0 in $h_n(X)$, there is a weakly compact space K' with $K \subset K' \subset X$ such that $i_*(x) = 0$, where $i : K \to K'$ is the inclusion.*

For example, homology (integral or with $\mathbb{Z}/2$-coefficients) is compactly supported since the image of a compact space under a continuous map is weakly compact.

A first comparison result is the following:

Proposition 9.1. *Let h and h' be compactly supported homology theories and $\tau : h \to h'$ be a natural transformation such that $\tau : h_n(pt) \to h'_n(pt)$ is an isomorphism for all k. Then τ is an isomorphism $\tau : h_n(U) \to h'_n(U)$ for all open $U \subset \mathbb{R}^k$.*

The proof is based on the **5-Lemma** in homological algebra.

Lemma 9.2. *Consider a commutative diagram of abelian groups and homomorphisms*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow\cong & & \downarrow f & & \downarrow\cong & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where the horizontal lines are exact sequences, the maps from B and D are isomorphisms, the map from A is surjective and the map from E is injective. Then the map $f : C \to C'$ is an isomorphism.

Proof: This is a simple diagram chasing argument. We demonstrate the principle by showing that $C \to C'$ is surjective and leave the injectivity as an exercise to the reader. For $c' \in C'$ consider the image $d' \in D'$ and the pre-image $d \in D$. Since E injects into E', the element d maps to 0 in E, and thus there is $c \in C$ mapping to d. By construction $f(c) - c'$ maps to 0 in D'. Thus there is $b' \in B'$ mapping to $f(c) - c'$. We take the pre-image $b \in B$ and replace c by $c - g(b)$, where g is the map from B to C. Then $f(c - g(b)) - c' = f(c) - fg(b) - c' = f(c) - g'(b') - c' = f(c) - f(c) + c' - c' = 0$, where g' is the map from B' to C'.
q.e.d.

With this lemma we can now prove the proposition.

Proof of Proposition 9.1: Let U_1 and U_2 be open subsets of a space X and suppose that τ is an isomorphism for U_1, U_2 and $U_1 \cap U_2$. Then the Mayer-Vietoris sequence together with the 5-Lemma imply that τ is an isomorphism for $U_1 \cup U_2$.

Now consider a finite union of s open cubes $(a_1, b_1) \times \cdots \times (a_k, b_k) \subset \mathbb{R}^k$. Since the intersection of two open cubes is again an open cube or empty, the intersection of the s-th cube U_s with $U_1 \cup ... \cup U_{s-1}$ is a union of $s-1$ open cubes. Since each cube is homotopy equivalent to a point pt, we conclude inductively over s that τ is an isomorphism for all $U \subset \mathbb{R}^k$ which are a finite union of s cubes.

Now consider an arbitrary $U \subset \mathbb{R}^k$ and $x \in h'_n(U)$. Since h'_n has compact support, there is a compact subspace $K \subset U$ such that $x = j_*(\beta)$ with $\beta \in h'_n(K)$. Cover K by a finite union V of open cubes such that $K \subset V \subset U$ and denote the inclusion from K to V

by i. Then, by the consideration above, $i_*(\beta)$ is in the image of $\tau : h_n(V) \to h'_n(V)$. Now consider the inclusion from V to U to conclude that x is in the image of $\tau : h_n(U) \to h'_n(U)$. Thus τ is surjective.

For injectivity, one argues similarly. Let $x \in h_n(U)$ such that $\tau(x) = 0$. Then we first consider a compact subspace K in U such that $x = j_*(\beta)$. Then, since $\tau(\beta) = 0$ in $h'_n(U)$, there is a compact set K' such that $K \subset K' \subset U$ and $j_*(\beta)$ maps to 0 in $h'_n(K')$. By covering K' by a finite number of cubes in U, we conclude that β maps to 0 in this finite number of cubes since τ is injective for this space. Thus $x = 0$.
q.e.d.

Applying the Mayer-Vietoris sequence and the 5-Lemma again, one concludes that in the situation of Proposition 9.1 one can replace U by a space which can be covered by a finite union of open subsets which are homeomorphic to open subsets of $\mathbb{R}^k$.

COROLLARY 9.3. *Let h and h' be compactly supported homology theories and $\tau : h \to h'$ be a natural transformation. Suppose that $\tau : h_n(pt) \to h'_n(pt)$ is an isomorphism for all n. Then for each topological manifold M (with or without boundary) admitting a finite atlas $\tau : h_n(M) \to h'_n(M)$ is an isomorphism for all n.*

In particular, this corollary applies to all compact manifolds. One can easily generalize this result by considering spaces $X = R \cup_f Y$ which are obtained by gluing a compact c-manifold R (i.e. a manifold together with a germ class of collars) via a continuous map f to space Y for which τ is an isomorphism. For then we decompose $R \cup_f Y$ into $U := R - \partial R$ and the union V of Y with the collar of ∂R in R. Then U is a manifold with finite atlas, $U \cap V$ is homotopy equivalent to ∂R, a manifold with finite atlas and V is homotopy equivalent to Y. Thus the result above together with the Mayer-Vietoris and the 5-Lemma argument implies that τ is an isomorphism $h_n(R \cup_f Y) \to h'_n(R \cup_f Y)$.

Definition: *We call a space X* **nice** *if it is either a topological manifold (with or without boundary) with finite atlas or obtained by gluing a compact topological manifold with boundary via a continuous map of the boundary to a nice space.*

COROLLARY 9.4. *Let h, h' and τ be as above. Then for each nice space X the homomorphism $\tau : h_n(X) \to h'_n(X)$ is an isomorphism.*

3. CW-complexes

Motivated by the definition of nice spaces, we introduce now another class of objects called (finite) CW-complexes which lead to nice spaces. Of course, CW-complexes are not only useful for comparing homology theories but in many aspects of algebraic topology.

Definition: *An* **m-dimensional finite CW-complex** *is a topological space X together with subspaces $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^m = X$. In addition we require that for*

$0 \leq j \leq m$, there are continuous maps $f_r^j = S_r^{j-1} \longrightarrow X^{j-1}$, where $D_r^j = D^j = \{x \in \mathbb{R}^j | \, ||x|| \leq 1\}$ and homeomorphisms

$$X^j \cong (+_{r=1}^{s_j} D_r^j) \cup_{(+f_r^j)} X^{j-1}$$

*We call X^0, X^1, ... a CW-**decomposition** of the topological space and call the open balls **cells**. We denote a CW-complex shortly by X.*

Thus a finite m-dimensional CW-complex can be obtained from a finite set of points with discrete topology by first attaching a finite number of 1-dimensional balls, followed by a finite number of 2-dimensional balls, ..., and finally a finite number of m-dimensional balls via a continuous map from the boundary of the balls to the already constructed space.

Examples:
1.) $X = S^m, X^0 = \cdots = X^{m-1} = pt, \; X^m = D^m \cup pt.$

2.) Let $f^j : S^{j-1} \longrightarrow \mathbb{RP}^{j-1}$ be the canonical projection. Then we have a homeomorphism

$$D^j \cup_{f^j} \mathbb{RP}^{j-1} \longrightarrow \mathbb{RP}^j$$

mapping $x \in D^j$ to $[x_1, \cdots, x_j, \sqrt{1 - \Sigma x_j^2}]$ and $[x] \in \mathbb{RP}^{j-1}$ to $[x, 0]$. Thus $X^j := \mathbb{RP}^j$ $(0 \leq j \leq m)$ gives a CW-decomposition of $\mathbb{RP}^m$.

3.) Similarly, $X^j := \mathbb{CP}^{[j/2]}$ gives a CW-decomposition of $\mathbb{CP}^n$.

Here is a first instance showing that it is useful to consider CW-decompositions.

THEOREM 9.5. *A finite CW-complex X is homologically and $\mathbb{Z}/2$-homologically finite. Denote the number of j-cells of a finite CW-complex X by β_j. Then:*

$$e(X) = \sum_{j=0}^{m} (-1)^j \cdot \beta_j$$

Proof: We prove the statement inductively over the cells. Suppose that Y is homologically finite and $\mathbb{Z}/2$-homologically finite. Let $f : S^{k-1} \to Y$ be a continuous map and consider $Z := D^k \cup_f Y$. We decompose $Z = U \cup V$ with $U = \overset{\circ}{D}{}^k$ and $V = Z - \{0\}$, where $0 \in D^k$. The space $U \cap V$ is homotopy equivalent to S^{k-1}, U is homotopy equivalent to a point, and V is homotopy equivalent to Y.

The Mayer-Vietoris sequence implies that Z is homologically and $\mathbb{Z}/2$-homologically finite, thus, and by Theorem 7.4

$$\begin{aligned} e(Z) &= e(Y) + e(pt) - e(S^{k-1}) \\ &= e(Y) + 1 - (1 + (-1)^{k-1}) \\ &= e(Y) + (-1)^k, \end{aligned}$$

which implies the statement.
q.e.d.

Remark: *In this case as well as in many other instances, it is enough to require that* X *is homotopy equivalent to a finite* CW*-complex.*

Remark: *One can generalize the definition to non-finite* CW*-complexes which are obtained from an arbitrary discrete set by attaching an arbitrary number of* 1*-cells,* 2*-cells and so on.*

CHAPTER 10

Künneth's theorem

Prerequisites: In this chapter we assume that the reader is familiar with tensor products of modules. The basic definitions and some results on tensor products relevant for our context are contained in Appendix C.

1. The $\times$-product

We want to compute the homology of $X \times Y$. To compare it with the homology of X and Y, we construct the $\times$-product $\mathbf{H}_i(X) \times \mathbf{H}_j(Y) \to \mathbf{H}_{i+j}(X \times Y)$. If $[\mathbf{S}, g] \in \mathbf{H}_k(X)$ and $[\mathbf{S}', g'] \in \mathbf{H}_l(Y)$ we construct an element

$$[\mathbf{S}, g] \times [\mathbf{S}', g'] \in \mathbf{H}_{k+l}(X \times Y)$$

and similar for $\mathbb{Z}/2$-homology.

For this we take the cartesian product of $\mathbf{S}$ and $\mathbf{S}'$ (considered as stratifold by example 6 in chapter 2) and the product of g and g'.

If $\mathbf{S}$ and $\mathbf{S}'$ are regular and $\mathbb{Z}/2-$oriented, then the product is regular and the $(k+l-1)-$dimensional stratum $+_{i+j=k+l-1}(\mathbf{S}^i \times (S')^j) = \mathbf{S}^k \times (\mathbf{S}')^{l-1} + \mathbf{S}^{k-1} \times (\mathbf{S}')^l$ is empty. Thus $[\mathbf{S} \times \mathbf{S}', g \times g']$ is an element of $\mathbf{H}_{k+l}(X \times Y; \mathbb{Z}/2)$. If $\mathbf{S}$ and $\mathbf{S}'$ are oriented then the $(k+l)$-dimensional stratum is $\mathbf{S}^k \times (\mathbf{S}')^l$ and so carries the product orientation. Thus $[\mathbf{S} \times \mathbf{S}', g \times g']$ is an element of $\mathbf{H}_{k+l}(X \times Y)$. This is the construction of the $\times$-products:

$$\mathbf{H}_i(X) \times \mathbf{H}_j(Y) \to \mathbf{H}_{i+j}(X \times Y)$$

and

$$\mathbf{H}_i(X; \mathbb{Z}/2) \times \mathbf{H}_j(X; \mathbb{Z}/2) \to \mathbf{H}_{i+j}(X \times Y; \mathbb{Z}/2)$$

which are defined as

$$[\mathbf{S}, g] \times [\mathbf{S}', g'] := [\mathbf{S} \times \mathbf{S}', g \times g'].$$

By construction the $\times$-products are well defined and bilinear. They are also by construction associative:

Proposition 10.1. *The $\times$-products are bilinear and associative.*

Since the $\times$-products are bilinear they induce maps from the tensor product

$$\mathbf{H}_i(X; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \mathbf{H}_j(Y; \mathbb{Z}/2) \longrightarrow \mathbf{H}_{i+j}(X \times Y; \mathbb{Z}/2)$$

and

$$\mathbf{H}_i(X) \otimes \mathbf{H}_j(Y) \longrightarrow \mathbf{H}_{i+j}(X \times Y).$$

(We denote the tensor product of abelian groups by $\otimes$ and of F-vector spaces by $\otimes_F$.)

We sum the left side over all i, j with $i + j = k$ to obtain homomorphisms

$$\times : \oplus_{i+j=k} \mathbf{H}_i(X) \otimes \mathbf{H}_j(Y) \to \mathbf{H}_k(X \times Y)$$

and

$$\times : \oplus_{i+j=k} \mathbf{H}_i(X; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \mathbf{H}_j(Y; \mathbb{Z}/2) \to \mathbf{H}_k(X \times Y; \mathbb{Z}/2).$$

It would be nice if these maps were isomorphisms. For $\mathbb{Z}/2$-homology, we will show this under some assumptions on X, but for integral homology these assumptions are not sufficient. The idea is to fix Y and to consider the functor

$$\mathbf{H}_k^Y(X) := \mathbf{H}_k(X \times Y)$$

where for $f : X \to X'$ we define

$$f_* : H_k^Y(X) \to H_k^Y(X')$$

by $(f \times Id)_*$. This is obviously a homology theory: The Mayer-Vietoris sequence holds since

$$(U_1 \times Y) \cup (U_2 \times Y) = (U_1 \cup U_2) \times Y,$$
$$(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y.$$

Furthermore this is a homology theory with compact support.

For X a point the maps $\times$ above are isomorphisms. Thus we could try to prove that they are always an isomorphism for nice spaces X by applying the comparison result Corollary 9.4 if

$$X \longmapsto \oplus_{i+j=k} \mathbf{H}_i(X) \otimes \mathbf{H}_j(Y) =: h_k^Y(X).$$

were also a homology theory and similarly if

$$X \longmapsto \oplus_{i+j=k} \mathbf{H}_i(X; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \mathbf{H}_j(Y; \mathbb{Z}/2) =: h_k^Y(X; \mathbb{Z}/2)$$

were a homology theory. Here, for $f : X \to X'$, we define $f_* = \oplus_{i+j=k}((f_* \otimes Id) : \mathbf{H}_i(X) \otimes \mathbf{H}_j(Y) \to \mathbf{H}_i(X') \otimes \mathbf{H}_j(Y))$.

The homotopy axiom is clear but the Mayer-Vietoris sequence is a problem. It would follow if for an exact sequence of abelian groups

$$A \xrightarrow{f} B \xrightarrow{g} C$$

and abelian group D the sequence

$$A \otimes D \xrightarrow{f \otimes Id} B \otimes D \xrightarrow{g \otimes Id} C \otimes D$$

were exact. But this is in general not the case. For example consider

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$$

and $D = \mathbb{Z}/2$ giving

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z}$$

which is not exact since $\cdot 2 : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is 0. If instead of abelian groups we work with vector spaces over a field F, the sequence

$$A \otimes_F D \overset{f \otimes_F Id}{\longrightarrow} B \otimes_F D \overset{g \otimes_F Id}{\longrightarrow} C \otimes_F D$$

is exact. It is enough to show this for short exact sequences $0 \to A \to B \to C \to 0$ by passing to the image in C and dividing out the kernel in A. Then there is a splitting $s : C \to B$ with $gs = Id$ and a splitting $p : B \to A$ with $pf = Id$. These splittings induce splittings of

$$A \otimes_F D \overset{f \otimes_F Id}{\longrightarrow} B \otimes_F D \overset{g \otimes_F Id}{\longrightarrow} C \otimes_F D$$

implying its exactness.

The sequence is also exact, if D is a torsion free finitely generated abelian group. Namely then $D \cong \mathbb{Z}^r$ for some r. It is enough to check exactness for $r = 1$, where it is trivial since $A \otimes \mathbb{Z} \cong A$. For larger r we use that $A \otimes (D \oplus D') \cong (A \otimes D) \oplus (A \otimes D')$.

Thus, if all homology groups of Y are finitely generated and torsion free, the functor $h_k^Y(X)$ is a generalized homology theory. And since $\mathbf{H}_k(X;\mathbb{Z}/2)$ is a $\mathbb{Z}/2$-vector space we conclude that for a fixed space Y the functor $h_k^Y(X;\mathbb{Z}/2)$ is a homology theory. To obtain some partial information about the integral homology groups of a product of two spaces, if $\mathbf{H}_k(Y)$ is not finitely generated and torsion free, we define rational homology groups.

Definition: $\mathbf{H}_m(X;\mathbb{Q}) := \mathbf{H}_m(X) \otimes \mathbb{Q}$. *For* $f : X \to Y$ *we define* $f_* : \mathbf{H}_m(X;\mathbb{Q}) \to \mathbf{H}_m(Y;\mathbb{Q})$ *by* $f_* \otimes Id : \mathbf{H}_m(X) \otimes \mathbb{Q} \to \mathbf{H}_m(Y) \otimes \mathbb{Q}$.

By the considerations above the rational homology groups are a homology theory called **rational homology**. Since the rational homology groups are $\mathbb{Q}$-vector spaces (scalar multiplication with $\lambda \in \mathbb{Q}$ is given by $\lambda(x \otimes \mu) := x \otimes \lambda\mu$), the functor

$$X \longmapsto \oplus_{i+j=k} \mathbf{H}_i(X;\mathbb{Q}) \otimes \mathbf{H}_j(Y;\mathbb{Q}) =: h_k^Y(X;\mathbb{Q}).$$

is a homology theory. By construction it has compact support.

2. The Künneth theorem

To apply Corollary 9.4 we have to check that the maps $\times : h_k^Y(X;\mathbb{Z}/2) \to \mathbf{H}_k^Y(X;\mathbb{Z}/2)$, $\times : h_k^Y(X) \to \mathbf{H}_k^Y(X)$ and $\times : h_k^Y(X;\mathbb{Q}) \to \mathbf{H}_k^Y(X;\mathbb{Q})$ commute with induced maps and the boundary operator in the Mayer-Vietoris sequence, in other words are natural transformations. The proof is the same in all three cases and so we only give it for $\mathbb{Z}/2$-homology:

LEMMA 10.2. *The map*

$$\times : h_k^Y(X;\mathbb{Z}/2) \to \mathbf{H}_k^Y(X;\mathbb{Z}/2).$$

is a natural transformation.

Proof: Everything is clear except the commutativity in the Mayer-Vietoris sequence. Let U_1 and U_2 be open subsets of X and consider for $i+j=k$ the element $[\mathbf{S}, f] \otimes [\mathbf{Z}, g] \in \mathbf{H}_i(U_1 \cup U_2) \otimes \mathbf{H}_j(Y)$. By our definition of the boundary operator in the Mayer-Vietoris sequence of $\mathbf{H}_i(X)$ we can decompose the stratifold $\mathbf{S}$ (after perhaps changing it by a bordism) as $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$ with $\partial \mathbf{S}_1 = \partial \mathbf{S}_2 =: \mathbf{Q}$, where $f(\mathbf{S}_1) \subset U_1$ and $f(\mathbf{S}_2) \subset U_2$. Then $d([\mathbf{S}, f]) = [\mathbf{Q}, f|_{\mathbf{Q}}]$. Thus $d([\mathbf{S}, f] \otimes [\mathbf{Z}, g]) = [\mathbf{Q}, f|_{\mathbf{Q}}] \otimes [\mathbf{Z}, g]$. On the other hand $[\mathbf{S}, f] \times [\mathbf{Z}, g] = [\mathbf{S} \times \mathbf{Z}, f \times g]$ and, since $\mathbf{S} \times \mathbf{Z} = (\mathbf{S}_1 \cup \mathbf{S}_2) \times \mathbf{Z}$, we conclude: $d([\mathbf{S} \times \mathbf{Z}, f \times g]) = [\mathbf{Q} \times \mathbf{Z}, f|_{\mathbf{T}} \times g]$. Thus the diagram

$$\begin{array}{ccc}
\mathbf{H}_i(U_1 \cup U_2) \otimes \mathbf{H}_j(Y) & \xrightarrow{\times} & \mathbf{H}_{i+j}((U_1 \cup U_2) \times Y) \\
\downarrow d & & \downarrow d \\
\mathbf{H}_{i-1}(U_1 \cap U_2) \otimes \mathbf{H}_j(Y) & \xrightarrow{\times} & \mathbf{H}_{i+j-1}((U_1 \cap U_2) \times Y)
\end{array}$$

commutes.
q.e.d.

Now the Künneth Theorem is an immediate consequence of Corollary 9.4:

THEOREM 10.3. **(Künneth Theorem**) *Let X be a nice space. Then for $F = \mathbb{Q}$ or $\mathbb{Z}/2\mathbb{Z}$*

$$\times : \oplus_{i+j=k} \mathbf{H}_i(X;F) \otimes_F \mathbf{H}_j(Y;F) \to \mathbf{H}_k(X \times Y;F)$$

is an isomorphism. The same holds for integral homology if for all j the groups $\mathbf{H}_j(Y)$ are torsion free and finitely generated.

We note that the Künneth theorem holds for all spaces X which are finite CW-complexes since all these spaces are nice. In a later chapter we will identify singular homology of CW-complexes with the homology groups defined in a traditional way using simplices. The world of simplices is more appropriate for dealing with the Künneth Theorem and one obtains there a general result computing the integral homology groups of a product of CW-complexes.

As an application we prove that for nice spaces X the Euler characteristic of $X \times Y$ is the product of the Euler characteristics of X and Y.

THEOREM 10.4. *Let X and Y be $\mathbb{Z}/2$ homologically finite and X a nice space. Then*

$$e(X \times Y) = e(X) \cdot e(Y).$$

Proof: By the previous theorem the proof follows from

LEMMA 10.5. *Let $A = (A_0, A_1, \ldots, A_k)$ and $B = (B_0, \ldots, B_r)$ be sequences of finite-dimensional $\mathbb{Z}/2$-vector spaces. Then for $C(A, B)$ with $C_s = \oplus_{i+j=s} A_i \otimes B_j$, we have*

$$e(C) = e(A) \cdot e(B).$$

Here $e(A) := \sum_i (-1)^i \dim A_i$.

Proof: We prove it by induction over k. Let A' be given by $A_0, A_1, \cdots, A_{k-1}$. Then $e(A') + (-1)^k \dim A_k = e(A)$. Define C' as $C(A', B)$. Then $C_s = C'_s$ for $s \leq k$ and $C_{k+j} = C'_{k+j} \oplus (A_k \otimes B_j)$.
Thus,

$$\begin{aligned} e(C) &= e(C') + (-1)^k \dim A_k \, e(B) \\ &= e(A') \cdot e(B) + (-1)^k \dim A_k \, e(B) \\ &= e(A) \cdot e(B). \end{aligned}$$

q.e.d.

Another application is the computation of the homology of a product of two spheres $S^n \times S^m$ for n and m positive. Since the homology groups of S^m are torsion free, the Künneth theorem implies $H_k(S^n \times S^m) = \oplus_{i+j=k} H_i(S^n) \otimes H_j(S^m)$.

$$H_k(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & k = 0, n+m \\ \mathbb{Z} & k = n, if\ n \neq m \\ \mathbb{Z} & k = m, if\ n \neq m \\ \mathbb{Z} \oplus \mathbb{Z} & k = n = m \\ 0 & else \end{cases}$$

Here we use that $A \otimes \mathbb{Z} \cong A$ for each abelian group A. We actually obtain with the $\times$-product a basis for the homology groups: Let x be a point in S^n and y be a point in S^m. Then $[(x, y), i]$ generates $H_0(S^n \times S^m)$, $[S^n \times y, i]$ generates $H_n(S^n \times S^m)$ and $[x \times S^m, i]$ generates $H_m(S^n \times S^m)$ for $n \neq m$, and these elements are a basis of $H_n(S^n \times S^n)$, if $n = m$, and finally the fundamental class $[S^n \times S^m]$ generates $H_{n+m}(S^n \times S^m)$. Here i always stands for the inclusion.

These examples are in agreement with our geometric intuition that the manifolds giving the homology classes "fish" the corresponding holes.

CHAPTER 11

Some lens spaces and quarternionic generalizations

1. Lens spaces

In this chapter we will construct a class of manifolds that, on the one hand, gives more fundamental examples to play with and, on the other hand, is the basis for some very interesting aspects of modern differential topology. Some of these aspects will be discussed in later chapters.

The manifolds under consideration have various geometric features. We will concentrate on one aspect: they are total spaces of smooth fibre bundles. A **smooth fibre bundle** is a smooth map $p : E \to B$ between smooth manifolds such that for each $x \in B$ there is an open neighbourhood U and a diffeomorphism $\varphi : p^{-1}(U) \to U \times F$ for some smooth manifold F with $p|_{p^{-1}(U)} = p_1\varphi$. Such a φ is called a **local trivialization**. For a point $x \in B$ we call $E_x := p^{-1}(x)$ the fibre over x.

We begin with some bundles over S^2 with fibre S^1. Let k be an integer. Decompose S^2 as $D^2 \cup_{S^1} D^2$ and define

$$L_k := D^2 \times S^1 \cup_{f_k} D^2 \times S^1$$

where $f_k : S^1 \times S^1 \to S^1 \times S^1$ is the diffeomorphism mapping (z_1, z_2) to $(z_1, z_1^k z_2)$. Here we consider S^1 as a subgroup of $\mathbb{C}^*$. The map is a diffeomorphism since $(z_1, z_2) \mapsto (z_1, z_1^{-k} z_2)$ is an inverse map. L_k is equipped with a smooth structure. It is called a **lens space**. Is it orientable? This is easily seen without deeper consideration for the following reason. Since $S^1 \times S^1 = \partial(D^2 \times S^1)$ is connected, f is either orientation preserving or orientation reversing (by continuity of the orientation and of df_x the orientation behaviour cannot jump). If it were orientation reversing, we are done by orienting both copies of $D^2 \times S^1$ in $D^2 \times S^1 \cup_f D^2 \times S^1$ equally. If it is orientation preserving, we are also done by orienting the second copy of $D^2 \times S^1$ in $D^2 \times S^1 \cup_f D^2 \times S^1$ opposite to the first one, making f artificially orientation reversing.

Of course, by computing $d(f_k)_x$ in our example we can decide if f_k is orientation preserving. For this consider $S^1 \times S^1$ as a submanifold of $\mathbb{C}^* \times \mathbb{C}^*$ and extend f_k to the map given by the same expression on $\mathbb{C}^* \times \mathbb{C}^*$. Then $(df_k)_{(z_1,z_2)}$ is given by the complex Jacobi matrix

$$\begin{pmatrix} 1 & 0 \\ k z_1^{k-1} z_2 & z_1^k \end{pmatrix}$$

To obtain the map on $T_{(z_1,z_2)}(S^1\times S^1)$ we have to restrict this map to $z_1^\perp \times z_2^\perp = T_{(z_1,z_2)}(S^1\times S^1)$. We give a basis of $T_{(z_1,z_2)}(S^1 \times S^1)$ by $(iz_1, 0)$ and $(0, iz_2)$ and use this basis as our standard orientation. We have to compare the orientation given by $d(f_k)_{(z_1,z_2)}(iz_1, 0)$ and $d(f_k)_{(z_1,z_2)}(0, iz_2)$ at the point $f_k(z_1, z_2) = (z_1, z_1^k z_2)$ with that given by $(iz_1, 0)$ and $(0, iz_1^k z_2)$. But $d(f_k)_{(z_1,z_2)}(iz_1, 0) = (iz_1, kiz_1^k z_2)$ and $d(f_k)_{(z_1,z_2)}(0, iz_2) = (0, iz_1^k z_2)$. The base change matrix is

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

and it has a positive determinant. Thus f_k is orientation preserving and to orient L_k we have to consider it as $D^2 \times S^1 \cup_{f_k} -D^2 \times S^1$. From now on, we consider L_k as an oriented 3-manifold with this orientation.

As mentioned above, there are different natural descriptions of lens spaces. Although we don't need this, we give another description of L_k for $k > 0$. For this we consider the 3-sphere S^3 as subspace of $\mathbb{C}^2$. The group of k-th roots of unity in S^1 is $G_k = \{z \in S^1 | z^k = 1\}$. We consider the space $S^3/_{G_k} := S^3/_\sim$, where $v \sim w$ if and only if there is $z \in G_k$ such that $zv = w$. For example, $S^3/_{G_2}$ is the projective space $\mathbb{RP}^3$. It is not difficult to identify $S^3/_{G_k}$ with L_k. As a hint one should start with the case $k = 1$ and identify $S^3 = S^3/_{G_1}$ with L_1. Once this is achieved, one can use this information to solve the case $k > 1$.

We consider the map $p : L_k \to S^2 = D^2 \cup -D^2$ mapping $(z_1, z_2) \in D^2 \times S^1$ to z_1 and $(z_1, z_2) \in -D^2 \times S^1$ to z_1. This is obviously well defined and by construction of the smooth structures on L_k and on $D^2 \cup -D^2 = S^2$ it is a smooth map. Actually, by construction $p : L_k \to S^2$ is a smooth fibre bundle.

We want to classify the manifolds L_k up to diffeomorphism. For this we first compute the homology groups. We prepare this by some general considerations. As above, consider two smooth c-manifolds W_1 and W_2 and a diffeomorphism $f : \partial W_1 \to \partial W_2$. Then consider the open covering of $W_1 \cup_f W_2$ given by the union of W_1 and the collar of ∂W_2 in W_2, denoted by U, and of W_2 and the collar of ∂W_1 in W_1, denoted by V. Obviously, the inclusions from W_1 to U and from W_2 to V as well as from ∂W_1 to $U \cap V$ are homotopy equivalences. With this information we consider the Mayer-Vietoris sequence and replace the homology group of U, V and $U \cap V$ by the isomorphic homology group of W_1, W_2 and ∂W_1:

$$\cdots \to \mathbf{H}_k(\partial W_1) \to \mathbf{H}_k(W_1) \oplus \mathbf{H}_k(W_2) \to \mathbf{H}_k(W_1 \cup_f W_2) \xrightarrow{d} \mathbf{H}_{k-1}(\partial W_1) \to \cdots$$

where the map from $\mathbf{H}_k(W_1) \oplus \mathbf{H}_k(W_2)$ to $\mathbf{H}_k(W_1 \cup W_2)$ is the difference of the maps induced by inclusions. The map from $\mathbf{H}_k(\partial W_1)$ to $\mathbf{H}_k(W_1)$ is $(j_1)_*$, where j_1 is the inclusion from ∂W_1 to W_1, and the map from $\mathbf{H}_k(\partial W_1)$ to $\mathbf{H}_k(W_2)$ is $(j_2)_* f_*$, where j_2 is the inclusion from ∂W_2 to W_2.

Applying this to L_k implies that $\mathbf{H}_r(L_k) = 0$ for $r > 3$ and we have an isomorphism $\mathbf{H}_3(L_k) \xrightarrow{d} \mathbf{H}_2(S^1 \times S^1) \cong \mathbb{Z}$. Since the fundamental class $[L_k] \in \mathbf{H}_3(L_k)$ is a primitive element, we conclude

$$\mathbf{H}_3(L_k) = \mathbb{Z}[L_k],$$

the free abelian group of rank 1 generated by the fundamental class$[L_k]$. The computation of $\mathbf{H}_2$ and $\mathbf{H}_1$ is given by the exact sequence:

$$0 \to \mathbf{H}_2(L_k) \to \mathbf{H}_1(S^1 \times S^1) \to \mathbf{H}_1(S^1) \oplus \mathbf{H}_1(S^1) \to \mathbf{H}_1(L_k) \to 0$$

in which the map from $\mathbf{H}_1(S^1 \times S^1)$ to the first component is $(p_2)_*$, where p_2 is the projection onto the second factor, and the map from $\mathbf{H}_1(S^1 \times S^1)$ to the second component is $(p_2)_*(f_k)_*$. By the Künneth Theorem 10.3 we have seen that $\mathbf{H}_1(S^1 \times S^1) = \mathbb{Z}[S^1, i_1] \oplus \mathbb{Z}[S^1, i_2]$, where $i_1(z) = (z, 1)$ and $i_2(z) = (1, z)$. If α is an element of $\mathbf{H}_1(S^1 \times S^1)$, the coefficients of α with respect to the basis $[S^1, i_1]$ and $[S^1, i_2]$ are $(p_1)_*(\alpha) \in \mathbf{H}_1(S^1) = \mathbb{Z}$ and $(p_2)_*(\alpha) \in \mathbf{H}_1(S^1) = \mathbb{Z}$.

Thus

$$(f_k)_*[S^1, i_1] = \deg(p_1 f_k i_1)[S^1, i_1] + \deg(p_2 f_k i_1)[S^1, i_2]$$

and

$$(f_k)_*[S^1, i_2] = \deg(p_1 f_k i_2)[S^1, i_1] + \deg(p_2 f_k i_2)[S^1, i_2]$$

From Proposition 8.4 we know the corresponding degrees and conclude that with respect to the basis $[S^1, i_1]$ and $[S^1, i_2]$ of $\mathbf{H}_1(S^1 \times S^1)$ the map $(f_k)_*$ is given by

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

With this information the exact sequence above gives

$$0 \to \mathbf{H}_2(L_k) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbf{H}_1(L_k) \to 0,$$

where the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ k & 1 \end{pmatrix}$$

The kernel of this linear map is 0 and the cokernel $\mathbb{Z}/|k|\mathbb{Z}$. Thus we have shown that $\mathbf{H}_2(L_k) = 0$ and $\mathbf{H}_1(L_k) \cong \mathbb{Z}/|k|\mathbb{Z}$ generated by the inclusions of the fibres into L_k:

Proposition 11.1. *The homology of L_k is*

$$\mathbf{H}_r(L_k) \cong \begin{cases} 0 & r > 3, r = 2 \\ \mathbb{Z} & r = 0, 3 \\ \mathbb{Z}/|k|\mathbb{Z} & r = 1 \end{cases}$$

where $\mathbf{H}_1(L_k)$ is generated by $[S^1, j]$ and $j : S^1 \to D^2 \times S^1 \subset L_k$ maps z to $(0, z)$.

As a consequence, $|k|$ is an invariant of the homeomorphism type or even the homotopy type of L_k. On the other hand, conjugation on the fibres S^1 induces a diffeomorphism from L_k to L_{-k}. Thus we conclude

Proposition 11.2. *L_k is diffeomorphic to L_q if and only if $|k| = |q|$.*

2. Milnor manifolds

Now, we generalize our construction by passing from the complex numbers to the **quaternions** $\mathbb{H}$. Recall that $\mathbb{H}$ is the skew field which as abelian group is $\mathbb{R}^4$ with basis $1, i, j, k$ and multiplication defined by the relations $i^2 = j^2 = k^2 = -1$ and $ij = -ji, ik = -ki$, $jk = -kj$ and $ij = k, jk = i, ki = j$. It is useful to consider $\mathbb{H}$ as $\mathbb{C} \times \mathbb{C}$ with $1 = (1,0), i = (i,0), j = (0,1)$ and $k = (0,i)$. Then the multiplication is given by the formula

$$(z_1, z_2) \cdot (y_1, y_2) = (z_1 y_1 - \bar{y}_2 z_2, y_2 z_1 + z_2 \bar{y}_1)$$

The unit vectors $S^3 = \{(z_1, z_2) | z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$ form a multiplicative subgroup. In contrast to $S^1 \subset \mathbb{C}$, this subgroup is not commutative. This is the reason why we have more possibilities if we generalize our construction of L_k to quaternions.

Let k, l be integers. Then we define a diffeomorphism

$$\begin{array}{rcccc} f_{k,l} & : & S^3 \times S^3 & \longrightarrow & S^3 \times S^3 \\ & & (x,y) & \longmapsto & (x, x^k y x^l) \end{array}$$

and denote

$$M_{k,l} := D^4 \times S^3 \cup_{f_{k,l}} -D^4 \times S^3$$

Here, as in the case of lens spaces, one can show that $f_{k,l}$ is orientation preserving, and thus one has to take the opposite orientation on the second copy of $D^4 \times S^3$ to orient $M_{k,l}$ in a consistent way. As for lens spaces the projection onto $D^4 \cup -D^4$ gives a smooth fibre bundle $p : M_{k,l} \to S^4$. We call these manifolds **Milnor manifolds**, since they were investigated by Milnor in his famous paper "On manifolds homeomorphic to the 7-sphere" [**Mi 1**].

We can compute $\mathbf{H}_r(M_{k,l})$ in the same way as $\mathbf{H}_r(L_k)$ once we know the induced map

$$(f_{k,l})_* : \mathbf{H}_3(S^3 \times S^3) \to \mathbf{H}_3(S^3 \times S^3).$$

To compute this, consider two maps $f, g : S^3 \to S^3$. We compute the degree of

$$\begin{array}{rcccc} f \cdot g & : & S^3 & \longrightarrow & S^3 \\ & & x & \longmapsto & f(x) \cdot g(x) \end{array}$$

Lemma 11.3. *For continuous maps $f, g : S^3 \to S^3$ the degree of $f \cdot g$ is*

$$\deg(f \cdot g) = \deg f + \deg g$$

Proof: Consider the diagonal map $\triangle : S^3 \to S^3 \times S^3$ mapping $x \to (x,x)$. The induced map in homology maps the fundamental class $[S^3]$ to $[S^3, i_1] + [S^3, i_2]$, where $i_1(q) = (q, 1)$ and $i_2(q) = (1, q)$. The map $S^3 \times S^3 \to S^3$ mapping (q_1, q_2) to $f(q_1) \cdot g(q_2)$ induces a map in homology mapping $[S^3, i_1]$ to $\deg f \cdot [S^3]$ and $[S^3, i_2]$ to $\deg g \cdot [S^3]$ (why?). Thus $\deg f \cdot g = \deg f + \deg g$.
q.e.d.

With this information one concludes that, with respect to the basis $[S^3, i_1]$ and $[S^3, i_2]$ of $\mathbf{H}_3(S^3 \times S^3)$, the induced map of $f_{k,l}$ on $\mathbf{H}_3(S^3 \times S^3)$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ k+l & 1 \end{pmatrix}$$

From this, as in the case of L_k, one can compute the homology of $M_{k,l}$ and obtains

Proposition 11.4. $\mathbf{H}_r(M_{k,l}) = 0$ *for* $r > 7, r = 1, 2, 5, 6$*;*
$\mathbf{H}_0(M_{k,l}) = \mathbb{Z}$
$\mathbf{H}_7(M_{k,l}) = \mathbb{Z} \cdot [M_{k,l}]$
$\mathbf{H}_3(M_{k,l}) \cong \mathbb{Z}/|k+l| \cdot \mathbb{Z}$
$\mathbf{H}_4(M_{k,l}) = \begin{cases} \mathbb{Z} & k+l=0 \\ 0 & \textit{else} \end{cases}$

Thus $|k + l|$ is an invariant of the homotopy type. In contrast to L_k, this is not enough to distinguish the manifolds $M_{k,l}$. In the next chapters we will develop various techniques of general interest which all have some implications for the manifolds $M_{k,l}$. Thus, these manifolds serve in a wonderful way as motivating examples for theories of high importance.

CHAPTER 12

Singular cohomology and Poincaré duality

1. Singular cohomology groups

Prerequisites: We assume that the reader knows what a smooth vector bundle is [**B-J**], [**Hi**].

In this chapter we consider another bordism group of stratifolds which at the first glance looks like singular homology. It is only defined for smooth manifolds (without boundary). Similar groups were first introduced by Quillen [**Q**] and Dold [**D**]. They consider bordism classes of smooth manifolds instead of stratifolds.

The main difference between the new groups and singular homology is that we consider bordism classes of non-compact stratifolds. To obtain something non-trivial we require that the map $g : \mathbf{T} \to M$ is a proper map. We recall that a map between paracompact spaces is **proper** if the preimage of each compact space is compact. A second difference is that we only consider smooth maps. For simplicity we only define these bordism groups for oriented manifolds. (Each m-dimensional manifold is in a canonical way homotopy equivalent to a $m+1$-dimensional oriented manifold, the total space of the so called orientation line bundle, so that one can extend the definition to non-oriented manifolds using this trick.)

Definition: *Let M be an oriented smooth m-dimensional manifold without boundary. Then we define the* **integral cohomology group** $H^k(M)$ *as the group of bordism classes of proper smooth maps $g : \mathbf{S} \to M$, where $\mathbf{S}$ is an oriented regular stratifold of dimension $m-k$ (of course we require also for bordisms that the maps are proper and smooth and the stratifolds are oriented and regular).*

Concerning the notation we have denoted cohomology groups by the letter $H^k(M)$ and not $\mathbf{H}^k(M)$, which seems to be more natural in analogy with the letter we used for homology groups. The reason is that our cohomology groups agree with ordinary singular cohomology groups of smooth manifolds (which are always denoted by $H^k(M)$) whereas our homology groups $\mathbf{H}_k(X)$ are in general different from the singular homology groups $H_k(X)$ (but equal for nice spaces like manifolds and CW-complexes). For this compare the discussion in §20.

The reader might wonder, why we required that M is oriented. The definition seems to work without this condition. This will become clear when we define induced maps.

Then we will better understand the relation between $H^k(M)$ and $H^k(-M)$, where we change the orientation of M.

As for homology groups the sum is given by disjoint union and the inverse of $[\mathbf{S}, g]$ is represented by $(-\mathbf{S}, g)$. The relation between the grade k of $H^k(M)$ and the dimension $m-k$ of representatives of the bordism classes looks strange but we will see that it is natural for various reasons.

If M is a point then $g : \mathbf{S} \to pt$ is proper if and only if $\mathbf{S}$ is compact. Thus

$$H^k(pt) = \mathbf{H}_{-k}(pt) \cong \mathbb{Z}, \text{ if } k = 0, \text{ and } 0 \text{ if } k \neq 0.$$

To get a first feeling for cohomology classes, we consider the following situation. Let $p : E \to N$ be a k-dimensional smooth oriented vector bundle over an n-dimensional oriented smooth manifold. Then the total space E is a smooth $(k+n)$-dimensional manifold. The 0-section $s : N \to E$ is a proper map since $s(N)$ is a closed subspace. Thus

$$[N, s] \in H^k(E)$$

is a cohomology class. This is the most important example we have in mind and will play an essential role when we define characteristic classes. A special case is given by a 0-dimensional vector bundle where $E = N$ and $p = Id$. Thus we have for each smooth oriented manifold N the class $[N, Id] \in H^0(N)$, which we call $1 \in H^0(N)$. Later we will define a multiplication on the cohomology groups and it will turn out that multiplication with $[N, id]$ is the identity, justifying the notation.

Is the class $[N, s]$ non-trivial? We will see that it is often non-trivial but it is zero if E admits a no-where vanishing section $v : N \to E$. Namely then we obtain a zero bordism by taking the smooth manifold $N \times [0, \infty)$ and the map $G : N \times [0, \infty) \to E$ mapping $(x, t) \mapsto tv(x)$. The fact that v is no-where vanishing implies that G is a proper map. Thus we have shown

PROPOSITION 12.1. *Let $p : E \to N$ be a smooth oriented k-dimensional vector bundle over a smooth oriented manifold N. If E has a no-where vanishing section v then $[N, s] \in H^k(E)$ vanishes.*

In particular, if $[N, s]$ is non-trivial, then E does not admit a no-where vanishing section.

In the following considerations and constructions it is helpful for the reader to look at the cohomology class $[N, s] \in H^k(E)$ and test the situation with this class.

2. Poincaré duality

Cohomology groups are, as indicated for example in Proposition 12.1, a useful tool. To apply this tool one has to find methods for their computation. We will do this in two completely different ways. The fact that they are so different is very useful since one can combine the information to obtain very surprising results like the vanishing of the Euler characteristic of odd-dimensional compact smooth manifolds.

The first tool, the famous Poincaré duality isomorphism, only works for compact oriented manifolds and relates their cohomology groups to the homology groups. Whereas in the classical approach to (co)homology the duality theorem is difficult to prove it is almost trivial in our context. The second tool is the Kronecker isomorphism which relates the cohomology groups to the dual space of the homology groups. This will be explained in a later chapter.

Let M be a compact oriented smooth m-dimensional manifold. (Here we recall that if we use the term manifold, then it is automatically without boundary; manifolds with boundary are in this book always c-manifolds. Thus a compact manifold is what in the literature is often called a **closed manifold**, a compact manifold without boundary.) If M is compact and $f : \mathbf{S} \to M$ is a proper map, then $\mathbf{S}$ is actually compact. Thus we obtain a homomorphism

$$P : H^k(M) \to \mathbf{H}_{m-k}(M)$$

which assigns to $[\mathbf{S}, g] \in H^k(M)$ the class $[\mathbf{S}, g]$ considered as element of $\mathbf{H}_{m-k}(M)$. Here we only "forget" that the map g is smooth and consider it as continuous map.

THEOREM 12.2. **(Poincaré duality):** *For a* **closed** *smooth oriented manifold M the map*

$$P : H^k(M) \to \mathbf{H}_{m-k}(M)$$

is an isomorphism

Proof: For the proof we apply the following useful approximation result for continuous maps from a stratifold to a smooth manifold. It is another nice application of partition of unity.

PROPOSITION 12.3. *Let $f : \mathbf{S} \to N$ be a continuous map, which in an open neighbourhood of a closed subset $A \subset \mathbf{S}$ is smooth. Then there is a smooth map $g : \mathbf{S} \to N$ which on A agrees with f and which is homotopic to f rel. A.*

Proof The proof is the same as for a map from a smooth manifold M to N in ([**B-J**] Satz 14.8). More precisely there it is proved that if we embed N as closed subspace into an Euclidean space $\mathbb{R}^n$ then we can find a smooth map g arbitrary close to f. The proof only uses for M smooth partition of unity. Finally arbitrary close maps are homotopic by ([**B-J**] Satz 12.9).
q.e.d.

As a consequence we obtain a similar result for c-stratifolds.

Proposition 12.4. *Let $f : \mathbf{T} \to M$ be a continuous map from a smooth c-stratifold $\mathbf{T}$ to a smooth manifold M, whose restriction to $\partial \mathbf{T}$ is a smooth map. Then f is homotopic relative boundary to a smooth map.*

The proof follows from 12.3 using as subset A an appropriate closed subset in the collar of $\overset{\circ}{\mathbf{T}}$.

We apply this result to finish our proof. If $g : \mathbf{S} \to M$ represents an element of $\mathbf{H}_{m-k}(M)$ we can apply Proposition 12.3 to replace g by a homotopic smooth map g' and so $[\mathbf{S}, g] = P([\mathbf{S}, g'])$. This gives surjectivity of P. Similarly we use the relative version 12.4 to show injectivity. Namely if for $[\mathbf{S}_1, g_1]$ and $[\mathbf{S}_2, g_2]$ in $H^k(M)$ we have $P([\mathbf{S}_1, g_1]) = P([\mathbf{S}_2, g_2])$ there is a bordism $(\mathbf{T}, G)$ between these two pairs, where G is a continuous map whose restriction to the boundary is smooth. We apply Proposition 12.4 to replace G by a smooth map G' which on the boundary agrees with the restriction of G. Thus $[\mathbf{S}_1, g_1] = [\mathbf{S}_2, g_2] \in H^k(M)$ and P is injective.
q.e.d.

In analogy to integral cohomology groups by considering bordism classes of proper maps on oriented regular stratifolds we can define **$\mathbb{Z}/2$-cohomology groups** of arbitrary (non-oriented) smooth manifolds. The only difference is that we replace oriented regular stratifolds by $\mathbb{Z}/2$-oriented regular stratifolds which means that $\mathbf{S}^{n-1} = \emptyset$ and no condition on the orientability of the top stratum. The corresponding cohomology groups are denoted:

$$H^k(M; \mathbb{Z}/2)$$

The proof of Poincaré duality works the same way for $\mathbb{Z}/2$-(co)homology:

Theorem 12.5. **(Poincaré duality for $\mathbb{Z}/2$-(co)homology):** *For a* **closed** *smooth oriented manifold M the map*

$$P : H^k(M; \mathbb{Z}/2) \to \mathbf{H}_{m-k}(M; \mathbb{Z}/2)$$

is an isomorphism

As mentioned above, we want to provide other methods for computing the cohomology groups. They are based on the same ideas as used for computing homology groups, namely to show that the cohomology groups fulfil axioms similar to the axioms of homology groups. One of the applications of these axioms will be an isomorphism between $H^k(M) \otimes \mathbb{Q}$ and $\mathrm{Hom}(\mathbf{H}_k(M), \mathbb{Q})$ and an isomorphism $H^k(M; \mathbb{Z}/2) \cong \mathrm{Hom}(\mathbf{H}_k(M; \mathbb{Z}/2), \mathbb{Z}/2)$. The occurrence of the dual spaces $\mathrm{Hom}(\mathbf{H}_k(M), \mathbb{Q})$ and $\mathrm{Hom}(\mathbf{H}_k(M; \mathbb{Z}/2), \mathbb{Z}/2)$ indicates a difference between the fundamental properties of homology and cohomology. The induced maps occurring should reverse their directions. We will see that this is the case.

3. The Mayer-Vietoris sequence

One of the most powerful tools for computing cohomology groups is, as for homology, the Mayer-Vietoris sequence. To formulate it we have to define for an open subset U of a smooth oriented manifold M the map induced by the inclusion $i : U \to M$. We equip U with the orientation induced from M. If $g : \mathbf{S} \to M$ is a smooth proper map we consider the open subset $g^{-1}(U) \subset \mathbf{S}$ and restrict g to this open subset. It is again a proper map (why ?) and thus we define

$$i^*[\mathbf{S}, g] := [g^{-1}(U), g|_{g^{-1}(U)}]$$

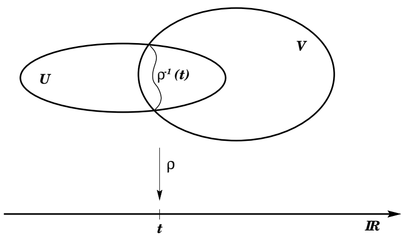
This is obviously well defined and a homomorphism $i^* : H^k(M) \to H^k(U)$. This map reverses the direction of the arrows, as was motivated above. If V is an open subset of U and $j : V \to U$ is the inclusion, then by construction

$$j^*i^* = (ij)^*.$$

The next ingredient for the formulation of the Mayer-Vietoris sequence is the boundary operator. We consider open subsets U and V in a smooth oriented manifold M, denote $U \cup V$ by X and define the boundary operator

$$\delta : H^k(U \cap V) \to H^{k+1}(U \cup V)$$

as follows. We introduce the disjoint closed subsets $A := X - V$ and $B := X - U$. We choose a smooth map $\rho : U \cup V \to \mathbb{R}$ mapping A to 1 and B to -1. Now we consider $[\mathbf{S}, f] \in H^k(U \cap V)$. Let $s \in (-1, 1)$ be a regular value of ρf. The preimage $\mathbf{D} := (\rho f)^{-1}(s)$ is an oriented regular stratifold of dimension $n-1$ sitting in $\mathbf{S}$. We define $\delta([\mathbf{S}, f]) := [\mathbf{D}, f|_{\mathbf{D}}] \in H^{k+1}(X)$. It is easy to check that $f|_{\mathbf{D}}$ is proper.



Similarly as in the definition of the boundary map for the Mayer-Vietoris sequence in homology one shows that δ is well defined and that one obtains an exact sequence. For details we refer to Appendix B.

At the first glance this definition of the boundary operator looks strange since $f(\mathbf{D})$ is contained in $U \cap V$. But considered as class in the cohomology of $U \cap V$ it is trivial. It is even zero in $H^{k+1}(U)$ as well as in $H^{k+1}(V)$. The reason is that in the construction of δ we can decompose $\mathbf{S}$ as $\mathbf{S}_+ \cup_{\mathbf{D}} \mathbf{S}_-$ with $\rho(\mathbf{S}_+) \geq s$ and $\rho(\mathbf{S}_-) \leq s$ (as for the boundary operator in homology we can assume up to bordism that there is a bicollar along $\mathbf{D}$). Then $(\mathbf{S}_-, f|_{\mathbf{S}_-})$ is a zero bordism of $(\mathbf{D}, f|_{\mathbf{D}})$ in U (note that $f|_{\mathbf{S}_-}$ is proper as map into U and not into V). Similarly $(\mathbf{S}_+, f|_{\mathbf{S}_+})$ is a zero bordism of $(\mathbf{D}, f|_{\mathbf{D}})$ in V. But in $H^{k+1}(U \cup V)$ it is in general non-trivial.

We summarize:

THEOREM 12.6. **(Mayer-Vietoris sequence for integral cohomology):** *The following sequence is exact and commutes with induced maps*

$$\cdots \to H^n(U \cup V) \to H^n(U) \oplus H^n(V) \to H^n(U \cap V) \xrightarrow{\delta} H^{n+1}(U \cup V) \to \cdots$$

The map from $H^n(U \cup V)$ to $H^n(U) \oplus H^n(V)$ is given by $\alpha \mapsto (j_U^*(\alpha), j_V^*(\alpha))$, the map from $H^n(U) \oplus H^n(V)$ to $H^n(U \cap V)$ by $(\alpha, \beta) \mapsto i_U^*(\alpha) - i_V^*(\beta)$

CHAPTER 13

Induced maps and the cohomology axioms

Prerequisites: In this chapter we apply one of the most powerful tools from differential topology, namely transversality. The necessary information can be found in [**B-J**], [**Hi**].

1. Transversality for stratifolds

We recall the basic definitions and results concerning transversality of manifolds. Let M, P and Q be smooth manifolds of dimensions m, p and q, and $f : P \to M$ and $g : Q \to M$ be smooth maps. Then we say that f is transversal to g if for all $x \in P$ and $y \in Q$ with $f(x) = g(y) = z$ we have $df(T_xP) + dg(T_yQ) = T_zM$. If $g : Q \to M$ is the inclusion of a point z in M then this condition means that z is a regular value of f. It is useful to note that the transversality condition is equivalent to the property that $f \times g : P \times Q \to M \times M$ is transversal to the diagonal $\Delta = \{(x,x)\} \subset M \times M$. Similar as for preimages of regular values one proves that $(P,f) \pitchfork (Q,g) := \{(x,y) \in (P \times Q) \,|\, f(x) = g(y)\}$ is a smooth submanifold of $P \times Q$ dimension $p + q - m$ [**B-J**], [**Hi**].

If all three manifolds are oriented then there is a canonical orientation on $(P,f) \pitchfork (Q,g)$. To define this we firstly consider the case, where g is an embedding. In this case we consider the normal bundle of $g(Q)$ and orient it in such a way that the juxtaposition of the orientations of TQ and the normal bundle give the orientation of M. In turn if an orientation of the normal bundle and of M is given we obtain an induced compatible orientation of Q. Now we note that in this case $(P,f) \pitchfork (Q,g)$ is a submanifold of P and the normal bundle of $(P,f) \pitchfork (Q,g)$ in P is the pull back of the normal bundle of Q in M and we equip it with the induced orientation, i.e. the orientation such the isomorphism between the fibres of the normal bundle induced by the differential of f is orientation preserving. By the considerations above this orientation of the normal bundle induces an orientation on $(P,f) \pitchfork (Q,g)$. If g is not an embedding we choose an embedding of Q into $\mathbb{R}^N$ (equipped with the canonical orientation) for some integer N and thicken M and P replacing them by $M \times \mathbb{R}^N$ and $P \times \mathbb{R}^N$ and replace f by $f \times \mathrm{id}$. The map given by g on the first component and by the embedding to $\mathbb{R}^n$ on the second gives an embedding of Q into $M \times \mathbb{R}^N$ and $f \times \mathrm{id}$ is transversal to this embedding and the preimage is canonically diffeomorphic to $(P,f) \pitchfork (Q,g)$. Thus the construction above gives an orientation on $(P,f) \pitchfork (Q,g)$. This orientation does not depend on the choice of N and the embedding to $\mathbb{R}^N$, since any two such embeddings are isotopic if we make N large enough by stabilization passing from $\mathbb{R}^N$ to $\mathbb{R}^{N+1}$. This definition of an induced orientation has the useful property that if $f' : P' \to P$ is another smooth map transversal to $(P, f) \pitchfork (Q, g)$, then the induced orientations on $(P', f') \pitchfork ((P', f) \pitchfork (Q, g))$ and $(P', ff') \pitchfork (Q, g)$ agree.

To shorten notation we often write $f \pitchfork g$ instead of $(P, f) \pitchfork (Q, g)$. If M, P and Q are oriented we mean this manifold with the induced orientation.

If we replace P by a smooth c-manifold with boundary and f is a smooth map transversal to g as well as $f|_{\partial P}$ transversal to g, then $(P, f) \pitchfork (Q, g) := \{(x, y) \in (P \times Q) \mid f(x) = g(y)\}$ is a smooth c-manifold of dimension $p+q-m$ with boundary $f|_{\partial P} \pitchfork g$. We obtain a similar statement if instead of admitting a boundary for P we replace Q by a c-manifold with boundary and require that f is transversal to the smooth c-map g as well as f transversal to $g|_{\partial Q}$.

The transversality theorem states that if $f : P \longrightarrow M$ and $g : Q \longrightarrow M$ are smooth maps then f is homotopic to f' such that f' is transversal to g [**B-J**], [**Hi**]. More generally, if $A \subset P$ is a closed subset and for some open neighbourhood U of A the maps $f|_U$ and g are transversal, then f is homotopic rel. A (i.e. the homotopy maps $(x, t) \in A \times T$ to $f(x)$) to f' such that f' is transversal to g.

The same argument implies the following statement:

THEOREM 13.1. *Let $f : P \longrightarrow M$ and $g_1 : Q_1 \longrightarrow M, \dots, g_r : Q_r \longrightarrow M$ be smooth maps such that for some closed subset $A \subset P$ and open neighbourhood U of A the maps $f|_U$ and g_i are transversal for $i = 1, \dots, r$. Then f is rel. A homotopic to f' such that f' is transversal to g_i for all i.*

We want to generalize this argument to maps $f : P \longrightarrow M$, where P is as before a smooth manifold, and $g : \mathbf{S} \longrightarrow M$ is a morphism from a stratifold to M. We say that f is **transversal** to g if and only if f is transversal to restrictions of g to all strata. If f is transversal to g, then we obtain a stratifold denoted by $g \pitchfork f$ whose underlying space is $\{(x, y) \in \mathbf{S} \times P | f(x) = g(y)\}$. The algebra is given by $\mathbf{C}(g \pitchfork f)$, the restriction of the functions in $\mathbf{S} \times P$ to this space. The argument for showing that this is a stratifold is the same as for the special case of the preimage of a regular value (Proposition 2.5). The strata of $g \pitchfork f$ are $g|_{\mathbf{S}^i} \pitchfork f$. The dimension of $g \pitchfork f$ is $\dim P + \dim \mathbf{S} - \dim M$. If $\mathbf{S}$ is a regular stratifold, then $g \pitchfork f$ is regular, the isomorphisms of appropriate local neighborhoods of the strata with a product are given by restrictions of the corresponding isomorphisms for $\mathbf{S} \times P$.

As a consequence of the transversality theorem for manifolds we see:

THEOREM 13.2. *Let $f : P \longrightarrow M$ be a smooth map from a smooth manifold P to M and $g : \mathbf{S} \longrightarrow M$ be a morphism from a stratifold $\mathbf{S}$ to M. Let A be a closed subset of P and U an open neighbourhood such that $f|_U$ is transversal to g. Then f is homotopic rel. A to f' such that f' is transversal to g.*

Proof: We simply apply Theorem 13.1 to replace f by f' (homotopic to f rel. A) such that f' is transversal to all $h|_{\mathbf{S}^i}$.
q.e.d.

2. The induced maps

We return to our construction of singular cohomology and define the induced maps. Let $f : N \to M$ be a smooth map between oriented manifolds and $[\mathbf{S}, g]$ be an element of $H^k(M)$. Then we replace f by a homotopic map $f' : N \to M$ which is transversal to g and consider $f' \pitchfork g$. This is a regular stratifold of dimension $n + \dim \mathbf{S} - m = n + m - k - m = n - k$. The stratum of dimension $n - k - 1$ is empty. The projection to N gives a map $g' : g \pitchfork f' \to N$. This is a proper map (why?). The orientations of M, N and $\mathbf{S}$ induce an orientation of $f' \pitchfork g$, as explained above. **This is the place where we used the orientation of the manifold** M. Thus the pair $(g \pitchfork f', g')$ represents an element of $H^k(N)$.

Using theorem 13.2 we see that the bordism class of $(g \pitchfork f', g')$ is unchanged if we choose another map f_1' homotopic to f and transversal to g. Namely then f' and f_1' are homotopic. We can assume that this homotopy is a smooth map, and that there is an $\epsilon > 0$ s.t. $h(x,t) = f'(x)$ for $t < \epsilon$ and $h(x,t) = f_1'(x)$ for $t > 1 - \epsilon$ (such a homotopy is often called a technical homotopy). By Theorem 13.2 we can further assume that this homotopy h is transversal to g. Then $(g \pitchfork h, g')$ is a bordism between $(g \pitchfork f', g')$ and $(g \pitchfork f_1', g')$.

For later use (Proof of Proposition 13.5) we note that this argument implies that the induced map is a homotopy invariant.

Next we show that if $(\mathbf{S}_1, g_1)$ and $(\mathbf{S}_2, g_2)$ are bordant, then $(g_1 \pitchfork f', g_1')$ is bordant to $(g_2 \pitchfork f', g_2')$, where f' is homotopic to f and transversal to g_1 and to g_2 simultaneously (by the argument above we are free in the choice of the map which is transversal to a given bordism class). Let $(\mathbf{T}, G)$ be a bordism between $(\mathbf{S}_1, g_1)$ and $(\mathbf{S}_2, g_2)$. Then using again the fact that we are free in the choice of f' we assume that f' is also transversal to G. Then $(G \pitchfork f', G')$ is a bordism between $(g_1 \pitchfork f', g_1')$ and $(g_2 \pitchfork f', g_2')$. Thus we obtain a well defined induced map

$$f^* : H^k(M) \to H^k(N)$$

mapping

$$[\mathbf{S}, g] \mapsto [g \pitchfork f', g']$$

where f' is transversal to g and g' is the restriction of the projection onto N. This construction respects disjoint unions and so we have defined the **induced homomorphism in cohomology** for a smooth map $f : N \to M$. As announced above this induced map in cohomology reverses its direction. By construction this definition agrees for inclusions with the previous definition used in the formulation of the Mayer-Vietoris sequence. Here one has to be careful with the orientation and we suggest that the reader checks that the conventions lead to the same definition.

The following is a useful special case for induced maps. Let $f : N \to M$ be an orientation preserving diffeomorphism. Then by construction $f^*([\mathbf{S}, g]) = [\mathbf{S}, f^{-1}g]$. If f reverses the orientation then $f^*([\mathbf{S}, g]) = -[\mathbf{S}, f^{-1}g]$. In particular if we consider as f the identity map from M to $-M$ equipped with opposite orientation, we see that $f^* = -\mathrm{id}$. (One should in this context not write id for the identity map, whose name in the oriented world should be reserved for the identity map from M to M, where both are equipped with the same orientation.)

If M and N are not oriented the same construction gives us an induced map

$$f^* : H^k(M; \mathbb{Z}/2) \to H^k(N; \mathbb{Z}/2)$$

mapping

$$[\mathbf{S}, g] \mapsto [g \pitchfork f', g']$$

A central case of an induced map is the situation considered in the previous chapter of a smooth oriented vector bundle $p : E \to N$ over an oriented manifold. We introduced the cohomology class $[N, s] \in H^k(E)$. We want to look at $s^*([N, s]) \in H^k(N)$. To obtain this class we have to approximate s by another map s' which is transversal to $s(N) \subset E$ (one can actually find s' which is again a section [**B-J**]). Then $s'(N) \pitchfork s(N)$ is a smooth submanifold of $s(N) = N$ of dimension $n - k$. Let $i : s'(N) \pitchfork s(N) \to s(N) = N$ be the inclusion then we obtain

$$s^*([N, s]) = [s'(N) \pitchfork s(N), i] \in H^k(N)$$

This class is called the **Euler class** of E and is abbreviated as

$$e(E) := s^*([N, s]) = [s'(N) \pitchfork s(N), i] \in H^k(N).$$

In the next chapter we will investigate this class in detail. From Proposition 12.1 we conclude:

PROPOSITION 13.3. *Let $p : E \to N$ be a smooth oriented k-dimensional vector bundle. If E has a no-where vanishing section v then $e(E) = 0$.*

3. The cohomology axioms

We will now formulate and prove properties of cohomology groups which are analogous to the axioms of a homology theory. Besides the fact that induced maps change direction the main difference is that we only defined cohomology groups of smooth manifolds and induced maps of smooth maps.

If $f : N \to M$ and $h : P \to N$ are smooth maps, such that f is transversal to $g : \mathbf{S} \to M$, where $\mathbf{S}$ is a regular stratifold, and h is transversal to $g' : f \pitchfork g \to N$, then

$fh : P \to M$ is transversal to $g : \mathbf{S} \to M$ and $fh \pitchfork g = h \pitchfork g'$ (with induced orientations as explained at the beginning of this chapter). This implies the following:

PROPOSITION 13.4. *Let $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ be smooth maps. Then*

$$f_1^* f_2^* = (f_2 f_1)^*$$

Furthermore by definition:

$$id^* = id$$

Here we stress again, that we reserved the name id for the identity map from M to M, both equipped with the same orientation!

Up to change of direction these are the properties of a functor assigning to a smooth manifold an abelian group and to a smooth map a homomorphism between these groups reversing its direction. To distinguish it from a functor we call it a **contravariant functor**. To make notation more symmetric, a functor is often also called a **covariant functor**.

To compare the Mayer-Vietoris sequence of different spaces it is useful to know that induced maps commute with the boundary operator. Since the construction of the boundary operator for cohomology is completely analogous to that for homology the same argument implies this statement.

The property of a contravariant functor (Proposition 13.4) is—in analogy to homology—the first fundamental property of a cohomology theory. The other two are the homotopy axiom and the Mayer-Vietoris sequence which he have already constructed. The homotopy axiom was also already shown when we proved that the induced map is well defined:

PROPOSITION 13.5. *Let $f : N \to M$ and $g : N \to M$ be homotopic smooth maps. Then*

$$f^* = g^* : H^k(M) \to H^k(M)$$

A contravariant functor $h^k(M)$ assigning to each smooth manifold abelian groups and to each smooth map an induced map such that the statements of Theorems 12.6, 13.4, 13.5 hold and where the boundary operator in the Mayer-Vietoris sequence commutes with induced maps, is called a **cohomology theory** for smooth manifolds and smooth maps. Thus singular cohomology is a cohomology theory.

As for homology one can use the cohomology axioms to compute the cohomology groups for many spaces like for example the spheres and complex projective spaces. For compact oriented manifolds without boundary one can use Poincaré duality and reduce it to the computation of homology groups.

CHAPTER 14

Products in cohomology and the Kronecker isomorphism

1. The $\times$-product and the Künneth theorem

So far the basic structure of cohomology is completely analogous to that of homology. The essential difference so far is the change of the direction of the maps between cohomology groups. We will in this chapter introduce a new structure called the cup product.

It is derived from the $\times$-product which is defined as the $\times$-product in homology. Let M and N be smooth oriented manifolds of dimension m and n respectively. If $[\mathbf{S}_1, g_1] \in H^k(M)$ and $[\mathbf{S}_2, g_2] \in H^r(N)$ we define

$$[\mathbf{S}_1, g_1] \times [\mathbf{S}_2, g_2] := (-1)^{nk}[\mathbf{S}_1 \times \mathbf{S}_2, g_1 \times g_2] \in H^{k+r}(M \times N).$$

The sign looks strange at the first glance but it is needed to give a comfortable expression when interchanging the factors, as we will discuss in the next paragraph. As in homology the **$\times$-product**

$$\times : H^k(M) \times H^r(N) \to H^{k+r}(M \times N)$$

is a bilinear and associative map. In the same way one defines the $\times$-product for $\mathbb{Z}/2$-cohomology (of course one can omit there the signs):

$$\times : H^k(M;\mathbb{Z}/2) \times H^r(N;\mathbb{Z}/2) \to H^{k+r}(M \times N;\mathbb{Z}/2)$$

As announced we study the behavior of the $\times$-product under a change of the factors. For this we consider the flip diffeomorphism $\tau : N \times M \to M \times N$ mapping (x, y) to (y, x), where M and N are m-resp. n-dimensional oriented manifolds. Then τ changes the orientation by $(-1)^{mn}$. Thus by the interpretation of induced map for diffeomorphisms, if $[\mathbf{S}, f] \in H^k(M)$ and $[\mathbf{S}', f'] \in H^r(N)$, then

$$\tau^*([\mathbf{S} \times \mathbf{S}', f \times f']) = (-1)^{mm'}[\mathbf{S} \times \mathbf{S}', \tau^{-1}(f \times f')].$$

To compare this with $[\mathbf{S}' \times \mathbf{S}, f' \times f]$ we consider the flip map τ' from $\mathbf{S} \times \mathbf{S}'$ to $\mathbf{S}' \times \mathbf{S}$ and note that $\tau^{-1}(f \times f') = (f' \times f)\tau'$. Since τ' changes the orientation by the factor $(-1)^{\dim \mathbf{S} \dim \mathbf{S}'} = (-1)^{(m-k)(n-r)}$, we conclude that

$$\tau^*([\mathbf{S}\times\mathbf{S}', f\times f']) = (-1)^{mn}(-1)^{(m-k)(n-r)}[\mathbf{S}'\times\mathbf{S}, (f'\times f)] = (-1)^{mr+nk+kr}[\mathbf{S}'\times\mathbf{S}, (f'\times f)].$$

Now we combine these signs with the sign occurring in the definition of the $\times$-product to obtain:

$$\tau^*([\mathbf{S}, f]\times[\mathbf{S}', f']) = \tau^*((-1)^{nk}([\mathbf{S}\times\mathbf{S}', f\times f']) = (-1)^{mr+kr}[\mathbf{S}'\times\mathbf{S}, (f'\times f)] = (-1)^{kr}[\mathbf{S}', f']\times[\mathbf{S}, f].$$

Thus we have the equality

$$\tau^*([\mathbf{S}, f] \times [\mathbf{S}', f']) = (-1)^{kr}([\mathbf{S}', f'] \times [\mathbf{S}, f])$$

The $\times$ product is a very useful tool. For example—as for homology—the $\times$-product is used in a Künneth theorem for rational cohomology and for $\mathbb{Z}/2$-cohomology. Here we define the **rational cohomology** groups $H^k(M;\mathbb{Q}) := H^k(M) \otimes \mathbb{Q}$. By elementary algebraic considerations similar to the arguments for rational homology groups one shows that rational cohomology fulfils the axioms of a cohomology theory. The proof of the Künneth Theorem would be the same as for homology if we had a comparison theorem like Corollary 9.4. The proof of Corollary 9.4 used the fact that homology groups are compactly supported. This is not the case for cohomology groups. But the inductive proof of Corollary 9.4 based on the 5-Lemma goes through in cohomology if we can cover M by finitely many open subsets U_i such that we know that the natural transformation is an isomorphism for all finite intersections of these subsets. This leads to the concept of a **good atlas** of a smooth manifolds M. This is an atlas $\varphi_i : U_i \to V_i$ such that all finite intersections of the U_i are diffeomorphic to $\mathbb{R}^m$. But $\mathbb{R}^m$ is homotopy equivalent to a point and, if we assume that for a point we have an isomorphism between the cohomology theories, the induction argument for the proof of Corollary 9.4 works for cohomology, if M has a finite good atlas:

Proposition 14.1. *Let M be a smooth oriented manifold admitting a finite good atlas. Let h and h' be cohomology theories and $\tau : h \to h'$ be a natural transformation which for a point is an isomorphism in all degrees. Then $\tau : h^k(M) \to (h')^k(M)$ is an isomorphism for all k.*

One can show that all smooth manifolds admit a good atlas (compare [**B-T**]). In particular all compact manifolds admit a finite good atlas.

If we combine Proposition 14.1 with the argument for the Künneth isomorphism in homology we obtain:

Theorem 14.2. **(Künneth Theorem for cohomology):** *Let M be a smooth oriented manifold admitting a finite good atlas. Then for F equal to $\mathbb{Z}/2$ or equal to $\mathbb{Q}$ the $\times$-product induces for each smooth oriented manifold N an isomorphism*

$$\times : \oplus_{i+j=k} H^i(M;F) \otimes_F H^j(N;F) \to H^k(M \times N;F)$$

If all cohomology groups of N are torsion free and finitely generated, then the same holds for integral cohomology.

2. The cup-product

The following construction with the $\times$-product is the main difference between homology and cohomology, since it can only be carried out for cohomology. Let $\Delta : M \to M \times M$

be the diagonal map $x \mapsto (x,x)$. Then we define the **cup product** as follows

$$\cup : H^k(M) \times H^r(M) \to H^{k+r}(M)$$

$$([\mathbf{S}_1, g_1], [\mathbf{S}_2, g_2]) \mapsto \Delta^*([\mathbf{S}_1, g_1] \times [\mathbf{S}_2, g_2]).$$

The properties of the $\times$-product imply that the cup product is bilinear and associative.

In addition it has the following property which one often calls **graded commutative**:

$$[\mathbf{S}_1, g_1] \cup [\mathbf{S}_2, g_2] = (-1)^{kr}[\mathbf{S}_2, g_2] \cup [\mathbf{S}_1, g_1].$$

This follows from the behavior of the $\times$ product under the flip map τ shown above together with the fact that $\tau\Delta = \Delta$.

There is also a neutral element, namely the cohomology class $[M, \mathrm{Id}] \in H^0(M)$. To see this we consider $[\mathbf{S}, g] \in H^k(M)$. Then $[M, \mathrm{Id}] \times [\mathbf{S}, g] = [M \times \mathbf{S}, \mathrm{Id} \times g]$. To determine $\Delta^*([M \times \mathbf{S}, \mathrm{Id} \times g])$ we note that $\mathrm{Id} \times g$ is transversal to Δ and so $\Delta^*([M \times \mathbf{S}, \mathrm{Id} \times g]) = [\mathbf{S}, g]$, i.e. $[M, \mathrm{Id}]$ is a neutral element. This property justifies our previous notation:

$$1 := [M, Id] \in H^0(M)$$

and we have

$$1 \cup [\mathbf{S}, g] = [\mathbf{S}, g].$$

Similarly, one shows

$$[\mathbf{S}, g] \cup 1 = [\mathbf{S}, g].$$

Furthermore we note that

$$f^*([\mathbf{S}_1, g_1] \cup [\mathbf{S}_2, g_2]) = f^*([\mathbf{S}_1, g_1]) \cup f^*([\mathbf{S}_2, g_2]),$$

i.e. the cup product commutes with induced maps.

For the computation of the $\cup$-product the following is a useful observation. Let M be an oriented manifold and suppose that $[N_1, g_1] \in H^k(M)$ and $[N_2, g_2] \in H^r(M)$ are cohomology classes with N_i smooth manifolds. Then we can obtain the cup product by considering as before $g := g_1 \times g_2$. But instead of making the diagonal transversal to g and then taking the transversal intersection we can keep the diagonal Δ unchanged, approximate g instead by a map g' transversal to Δ and take the transversal intersection. It is easy to use the transversality theorem to prove the existence of a bordism between the two cohomology classes obtained by making Δ transversal to g or by making g transversal to Δ. Furthermore we can interpret the latter transversal intersection as the transversal intersection of g_1 and g_2, i.e. we approximate g_1 by g_1' transversal to g_2 and then we obtain:

Let $[N_1, g_1] \in H^k(M)$ and $[N_2, g_2] \in H^r(M)$ be cohomology classes with N_i smooth manifolds such that g_1 is transversal to g_2. Then

$$[N_1, g_1] \cup [N_2, g_2] = [g_1 \pitchfork g_2, g_1 p_1],$$

where $m = \dim M$.

A priory this identity is only clear up to sign and we have to show that the sign is $+$. To do this it is enough to consider the case, where g_1 and g_2 are embeddings and after identifying N_i with their image under g_i we assume that N_i are submanifolds of M. The orientation of $N_1 \cap N_2 \subset N_1$ (which with the inclusion to M represents $[g_1 \pitchfork g_2, g_1 p_1]$) at $x \in N_1 \cap N_2$ is given by requiring that $T_x N_1 = T_x(N_1 \cap N_2) \oplus \nu_x(N_2, M)$ (where $\nu(N_2, M)$ is the normal bundle of N_2 in M) preserves the orientations induced from the orientation of N_i and M. On the other hand $\Delta^*([N_1 \times N_2, g_1 \times g_2])$ is represented by $N_1 \cap N_2$ together with the inclusion to M which we identify with $\Delta(M)$. The orientation at $x \in N_1 \cap N_2$ of $N_1 \cap N_2 \subset M$ is given by requiring that the decomposition $T_x(N_1 \cap N_2) \oplus \nu_x(N_1 \times N_2, M \times M) = T_x \Delta(M) = T_x M$ preserves the orientation. We have to determine the orientation of $\nu_x(N_1 \times N_2, M \times M)$ in terms of the orientations of $\nu_x(N_1, M)$ and $\nu_x(N_2, M)$. Comparing the orientations of $T_x N_1 \oplus \nu_x(N_1, M) \oplus T_x N_2 \oplus \nu_x(N_2, M) = T_x M \oplus T_x M$ and $T_x N_1 \oplus T_x N_2 \oplus \nu_x(N_1, M) \oplus \nu_x(N_2, M) = T_{(x,x)}(M \times M)$ we see that as oriented vector spaces $\nu_x(N_1 \times N_2, M \times M) = (-1)^{(m-n_1)n_2} \nu_x((N_1, M) \oplus \nu_x(N_2, M)$. Combining this with the identity $T_x(N_1 \cap N_2) \oplus \nu_x(N_1, M) \oplus \nu_x(N_2, M) = T_x M$ we obtain $(-1)^{(m-n_1)n_2}(T_x(N_1 \cap N_2) \oplus \nu_x(N_1, M) \oplus \nu_x(N_2, M)) = T_x M = T_x N_1 \oplus \nu_x(N_1, M)$. Comparing this with the orientation of $N_1 \cap N_2 \subset N_1$ we conclude that $(-1)^{(m-n_1)n_2}(T_x(N_1 \cap N_2) \oplus \nu_x(N_1, M) \oplus \nu_x(N_2, M) = T_x(N_1 \cap N_2) \oplus \nu_x(N_2, M) \oplus \nu_x(N_1, M)$ and so we conclude that the orientations differ by $(-1)^{(m-n_1)n_2}(-1)^{(m-n_1)(m-n_2)} = (-1)^{m(m-n_1)} = (-1)^{mk}$, where $k = m - n_1$. This is the sign we introduced when defining the $\times$-product and so we have shown that the sign in the formula is correct.

For example we can use this to compute the cup product structure for the complex projective spaces $\mathbb{CP}^n$. Since this is a closed oriented smooth manifold we have the tautology $H^{2k}(\mathbb{CP}^n) = \mathbf{H}_{2n-2k}(\mathbb{CP}^n)$ which by Theorem 8.8 is $\mathbb{Z}$ generated by $[\mathbb{CP}^{n-k}, i]$, where i is the inclusion $[z_0, ..., z_{n-k}] \mapsto [z_0, ..., z_{n-k}, 0..., 0]$. To compute the cup product $[\mathbb{CP}^{n-k}, i] \cup [\mathbb{CP}^{n-l}, j]$ we have to replace i by a map which is transversal to j. This can easily be done by choosing an appropriate other embedding, namely $i'([z_0, ..., z_{n-k}]) := [0, ..., 0, z_0, ..., z_{n-k}]$. This is the same homology class since the inclusions are homotopic. The map i' is transversal to j and so the cup product is represented by $[i'(\mathbb{CP}^{n-k}) \cap j(\mathbb{CP}^{n-l}), s]$, where s is again the inclusion. The intersection is $\mathbb{CP}^{n-k-l}$ and the map is up to a permutation the standard embedding. We conclude

$$[\mathbb{CP}^{n-k}, i] \cup [\mathbb{CP}^{n-l}, i] = [\mathbb{CP}^{n-k-l}, i]$$

As a consequence we look at $x := [\mathbb{CP}^{n-1}, i] \in H^2(\mathbb{CP}^n)$ and conclude:

$$x^r = [\mathbb{CP}^{n-r}, i] \in H^{2r}(\mathbb{CP}^n),$$

where x^r stands for the r-fold cup product. In particular $x^n = [\mathbb{CP}^0, i]$, the canonical generator of $H^{2n}(\mathbb{CP}^n)$.

It is useful to collect all cohomology groups into a direct sum and denote it by

$$H^*(M) := \oplus_k H^k(M)$$

The cup product induces a ring structure on this:

$$(\sum_i \alpha_i)(\sum_j \beta_j) := \sum_k (\sum_{i+j=k} \alpha_i \cup \beta_j),$$

where $\alpha_i \in H^i(M)$ and $\beta_j \in H^j(M)$. This way we consider $H^*(M)$ a ring called the **cohomology ring**. The computation above for the complex projective spaces can be reformulated as:

$$H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/x^{n+1},$$

This ring is called a truncated polynomial ring.

We also introduce the $\mathbb{Z}/2$-cohomology ring as

$$H^*(M;\mathbb{Z}/2) := \oplus_k H^k(M;\mathbb{Z}/2)$$

By a similar argument one shows:

$$H^*(\mathbb{RP}^n;\mathbb{Z}/2) = (\mathbb{Z}/2)[x]/x^{n+1},$$

where $x \in H^1(\mathbb{RP}^n;\mathbb{Z}/2)$ is the non-trivial element.

3. The Kronecker isomorphism

Now we can prove the announced relation between cohomology and homology groups. Let M be an oriented smooth m-dimensional manifold. The first step is the construction of the so called Kronecker homomorphism from $H^k(M)$ to $\mathrm{Hom}\,(\mathbf{H}_k(M),\mathbb{Z})$. The map is induced by a bilinear map $H^k(M) \times \mathbf{H}_k(M) \to \mathbb{Z}$, where M is an oriented smooth manifold. To describe this let $[\mathbf{S}_1, g_1] \in H^k(M)$ be a cohomology class and $[\mathbf{S}_2, g_2] \in \mathbf{H}_k(M)$ be a homology class. Applying Proposition 12.4 we can approximate g_2 by a smooth map and so we assume from now on that g_2 is smooth. We consider $g = g_1 \times g_2 : (-1)^{mk}\mathbf{S}_1 \times \mathbf{S}_2 \to M \times M$. The sign changing the orientation of $\mathbf{S}_1 \times \mathbf{S}_2$ is compatible with the sign introduced in the definition of the $\times$-product.

Let $\Delta : M \to M \times M$ be the diagonal map. We want to approximate Δ by a smooth map Δ' with is transversal to $g_1 \times g_2$ in such a way that the transversal intersection $\Delta' \pitchfork (g_1 \times g_2)$ is compact. To achieve this we note that since g_1 is proper and $\mathbf{S}_2$ is compact the intersection $\mathrm{im}(g_1 \times g_2) \cap \mathrm{im}(\Delta)$ is compact. Namely, we define $C_0 := \{x \in \mathbf{S}_1 |\, g_1(x) \in \mathrm{im}(g_2)\}$, which is compact since $\mathbf{S}_2$ is compact and g_1 is proper. Thus $g_1 \times g_2(C_0 \times \mathbf{S}_2)$ is compact. But $\mathrm{im}(g_1 \times g_2) \cap \mathrm{im}(\Delta)$ is a closed subset of $(g_1 \times g_2)(C_0 \times \mathbf{S}_2)$ and so is compact. Since Δ is proper $C_1 := \Delta^{-1}(\mathrm{im}(g_1 \times g_2) \cap \mathrm{im}(\Delta))$ is compact. We choose compact subsets $C_2 \subset C_3 \subset M$ such that $C_1 \subset \mathring{C}_2$ and $C_2 \subseteq \mathring{C}_3$. Then $A := M - \mathring{C}_2$ is a closed subset which is contained in the open subset $U := M - C_1$. Since $\mathrm{im}(g_1 \times g_2) \cap \Delta(U) = \emptyset$, the map $\Delta|_U$ is transversal to $g_1 \times g_2$. We approximate Δ by a transversal map Δ'

which on A agrees with Δ. By construction $\Delta' \pitchfork (g_1 \times g_2) \subset C_2 \times \mathbf{S}_1 \times \mathbf{S}_2$. The set $D := \{x \in \mathbf{S}_1 | \, g_1(x) \in \mathrm{im}(p_1 \Delta'(C_2))\}$ is compact since $p_1(\Delta'(C_2))$ is compact and g_1 is proper. But $\Delta' \pitchfork (g_1 \times g_2) \subset C_2 \times D \times \mathbf{S}_2$ is a closed subset of a compact space and so compact. It is a zero-dimensional stratifold and oriented. We consider the sum of the orientations of this stratifold, where we recall that we equipped $\mathbf{S}_1 \times \mathbf{S}_2$ with $(-1)^{mk}$ times the product orientation. This way we attach to a cohomology class $[\mathbf{S}_1, g_1] \in H^k(M)$ and a homology class $[\mathbf{S}_2, g_2] \in \mathbf{H}_k(M)$ an integer denoted:

$$\langle [\mathbf{S}_1, g_1], [\mathbf{S}_2, g_2] \rangle \in \mathbb{Z}$$

A similar transversality argument as used for showing that f^* is well defined implies that this number is well defined, if we assume the same transversality condition for the bordisms.

The construction gives a bilinear map which we call the **Kronecker pairing** or **Kronecker product**:

$$\langle ..., ... \rangle : H^k(M) \times \mathbf{H}_k(M) \to \mathbb{Z}.$$

If M is a compact m-dimensional smooth manifold there is the following relation between the cup-product, Poincaré duality and the Kronecker pairing:

PROPOSITION 14.3. *Let* $[\mathbf{S}_1, g_1] \in H^k(M)$ *and* $[\mathbf{S}_2, g_2] \in H^{m-k}(M)$ *be cohomology classes. Then*

$$\langle [\mathbf{S}_1, g_1], P([\mathbf{S}_2, g_2]) \rangle = \langle [\mathbf{S}_1, g_1] \cup [\mathbf{S}_2, g_2], [M] \rangle$$

This useful identity follows from the definitions.

The Kronecker pairing gives a homomorphism $H^k(M) \to \mathrm{Hom}\,(\mathbf{H}_k(M), \mathbb{Z})$ by mapping $[\mathbf{S}_1, g_1] \in H^k(M)$ to the homomorphism assigning to $[\mathbf{S}_2, g_2] \in \mathbf{H}_k(M)$ the Kronecker pairing $\langle [\mathbf{S}_1, g_1], [\mathbf{S}_2, g_2] \rangle$. We call this the **Kronecker homomorphism**:

$$\kappa : H^k(M) \to \mathrm{Hom}(\mathbf{H}_k(M), \mathbb{Z})$$

The Kronecker homomorphism $H^k(M)$ to $\mathrm{Hom}\,(\mathbf{H}_k(M), \mathbb{Z})$ commutes with induced maps $f : N \to M$:

$$\langle f^*([\mathbf{S}_1, g_1]), [\mathbf{S}_2, g_2] \rangle = \langle [\mathbf{S}_1, g_1], f_*([\mathbf{S}_2, g_2]) \rangle$$

for all $[\mathbf{S}_1, g_1] \in H^k(M)$ and $[\mathbf{S}_2, g_2] \in \mathbf{H}_k(N)$.

The Kronecker homomorphism also commutes with the boundary operators in the Mayer-Vietoris sequence. The argument is the following. Choose a separating function $\rho : U \cup V \to \mathbb{R}$ as in the definition of δ. For $[\mathbf{S}_1, g_1] \in H^k(U \cap V)$ and $[\mathbf{S}_2, g_2] \in \mathbf{H}_{k-1}(U \cup V)$ we choose a common regular value s of ρg_1 and ρg_2. This gives a decomposition of $\mathbf{S}_1 = (\mathbf{S}_1)_+ \cup (\mathbf{S}_1)_-$ and $\mathbf{S}_2 = (\mathbf{S}_2)_+ \cup (\mathbf{S}_2)_-$ as in the definition of δ in §10. Then $\delta([\mathbf{S}_1, g_1]) = [\partial(\mathbf{S}_1)_+, g_1|_{\partial(\mathbf{S}_1)_+}]$ and $d([\mathbf{S}_2, g_2]) = [\partial(\mathbf{S}_2)_+, g_2|_{\partial(\mathbf{S}_2)_+}]$. We consider the oriented regular stratifold $(\mathbf{S}_1)_+ \times (\mathbf{S}_2)_+$ with boundary $(\partial(\mathbf{S}_1)_+ \times (\mathbf{S}_2)_+) \cup_{\partial(\mathbf{S}_1)_+ \times \partial(\mathbf{S}_2)_+} -((\mathbf{S}_1)_+ \times \partial((\mathbf{S}_2)_+))$.

(The product of two bounded stratifolds has—as the product of two bounded smooth manifolds—corners. There is a standard method for smoothing the corners which is based on collars. Thus the same can be done for stratifolds. Smoothing of corners is in a different context explained in appendix A.) Now we approximate the diagonal map $\Delta : X \to X \times X$ by a map Δ' which is transversal to $g_1 \times g_2 : (\mathbf{S}_1)_+ \times (\mathbf{S}_2)_+ \to X \times X$ and the restriction of $g_1 \times g_2$ to $\partial((\mathbf{S}_1)_+ \times (\mathbf{S}_2)_+)$ and to $\partial(\mathbf{S}_1)_+ \times \partial(\mathbf{S}_2)_+$. We consider the bounded stratifold $(\mathbf{S}_1 \times \mathbf{S}_2, g_1 \times g_2) \pitchfork (X, \Delta')$. This is a 1-dimensional stratifold with boundary $(\mathbf{S}_1 \times \mathbf{S}_2, g_1 \times g_2)|_{\partial((\mathbf{S}_1)_+ \times (\mathbf{S}_2)_+)} \pitchfork (X, \Delta')$. Since Δ' is transversal to $\partial(\mathbf{S}_1)_+ \times \partial(\mathbf{S}_2)_+$ the dimension of $(\partial(\mathbf{S}_1)_+ \times \partial(\mathbf{S}_2)_+, g_1|_{\partial(\mathbf{S}_1)_+} \times g_2|_{\partial(\mathbf{S}_2)_+}) \pitchfork (X, \Delta')$ is -1 implying that the boundary of $(\mathbf{S}_1 \times \mathbf{S}_2, g_1 \times g_2) \pitchfork (X, \Delta')$ is $(\partial(\mathbf{S}_1)_+ \times \mathbf{S}_2, g_1|_{\partial(\mathbf{S}_1)_+} \times g_2) \pitchfork (X, \Delta') + (-\mathbf{S}_1 \times \partial(\mathbf{S}_2)_+, g_1 \times g_2|_{\partial(\mathbf{S}_2)_+}) \pitchfork (X, \Delta')$ (the sign comes from the sign in the decomposition of the boundary of $(\mathbf{S}_1)_+ \times (\mathbf{S}_2)_+$). The number of oriented intersection points of $(\partial(\mathbf{S}_1)_+ \times \mathbf{S}_2, g_1|_{\partial(\mathbf{S}_1)_+} \times g_2) \pitchfork (X, \Delta')$ is the Kronecker pairing of $\delta([\mathbf{S}_1, g_1])$ and $[\mathbf{S}_2, g_2]$. The number of oriented intersection points of $(\mathbf{S}_1 \times \partial(\mathbf{S}_2)_+, g_1 \times g_2|_{\partial(\mathbf{S}_2)_+}) \pitchfork (X, \Delta')$ is the Kronecker pairing of $[\mathbf{S}_1, g_1]$ and $d([\mathbf{S}_2, g_2])$. Thus these two numbers agree.

These considerations imply that the Kronecker homomorphism gives a natural transformation from

$$H^k(M) \to \operatorname{Hom}(\mathbf{H}_k(M), \mathbb{Z}).$$

Unfortunately $\operatorname{Hom}(\mathbf{H}_k(M), \mathbb{Z})$ is not a cohomology theory. The reason is that if $A \to B \to C$ is an exact sequence then in general the induced sequence $\operatorname{Hom}(C, \mathbb{Z}) \to \operatorname{Hom}(B, \mathbb{Z}) \to \operatorname{Hom}(A, \mathbb{Z})$ is not exact. But by a similar argument as for taking the tensor product with $\mathbb{Q}$ the induced sequence $\operatorname{Hom}(C, \mathbb{Q}) \to \operatorname{Hom}(B, \mathbb{Q}) \to \operatorname{Hom}(A, \mathbb{Q})$ is exact. Thus $\operatorname{Hom}(\mathbf{H}_k(X), \mathbb{Q})$ is a cohomology theory.

We can in a similar way define the Kronecker pairing for the $\mathbb{Z}/2$-(co)homology groups of a (not necessarily orientable) manifold M. The only difference is that we have to take the number of points $\bmod 2$ in the transversal intersection instead of the sum of the orientations as before. From the Kronecker product we obtain as before a natural transformation

$$H^k(M; \mathbb{Z}/2) \to \operatorname{Hom}(\mathbf{H}_k(M; \mathbb{Z}/2), \mathbb{Z}/2),$$

where now both sides are cohomology theories.

For M a point both these natural transformations are obviously an isomorphism. Thus we obtain from Proposition 14.1:

THEOREM 14.4. *The Kronecker homomorphism is for all smooth oriented manifolds M admitting a finite good atlas an isomorphism:*

$$\kappa : H^k(M; \mathbb{Q}) \cong \mathit{Hom}(\mathbf{H}_k(M), \mathbb{Q})$$

and if M is not oriented:

$$\kappa : H^k(M; \mathbb{Z}/2) \cong \mathit{Hom}(\mathbf{H}_k(M; \mathbb{Z}/2), Z/2)$$

In particular this theorem applies to all compact oriented manifolds. There is also a version of the Kronecker Theorem for integral cohomology, but the Kronecker homomorphism is not an isomorphism any more. It is still surjective and the kernel is isomorphic to the torsion subgroup of $\mathbf{H}_{k-1}(M)$. We will not give a proof of this result. One way to prove it is to use the isomorphism between our (co)homology groups and the classical groups defined using chain complexes. This will be explained in a later chapter. The world of chain complexes is closely related to homological algebra and in this context the integral Kronecker Theorem is rather easy to prove. One can also give a more direct prove using linking numbers, but this would lead to far in our present context.

As announced before we want to combine for closed (oriented) manifolds Poincaré duality with other ways to determine (co)homology groups, like for example the Kronecker Theorem. Here are examples of such applications.

If we compose the Kronecker isomorphism with Poincaré duality we obtain the following non-trivial consequence:

Corollary 14.5. *Let M be a closed smooth oriented m-dimensional manifold. Then the Kronecker pairing induces an isomorphism which we denote by $\langle ..., ... \rangle$:*

$$\langle ..., ... \rangle : \mathbf{H}_{m-k}(M;\mathbb{Q}) \cong Hom\,(\mathbf{H}_k(M),\mathbb{Q}).$$

Similarly if M is not necessarily oriented:

$$\langle ..., ... \rangle : \mathbf{H}_{m-k}(M;\mathbb{Z}/2) \cong Hom\,(\mathbf{H}_k(M;\mathbb{Z}/2),\mathbb{Z}/2)$$

An important consequence of this result is that the Euler characteristic of an odd-dimensional closed smooth manifold M vanishes. For, the Betti numbers $b_k(M;\mathbb{Z}/2)$ are equal to $b_{m-k}(M;\mathbb{Z}/2)$ and so $(-1)^k b_k(M;\mathbb{Z}/2) + (-1)^{m-k} b_{m-k}(M;\mathbb{Z}/2) = 0$ implying:

Corollary 14.6. *The Euler characteristic of a smooth closed odd-dimensional manifold vanishes.*

We earlier quoted a result from differential topology that there is a nowhere vanishing vector field on a closed smooth manifold if and only if the Euler characteristic vanishes. As a consequence of the corollary we conclude that each closed odd-dimensional smooth manifold has a nowhere vanishing vector field.

CHAPTER 15

The signature

As an application of the cup product, we define the signature of a closed smooth oriented $4k$-dimensional manifold and prove an important property. We recall from linear algebra the definition of the signature or index of a symmetric bilinear form over a finite dimensional $\mathbb{Q}$-vector space

$$b : V \times V \longrightarrow \mathbb{Q}.$$

The signature $\tau(b)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of a matrix representation of b. Equivalently, one chooses a basis $e_1, \cdots, e_r$ of V such that $b(e_i, e_j) = 0$ for $i \neq j$ and defines $\tau(b)$ as the number of e_i with $b(e_i, e_i) > 0$ minus the number of e_j with $b(e_j, e_j) < 0$. This is independent of any choices and a fundamental algebraic invariant. If we replace b by $-b$ the signature changes its sign:

$$\tau(-b) = -\tau(b)$$

Now we define the signature of a closed smooth oriented $4k$-dimensional manifold M. We know from §8 that the cohomology group $H^{2k}(M) \cong \mathbf{H}_{2k}(M)$ is finitely generated. Recall that we abbreviated the fundamental class $[M, \mathrm{id}] \in \mathbf{H}_{4k}(M)$ by $[M]$. The **intersection form** of M is the bilinear form

$$S(M) : H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z}$$

mapping

$$(\alpha, \beta) \mapsto \langle \alpha \cup \beta, [M] \rangle,$$

the Kronecker pairing between $\alpha \cup \beta$ and the fundamental class. Since $\alpha \cup \beta = (-1)^{(2k)^2} \beta \cup \alpha$ the intersection form is symmetric. Thus we can consider after taking the tensor product with $\mathbb{Q}$ the signature $\tau(S(M) \otimes \mathbb{Q})$ and define the **signature** of M as

$$\tau(M) := \tau(S(M) \otimes \mathbb{Q}).$$

This is an important invariant of manifolds as we will see. If we replace M by $-M$ then we only replace $[M]$ by $-[M]$ and thus S changes its sign implying

$$\tau(-M) = -\tau(M).$$

Since $H^{2k}(S^{4k}) = 0$, it is zero on spheres. We have computed the cohomology ring of $\mathbb{CP}^{2k}$ and know that $H^{2k}(\mathbb{CP}^{2k}) = \mathbb{Z}x^k$ and that $\langle x^{2k}, [\mathbb{CP}^{2k}] \rangle = 1$. Thus we have:

$$\tau(\mathbb{CP}^{2k}) = 1.$$

The relevance of the signature is demonstrated by the fact that it is bordism invariant:

Theorem 15.1. *If a compact oriented smooth manifold M is the boundary of a compact oriented smooth c-manifold, then its signature vanishes:*

$$\tau(M) = 0$$

The main ingredient of the proof is the following:

Lemma 15.2. *Let W be a compact smooth oriented c-manifold of dimension $2k+1$. Let $j : \partial W \to \overset{\circ}{W}$ be the map given by $j(x) := \varphi(x, \epsilon/2)$, where φ is the collar of W. Then*

$$\ker(j_* : \mathbf{H}_k(\partial W) \to \mathbf{H}_k(\overset{\circ}{W})) = im(j^* : H^k(\overset{\circ}{W}) \to H^k(\partial W) = \mathbf{H}_k(\partial W)).$$

Proof: If $[\mathbf{S}, g] \in \mathbf{H}_k(\partial W)$ maps to 0 under j_* there is a compact regular c-stratifold $\mathbf{T}$ with $\partial \mathbf{T} = \mathbf{S}$ and a map $G : \mathbf{T} \to \overset{\circ}{W}$ extending $j \circ g$. Now we consider $P := \mathbf{T} \cup_{\partial \mathbf{T} \times \epsilon/2} \partial \mathbf{T} \times (0, \epsilon/2]$ and extend G to a smooth proper map $\bar{G} : P \to \overset{\circ}{W}$ in such a way that for t small enough (x,t) is mapped to $\varphi(g(x), t)$. For some fixed $\delta > 0$ we consider $j_\delta : \partial W \to \overset{\circ}{W}$ by mapping x to $\varphi(x, \delta)$. For δ small enough (so that the intersection of the image of $\mathbf{T}$ with the image of j_δ is empty) we have by construction of $[P, \bar{G}]$ that $j_\delta^*([P, \bar{G}]) = \pm[\mathbf{S}, g]$. Since j_δ is homotopic to j we have shown $\ker j_* \subset \operatorname{im} j^*$.

To show the other inclusion, we consider $[P, h] \in H^k(\overset{\circ}{W})$. By Sard's Theorem h is transversal to $\varphi(\partial W, \delta)$ for some $\delta > 0$. We denote $\mathbf{S} = h \pitchfork \varphi(\partial W \times \delta)$. Then $j_\delta^*([P, h]) = [\mathbf{S}, h|_{\mathbf{S}}]$ and—since j_δ is homotopic to j—we have $j^*([P, h]) = [\mathbf{S}, h|_{\mathbf{S}}]$. To show that $j_*([\mathbf{S}, h|_{\mathbf{S}}]) = 0$ we consider $h^{-1}(\overset{\circ}{W} - (\partial W \times (0, \delta))$. We are finished if this is a regular c-stratifold $\mathbf{T}$ with boundary $\mathbf{S}$. Namely then $(\mathbf{T}, h|_{\mathbf{T}})$ is a zero bordism of $(\mathbf{S}, h|_{\mathbf{S}})$. Now we assume that $\mathbf{S}$ has a bicollar in P (for this we have to replace P by a bordant regular stratifold as explained in Appendix B (see Lemma B.1 in the detailed proof of the Mayer-Vietoris sequence). Then it is clear that $h^{-1}(\overset{\circ}{W} - (\partial W \times (0, \delta))$ is a oriented regular c-stratifold $\mathbf{T}$ with boundary $\mathbf{S}$ finishing the argument.
q.e.d.

This Lemma is normally obtained from the generalization of Poincaré duality to compact oriented manifolds with boundary, the Lefschetz duality Theorem. But one only needs this partial elementary information for the proof of Theorem 15.1.

Combining this Lemma with the Kronecker isomorphism (which implies that after passing to rational (co)homology we have: $j^* = (j_*)^*$, where the last $*$ denotes the dual map) we conclude that for $j_* : \mathbf{H}_k(\partial W) \to \mathbf{H}_k(\overset{\circ}{W})$:

$$\operatorname{rank}(\ker j_*) = \operatorname{rank}(\operatorname{im}((j_*)^*)).$$

From linear algebra we know that $\operatorname{rank}(\operatorname{im} j_*) = \operatorname{rank}(\operatorname{im}((j_*)^*))$ and we obtain:

$$\operatorname{rank}(\ker j_*) = \operatorname{rank}(\operatorname{im} j_*)$$

and by the dimension formula:

$$\operatorname{rank}(\ker j_*) = 1/2 \operatorname{rank} \mathbf{H}_k(\partial W)$$

Applying Lemma 15.2 again we finally note:

$$\operatorname{rank}(\operatorname{im} j^*) = 1/2 \operatorname{rank} H^k(\partial W).$$

As a last preparation for the proof of Theorem 15.1 we need the following observation from linear algebra. Let $b : V \times V \to \mathbb{Q}$ be a symmetric non-degenerate bilinear form on a finite dimensional $\mathbb{Q}$-vector space. Suppose that there is a subspace $U \subset V$ with $\dim U = \frac{1}{2} \dim V$ such that, for all $x, y \in U$, we have $b(x, y) = 0$. Then $\tau(b) = 0$. The reason is the following. Let $e_1, \dots, e_n$ be a basis of U. Since the form is non-degenerate, there are elements $f_1, \dots, f_n$ in V such that $b(f_i, e_j) = \delta_{ij}$ and $b(f_i, f_j) = 0$. This implies that $e_1, \dots, e_n, f_1, \dots, f_n$ are linear independent and thus form a basis of V. Now consider $e_1 + f_1, \dots, e_n + f_n, e_1 - f_1, \dots, e_n - f_n$ and note that, with respect to this basis, b has the form

$$\begin{pmatrix} 2 & & & & & \\ & \ddots & & & & \\ & & 2 & & & \\ & & & -2 & & \\ & & & & \ddots & \\ & & & & & -2 \end{pmatrix}$$

and thus

$$\tau(b) = 0.$$

Proof of Theorem 12.6: We first note that for $\alpha \in \operatorname{im} j^*$ and $\beta \in \operatorname{im} j^*$ the intersection form $S(\partial W)(\alpha, \beta)$ vanishes. For if, $\alpha = j^*(\bar{\alpha})$ and $\beta = j^*(\bar{\beta})$ then

$$S(\partial W)(\alpha, \beta) = \langle j^*(\bar{\alpha}) \cup j^*(\bar{\beta}), [\partial W] \rangle = \langle \bar{\alpha} \cup \bar{\beta}, j_*([\partial W] \rangle = 0$$

since $j_*([\partial W]) = 0$ (note that W is a zero bordism).

Thus the intersection form vanishes on $\operatorname{im} j^*$. By Poincaré duality the intersection form $S(\partial W) \otimes \mathbb{Q}$ is non-degenerate. Since the rank of $\operatorname{im} j^*$ is $1/2 \operatorname{rank} H^k(\partial W)$ the proof is finished by the considerations above from linear algebra.
q.e.d.

The relevance of Theorem 15.1 becomes more visible if we define bordism groups of compact oriented smooth manifolds. They were introduced by Thom [**Th 1**] who computed them and provided with this the ground for very interesting applications (for example the signature theorem, which in a special case we will discuss later). The group Ω_n is defined as the bordism classes of compact oriented smooth manifolds. More precisely the elements in Ω_n are represented by a compact smooth n-dimensional manifold M and two such manifolds M and M' are equivalent if there is a compact oriented manifold W with boundary $M + -M'$. The sum is given by disjoint union, the inverse of a bordism class M is $-M$. Thus the definition is analogous to the definition of $\mathbf{H}_n(pt)$, the difference is that we only consider manifolds instead of regular stratifolds.

Whereas it was simple to determine $\mathbf{H}_n(pt)$ it is very difficult to compute the groups Ω_n. This difficulty is indicated by the following consequence of Theorem 15.1.

The signature of a disjoint union of manifolds is the sum of the signatures, and $\tau(-M) = -\tau(M)$. Thus we conclude from Theorem 15.1, that

$$\tau : \Omega_{4k}(pt) \to \mathbb{Z}$$

is a homomorphism. This homomorphism $\tau : \Omega_{4k}(pt) \to \mathbb{Z}$ is a surjective map. The reason is that $\tau(\mathbb{CP}^{2k}) = 1$.

Thus we obtain:

Corollary 15.3. *For each $k \geq 0$ the groups $\Omega_{4k}(pt)$ are non-trivial.*

It is natural to ask what the signature of a product of two manifolds is. It is the product of the signatures of the two manifolds:

Theorem 15.4. *Let M and N be closed oriented smooth manifolds. Then*

$$\tau(M \times N) = \tau(M)\tau(N).$$

The proof is based on the Künneth theorem for rational cohomology and Poincaré duality and we refer to Hirzebruch's original proof [**Hir**], p. 85.

CHAPTER 16

The Euler class

1. The Euler class

We recall the definition of the Euler class. Let $p : E \to M$ be a smooth oriented k-dimensional vector bundle over a smooth oriented manifold M. Let $s : M \to E$ be the zero section. Then $e(E) := s^*[M, s] \in H^k(M)$ is the **Euler class** of E. The Euler class is called a **characteristic class**. We will define other characteristic classes like the Chern, Pontrjagin and Stiefel-Whitney classes.

By construction the Euler classes of bundles $p : E \to M$ and $p' : E' \to M$, which are orientation preserving isomorphic, are equal. Thus the Euler class is an invariant of the oriented isomorphism type of a smooth vector bundle. We also recall Proposition 13.3, that if a smooth oriented bundle E has a nowhere vanishing section v then $e(E) = 0$. In particular the Euler class of a positive dimensional trivial bundle is 0.

The following properties of the Euler class are a fundamental tool.

THEOREM 16.1. *Let $p : E \to M$ be a smooth oriented vector bundle. Then, if $-E$ is E with opposite orientation :*

$$e(-E) = -e(E)$$

If $f : N \to M$ is a smooth map, then the Euler class is natural:

$$e(f^*E) = f^*(e(E)).$$

If $q : F \to M'$ is another smooth oriented vector bundle then

$$e(E \times F) = e(E) \times (F),$$

and if $M = M'$,

$$e(E \oplus F) = e(E) \cup e(F).$$

Here we recall that the Whitney sum $E \oplus F := \Delta^*(E \times F)$ is the pull back of $E \times F$ under the diagonal map. The fibre of $E \oplus F$ at x is $E_x \oplus F_x$.

Proof: The first property follows from the definition of the Euler class. For the second property we first consider the case where $N \subset M$ is a submanifold of M and f is the inclusion. In this case it is clear from the definition that $e(f^*E) = f^*e(E)$. Next we assume that f is a diffeomorphism and note that the property follows again from the definition. Combining these two cases we conclude that the statement holds for embeddings $f : N \to M$. A next obvious case is given by considering for an arbitrary manifold N the projection $p : M \times N \to M$ and see that $e(p^*(E)) = p^*(E)$. Now we consider the general case of a smooth map $f : N \to M$. Let $g : N \to M \times N$ be the map $x \mapsto (f(x), x)$. This is an embedding and $pg = f$. Thus from the cases above we see:

$$e(f^*(E)) = e((pg)^*(E)) = e(g^*(p^*(E))) = g^*(e(p^*(E))) = g^*(p^*(e(E))) = (pg)^*e((E)) = f^*(e(E))$$

The property $e(E\times F) = e(E)\times(F)$ follows again from the definition. Combining this with the definition of the Whitney sum and the naturality we conclude $e(E\oplus F) = e(E)\cup e(F)$.
q.e.d.

The following is a useful observation.

Corollary 16.2. *Let $p : E \to M$ be a smooth oriented vector bundle. If E is odd-dimensional, then*

$$2e(E) = 0$$

Proof: If E is odd-dimensional $-Id : E \to E$ is an orientation reversing bundle isomorphism and thus we conclude $e(E) = -e(E)$.
q.e.d.

Remark: *The name Euler class was chosen since there is a close relation between the Euler class of a closed oriented smooth manifold M and the Euler characteristic. Namely:*

$$e(M) = \langle e(TM), [M]\rangle,$$

the Euler characteristic is the Kronecker product between the Euler class of the tangent bundle and the fundamental class of M. By definition of the Euler class and the Kronecker product this means that if $v : M \to TM$ is a section, which is transversal to the 0-section, then the Euler characteristic is the sum of the orientations of the intersection of v with the 0-section. This identity is the Poincaré-Hopf Theorem.

In special cases one can compute $\langle e(TM), [M]\rangle$ directly and verify the Poincaré-Hopf Theorem. We have done this already for the spheres. For complex projective spaces one has:

$$\langle e(T\mathbb{CP}^m), [\mathbb{CP}^m]\rangle = m + 1$$

We leave this as an exercise to the reader. Combining it with Proposition 9.5 we conclude:

Theorem 16.3. *Each vector field on $\mathbb{CP}^n$ has a zero.*

2. Euler classes of some bundles

Now we compute the Euler class of some bundles. As a first example we consider the **tautological bundle**

$$p : L = \{([x], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | v \in \mathbb{C} \cdot x\} \to \mathbb{CP}^n.$$

This is a complex vector bundle of complex dimension 1, whose fibre over $[x]$ is the vector space generated by x. By construction the restriction of the tautological bundle over $\mathbb{CP}^n$ to $\mathbb{CP}^k$ for some $k < n$ is the tautological bundle over $\mathbb{CP}^k$. This is the reason which allows by abuse of notation to use the same name for bundles over different spaces. A complex vector space V considered as a real vector space has a canonical orientation. Namely choose a basis $(v_1, \dots, v_n)$ and consider the basis of the real vector space $(v_1, iv_1, v_2, iv_s, \dots, v_n, iv_n)$. The orientation given by this basis is independent of the choice of the basis $(v_1, \dots, v_n)$ (why?). Using this orientation fibrewise we can consider L as a 2-dimensional oriented real vector bundle. To compute the Euler class we first consider the case $p : L \to \mathbb{CP}^1$ and consider the section given by

$$s : [x_0, x_1] \mapsto ([x_0, x_1], x_0 \cdot \bar{x}_1, x_1 \cdot \bar{x}_1)$$

Then

$$s([x_0, x_1]) = 0 \Leftrightarrow x_1 = 0$$

To check whether the section is transversal and to compute $\epsilon([1,0])$, we choose local coordinates around this point: $\varphi : \{([x], v) \in \mathbb{CP}^1 \times \mathbb{C}^2 | v \in \mathbb{C}x \text{ and } x_0 \neq 0\} \to \mathbb{C} \times \mathbb{C}$ mapping $([x], v)$ to $(\frac{x_1}{x_0}, \mu)$, where $v = \mu(1, \frac{x_1}{x_0})$. This map is an isomorphism. With respect to this trivialization, we have $p_2 \varphi s([1, x_1]) = p_2 \varphi([1, x_1], (\bar{x}_1, x_1 \cdot \bar{x}_1)) = \bar{x}_1$. Thus s is transversal and $\epsilon([1,0]) = -1$. We conclude:

PROPOSITION 16.4. $\langle e(L), [\mathbb{C}P^1] \rangle = -1$.

We return to the tautological bundle $p : L \to \mathbb{CP}^n$ over $\mathbb{CP}^n$. The restriction of $p : L \to \mathbb{CP}^n$ to $\mathbb{CP}^1$ is $p : L \to \mathbb{CP}^1$. Using the naturality of the Euler class the statement above implies $\langle e(L), [\mathbb{CP}^1, i] \rangle = -1$. We recall that we defined $x := [\mathbb{CP}^{n-1}, i] \in H^2(\mathbb{CP}^n)$ and showed that $\langle x, [\mathbb{CP}^1, i] \rangle = 1$. Thus Proposition 16.4 implies:

$$e(L) = -x.$$

As another example we consider the complex line bundle

$$E_k := D^2 \times \mathbb{C} \cup_{f_k} -D^2 \times \mathbb{C} \xrightarrow{p_1} D^2 \cup -D^2 = S^2,$$

where $f_k : S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}$ maps $(z, v) \mapsto (z, z^k \cdot v)$. This bundle is closely related to the lens spaces. If we equip E_k with the Riemannian metric induced from the standard Euclidean metric on $\mathbb{C} = \mathbb{R}^2$, the lens space L_k is the sphere bundle SE_k. The bundle E_k can naturally be equipped with the structure of a smooth vector bundle by describing it as:

$$\mathbb{C} \times \mathbb{C} \cup_{g_k} \mathbb{C} \times \mathbb{C}$$

with

$$\begin{aligned} g_k \ : \ & \mathbb{C}^* \times \mathbb{C} \longrightarrow \mathbb{C}^* \times \mathbb{C} \\ & (x, y) \longmapsto (1/x, x^k y) \end{aligned}$$

If we consider E_k above as an oriented bundle over $D^2 \cup_{\bar{z}} D^2$ instead of over the diffeomorphic oriented manifold $D^2 \cup -D^2$, we have to describe $E_k = D^2 \times \mathbb{C} \cup_{f'_k} D^2 \times \mathbb{C} \xrightarrow{p_1}$

$D^2 \cup_{\bar z} D^2 = S^2$, where $f_k'(z,v) = (\bar z, z^k \cdot v) = (1/z, z^k \cdot v)$.

This describes E_k as a smooth (even holomorphic) vector bundle over $\mathbb{C} \cup_{\frac{1}{x}} \mathbb{C} = S^2$. Now we first compute $e(E_1)([S^2])$ by choosing a transversal section. For $||x|| < 2$ and $x \in \mathbb{C}$, the first copy of $\mathbb{C}$ in $\mathbb{C} \cup_{\frac{1}{x}} \mathbb{C}$, we define the section as $s(x) := (x, \bar x)$ and for z in the second copy we define $s(z) := (z, \rho(||z||)^2)$, where $\rho : [0,\infty) \to (0,\infty)$ is a smooth function with $\rho(s) = 1/s$ for $s > 1/2$. This smooth section has a single 0 at 0 in the first summand and there it intersects transversally with local orientation -1.

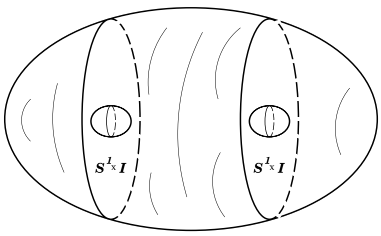
We conclude:

$$\langle e(E_1), [S^2]\rangle = -1.$$

From this we compute $\langle e(E_k), [S^2]\rangle$ for all k by showing

$$\langle e(E_{k+l}), [S^2]\rangle = \langle e(E_k), [S^2]\rangle + \langle e(E_l), [S^2]\rangle.$$

Namely consider D^3 with two holes as in the following picture and denote this 3-dimensional oriented manifold by M:



Decompose M along the two embedded $S^1 \times I$'s and denote the three resulting areas M_1, M_2 and M_3. Now construct a bundle over M by gluing $M_1 \times \mathbb{C}$ to $M_2 \times \mathbb{C}$ via $f_k \times Id : S^1 \times \mathbb{C} \times I \to S^1 \times \mathbb{C} \times I$ and $M_2 \times \mathbb{C}$ to $M_3 \times \mathbb{C}$ via $f_l \times Id : S^1 \times \mathbb{C} \times I \to S^1 \times \mathbb{C} \times I$ to obtain

$$E := M_1 \times \mathbb{C} \cup_{f_k \times Id} M_2 \times \mathbb{C} \cup_{f_l \times Id} M_3 \times \mathbb{C} \xrightarrow{p_1} M_1 \cup M_2 \cup M_3 = M.$$

Orient M so that $\partial M = S^2 + (-S_1^2) + (-S_2^2)$, where S_i^2 are the boundaries of the two holes. Then the reader should convince himself that

$$E|_{S^2} = E_{k+l}$$

since we can combine the two gluings by f_k and f_l along the two circles into one gluing by $f_l \circ f_k = f_{l+k}$. By construction $E|_{S_1^2} = E_k$ and $E|_{S_2^2} = E_l$. Next we note that

$$\langle e(E), [\partial M]\rangle = 0$$

since $[\partial M]$ is zero in $\mathbf{H}_2(M)$ (M itself is a zero bordism of ∂M). But

$$\begin{aligned}\langle e(E), [\partial M]\rangle &= \langle e(E), ([S^2] + [-S_1^2] + [-S_2^2])\rangle \\ &= \langle e(E|_{S^2}), [S^2]\rangle - \langle e(E|_{S_1^2}), [S_1^2]\rangle - \langle e(E|_{S_2^2}), [S_2^2]\rangle \\ &= \langle e(E_{k+l}), [S^2]\rangle - \langle e(E_k), [S^2]\rangle - \langle e(E_l), [S^2]\rangle\end{aligned}$$

Since $\langle e(E), [\partial M]\rangle = 0$ we have shown:

LEMMA 16.5. *The map $\mathbb{Z} \to \mathbb{Z}$ mapping k to $\langle e(E_k), [S^2]\rangle$ is a homomorphism.*

Combining this with the fact $\langle e(E_1), [S^2]\rangle = -1$, we conclude

PROPOSITION 16.6.

$$\langle e(E_k), [S^2]\rangle = -k$$

In particular: There is an orientation preserving bundle isomorphism between E_k and E_r if and only if $k = r$.

In complete analogy we study the bundle $E_{k,l}$ over S^4 given as

$$D^4 \times \mathbb{H} \cup_{f_{k,l}} -D^4 \times \mathbb{H} \xrightarrow{p_1} D^4 \cup -D^4 = S^4$$

where

$$f_{k,l}(z, v) = (z, z^k \cdot v \cdot z^l)$$

and we use quarternionic multiplication ($z \in S^3$). As in the case of E_k over S^4, we show that

$$\langle e(E_{1,0}), [S^4]\rangle = -1.$$

By the same argument as in the case of E_k, one shows

$$\langle e(E_{k+k',l+l'}), [S^4]\rangle = \langle e(E_{k,l}), [S^4]\rangle + \langle e(E_{k',l'}), [S^4]\rangle$$

or, in other words, that the map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ mapping (k, l) to $\langle e(E_{k,l}), [S^4]\rangle$ is a homomorphism.

Next we consider the following isomorphism of $\mathbb{H}$, considered as a real vector space:

$$(z_1, z_2) \mapsto (\bar{z}_1, -z_2) =: \overline{(z_1, z_2)}$$

and note that, for $z \in S^3$, we have $\bar{z} = z^{-1}$. Further $\overline{x \cdot y} = \bar{y} \cdot \bar{x}$. Now consider the bundle isomorphism

$$E_{k,l} \to E_{-l,-k}$$

mapping $(x, v) \mapsto (x, \bar{v})$. Since $v \mapsto \bar{v}$ is orientation reversing, this implies

$$E_{k,l} \cong -E_{-l,-k}$$

and so

$$-\langle e(E_{k,l}), [S^4]\rangle = \langle e(E_{-l,-k}), [S^4]\rangle.$$

This implies

$$\langle e(E_{k,l}), [S^4]\rangle = c(k + l)$$

for some constant c. Since $\langle e(E_{1,0}), [S^4]\rangle = -1$, we conclude $c = -1$ and thus

PROPOSITION 16.7. $\langle e(E_{k,l}), [S^4] \rangle = -k - l.$

CHAPTER 17

The Chern classes

Now we define the Chern classes of a complex vector bundle $p : E \to M$. We remind the reader that a smooth **k-dimensional complex vector bundle** is a smooth map $p : E \to M$ together with a $\mathbb{C}$-vector space structure on the fibres which is locally isomorphic to $U \times \mathbb{C}^k$, where the isomorphism means diffeomorphism and fibrewise $\mathbb{C}$-linear. For example we know that the tautological bundle $p : L \to \mathbb{CP}^n$ is a 1-dimensional complex vector bundle. If E and F are complex vector bundles the Whitney sum $E \oplus F$ is a complex vector bundle. Given two complex vector bundles E and F one can consider their tensor product $E \otimes_{\mathbb{C}} F$ which is obtained by taking fibrewise the tensor product to obtain a new complex vector bundle [**Mi-St**]. If E and F are smooth vector bundles then $E \otimes_{\mathbb{C}} F$ is again smooth.

To prepare the definition of Chern classes we consider for a smooth manifold M the homology of $M \times \mathbb{CP}^N$, for some N. By the Künneth Theorem and the fact that $H^*(\mathbb{CP}^N) = \mathbb{Z}[e(L)]/_{e(L)^{N+1}}$(implying that the cohomology of $\mathbb{CP}^N$ is free) we have for $k \leq N$ if M admits a finite good atlas:

$$H^k(M \times \mathbb{CP}^N) \cong H^k(M) \otimes \mathbb{Z} \cdot 1 \oplus H^{k-2}(M) \otimes \mathbb{Z} \cdot e(L) \oplus H^{k-4}(M) \otimes \mathbb{Z} \cdot (e(L) \cup e(L)) \oplus \cdots$$

Actually the same result is true for arbitrary manifolds M as one can show inductively over N using the Mayer-Vietoris sequence. Now let $p : E \to M$ be a smooth k-dimensional complex vector bundle and consider $p_1^* E \otimes_{\mathbb{C}} p_2^* L$, a complex vector bundle over $M \times \mathbb{CP}^N$ for some $N > k$. Since every complex vector bundle considered as a real bundle has a canonical orientation, we can consider the Euler class $e(p_1^* E \otimes_{\mathbb{C}} p_2^* L) \in H^{2k}(M \times \mathbb{CP}^N)$. Using the isomorphism above we define the **Chern classes** $c_i(E) \in H^{2i}(M)$ by the equation

$$e(p_1^* E \otimes_{\mathbb{C}} p_2^* L) = \sum_{i=0}^{k} c_i(E) \times e(L)^{k-i}.$$

With other words the Chern classes are the coefficients of $e(p_1^* E \otimes_{\mathbb{C}} p_2^* L)$ if we consider the Euler class as a "polynomial" in $e(L)$.

Since for the inclusion $i : \mathbb{CP}^N \to \mathbb{CP}^{N+1}$ we know that $i^* L$ is the tautological bundle over $\mathbb{CP}^N$, if L was the tautological bundle over $\mathbb{CP}^{N+1}$, this definition does not depend on N so long as N is larger than k.

We prove some basic properties of the Chern classes. The naturality of the Euler class implies that the Chern classes are natural, i.e. if $f : N \to M$ is a smooth map, then

$$c_k(f^*(E)) = f^*(c_k(E)).$$

The Chern classes depend only on the isomorphism type of the bundle. Both implies that the Chern classes of a trivial bundle are zero except $c_0 = 1$. Comparing with a point we conclude that for arbitrary bundles E we have

$$c_0(E) = 1.$$

By construction $c_i(E) = 0$ for $i > k$, where k is the complex dimension of E. Next we note that $c_k(E) = e(E)$. To see this, fix a point $x_0 \in \mathbb{CP}^N$ and consider the inclusion

$$\begin{aligned} j: \quad M &\longrightarrow M \times \mathbb{CP}^N \\ x &\longmapsto (x, x_0). \end{aligned}$$

Then $j^*(p_1^*E \otimes_{\mathbb{C}} p_2^*L) = j^*p_1^*E \otimes_{\mathbb{C}} j^*p_2^*L \cong j^*p_1^*E = E$, since $p_2 j$ is the constant map and so $j^*p_2^*L$ is the product bundle $M \times \mathbb{C}$. On the other hand $j^* : H^{2k}(M \times \mathbb{CP}^N) \to H^{2k}(M)$ maps $H^{2k}(M) \otimes \mathbb{Z} \cdot e(L)^0 = H^{2k}(M)$ identically to $H^{2k}(M)$ and the other summands in the decomposition to 0. Thus $e(E) = e(j^*(p_1^*E \otimes_{\mathbb{C}} p_2^*L)) = j^*e(p_1^*E \otimes_{\mathbb{C}} p_2^*L) = j^*c_k(E) \times e(L)^0$, and therefore

$$e(E) = c_k(E).$$

This property together with the following product formula is basic for the Chern classes. We would like to know $c_r(E \oplus F)$ for k and l-dimensional complex vector bundles E and F over M. For this we choose $N \geq k + l$ and note that

$$p_1^*(E \oplus F) \otimes_{\mathbb{C}} p_2^*L = (p_1^*E \otimes_{\mathbb{C}} p_2^*L) \oplus (p_1^*F \otimes_{\mathbb{C}} p_2^*L).$$

Then we conclude from

$$e((p_1^*E \otimes_{\mathbb{C}} p_2^*L) \oplus (p_1^*F \otimes_{\mathbb{C}} p_2^*L)) = e(p_1^*E \otimes_{\mathbb{C}} p_2^*L) \cup e(p_1^*F \otimes_{\mathbb{C}} p_2^*L)$$

and the definition of the Chern classes:

$$\sum_{i=0}^{k+l} c_i(E \oplus F) \times e(L)^{k+l-i} = (\sum_{j=0}^{k} c_j(E) \times e(L)^{k-j}) \cup (\sum_{s=0}^{l} c_s(F) \times e(L)^{l-s}),$$

that

$$c_i(E \oplus F) = \sum_{j+s=i} c_j(E) \cup c_s(F)$$

A convenient way to write the product formula is to consider the Chern classes as elements of the cohomology ring $H^*(M) = \oplus_k H^k(M)$. We define the **total Chern class** as

$$c(E) := \sum_k c_k(E) \in H^*(M)$$

Then the product formula translates to:

$$c(E \oplus F) = c(E) \cup c(F)$$

We summarize these properties as

THEOREM 17.1. *Let E be a k-dimensional smooth complex vector bundle over M. The Chern classes are natural, i.e. if $f : N \to M$ is a smooth map, then*

$$c_k(f^*(E)) = f^*(c_k(E)).$$

The Chern classes depend only on the isomorphism type of the bundle.

$$c_0(E) = 1.$$

For $i > k$:

$$c_i(E) = 0$$
$$c_k(E) = e(E)$$

*If E and F are smooth complex vector bundles over M, then (***Whitney formula***):*

$$c_r(E \oplus F) = \sum_{i+j=r} c_i(E) \cup c_j(F),$$

or using the total Chern class:

$$c(E \oplus F) = c(E) \cup c(F)$$

One can show that these properties characterize the Chern classes uniquely [**Mi-St**].

At the end of this chapter we shortly introduce Stiefel-Whitney classes (although we will not apply them in this book). The definition is completely analogous to the definition of the Euler class and the Chern classes. The main difference is that we will replace oriented or even complex vector bundles by arbitrary vector bundles.

If E is a k-dimensional vector bundle (not oriented) we can make the same construction as for the Euler class with $\mathbb{Z}/2$-cohomology instead of integral cohomology and define $w_k(E) := s^*[M, s] \in H^k(M; \mathbb{Z}/2)$ the k-th **Stiefel-Whitney class** of E. This class fulfills the analogous properties as were shown for the Euler class in Theorem 16.1. Perhaps a better name for the k-th Stiefel-Whitney class would be to call it mod2 Euler class, since it is the version of the Euler class for $\mathbb{Z}/2$-cohomology.

Now we define the lower Stiefel-Whitney classes. This is done in complete analogy to the Chern classes, where we replace the Euler class by the k-th Stiefel-Whitney class and the tautological bundle over the complex projective space by the tautological bundle L over $\mathbb{RP}^N$. This is a 1-dimensional real bundle. The $\mathbb{Z}/2$-cohomology of $M \times \mathbb{RP}^N$ is:

$$H^k(M \times \mathbb{RP}^N; \mathbb{Z}/2) \cong H^k(M; \mathbb{Z}/2) \otimes \mathbb{Z}/2 \cdot 1 \oplus H^{k-1}(M; \mathbb{Z}/2) \otimes \mathbb{Z}/2 \cdot w_1(L) \oplus H^{k-2}(M; \mathbb{Z}/2) \otimes \mathbb{Z}/2 \cdot (w_1(L) \cup w_1(L)) \oplus \cdots$$

Then we define the **Stiefel-Whitney classes** of a real smooth vector bundle E of dimension k over M denoted $w_i(E) \in H^i(M)$ by the equation

$$w_k(p_1^* E \otimes_{\mathbb{R}} p_2^* L) = \sum_{i=0}^{k} w_i(E) \times w_1(L)^{k-i}.$$

With other words the Stiefel-Whitney classes are the coefficients of $e(p_1^* E \otimes_{\mathbb{R}} p_2^* L)$ if we consider the Euler class as a "polynomial" in $w_1(L)$. By a similar argument as for Theorem 17.1 one proves:

THEOREM 17.2. *Let E be a k-dimensional smooth real vector bundle over M. Then the analogous statement as in the previous theorem hold. In particular for $i > k$:*

$$w_i(E) = 0$$

*If E and F are smooth vector bundles over M, then (***Whitney formula***):*

$$w_r(E \oplus F) = \sum_{i+j=r} w_i(E) \cup w_j(F).$$

CHAPTER 18

Pontrjagin classes and applications to bordism

1. Pontrjagin classes

To obtain an other invariant for k-dimensional real vector bundles E we simply complexify the bundle considering

$$E \otimes_{\mathbb{R}} \mathbb{C}.$$

This means that we replace the fibres of E_x of E by the complex vector spaces $E_x \otimes_{\mathbb{R}} \mathbb{C}$ or equivalently by $E_x \oplus E_x$ with complex vector space structure given by $i \cdot (v, w) := (-w, v)$. This is a complex vector bundle of complex dimension k and we define the **Pontrjagin classes**

$$p_r(E) := (-1)^r c_{2r}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4r}(M).$$

Here one might wonder why we have not taken $c_{2r+1}(E \otimes_{\mathbb{R}} \mathbb{C})$ into account. The reason is that these classes are 2-torsion, as we will discuss. Also the sign convention asks for an explanation. One could leave out the sign without any problem. Probably the historical reason for the sign convention is that for $2n$-dimensional oriented bundles one can show:

$$p_n(E) = e(E) \cup e(E).$$

We prepare the argument that the classes $c_{2r+1}(E \otimes_{\mathbb{R}} \mathbb{C})$ are 2-torsion by some general considerations. If V is a complex k-dimensional vector space, we consider its conjugate complex vector space $\bar{V}$ with new scalar multiplication $\lambda \diamond v := \bar{\lambda} \cdot v$. Note that the orientation of $\bar{V}$, as a real vector space, is $(-1)^k$ times the orientation of V (why?). Taking the conjugate complex structure fibrewise we obtain for a complex bundle E the **conjugate bundle** $\bar{E}$. The change of orientation of vector spaces translates to complex vector bundles giving for a k-dimensional complex vector bundle that as oriented bundles $\bar{E} \cong (-1)^k E$. From this one concludes:

$$c_i(\bar{E}) = (-1)^i c_i(E)$$

Now we note that since $\mathbb{C}$ is as a complex vector space isomorphic to its conjugate this isomorphism induces an isomorphism:

$$E \otimes_{\mathbb{R}} \mathbb{C} \cong \overline{E \otimes_{\mathbb{R}} \mathbb{C}}.$$

Thus $c_{2r+1}(E \otimes_{\mathbb{R}} \mathbb{C}) = -c_{2r+1}(E \otimes_{\mathbb{R}} \mathbb{C})$ implying $2c_{2r+1}(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$.

Since $2c_{2r+1}(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$, the product formula for the Chern classes gives the corresponding **product formula for the Pontrjagin classes**:

$$p_r(E \oplus F) = \sum_{i+j=r} p_i(E) \cup p_j(F) + \beta,$$

where $2\beta = 0$.

Again we introduce the **total Pontrjagin class**:

$$p(E) := \sum_k p_k(E) \in H^*(M)$$

and rewrite the product formula as:

$$p(E \oplus F) = p(E) \cup p(F) + \beta,$$

where $2\beta = 0$.

For the computation of the Pontrjagin classes of a complex vector bundle the following considerations are useful. Let V be a complex vector space. If we forget that V is a complex vector space and complexify it to obtain $V \otimes_{\mathbb{R}} \mathbb{C}$, we see that $V \otimes_{\mathbb{R}} \mathbb{C}$ is, as a complex vector space, isomorphic to $V \oplus \bar{V}$. Namely, $V \otimes_{\mathbb{R}} \mathbb{C}$ is, as a real vector space, equal to $V \oplus V$ and - with respect to this decomposition - the multiplication by i maps (x, y) to $(-y, x)$. With this we write down an isomorphism

$$\begin{array}{ccccc} V \otimes_{\mathbb{R}} \mathbb{C} & = & V \oplus V & \longrightarrow & V \oplus \bar{V} \\ & & (x, y) & \longmapsto & (x + iy, ix + y) \end{array}$$

This extends to vector bundles. For a complex vector bundle E the fibrewise isomorphism above gives an isomorphism :

$$E \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$$

Using the product formula for Chern classes one can express the Pontrjagin classes of a complex vector bundle E in terms of the Chern classes of E. For example:

$$p_1(E) = -c_2(E \oplus \bar{E}) = -(c_1(E) \cup c_1(\bar{E}) + c_2(E) + c_2(\bar{E})) = c_1^2(E) - 2c_2(E)$$

Now, we compute $\langle p_1(E_{k,l}), [S^4] \rangle$, where $p : E_{k,l} \to S^4$ is the $\mathbb{R}^4$-bundle considered above. As for the Euler class one shows that

$$(k, l) \longmapsto \langle p_1(E_{k,l}), [S^4] \rangle$$

is a homomorphism. Next we observe that $p_1(E_{k,l})$ does not depend on the orientation of $E_{k,l}$ and, since $E_{k,l}$ is isomorphic to $E_{-l,-k}$ (reversing orientation), we conclude

$$\langle p_1(E_{k,l}), [S^4] \rangle = \langle p_1(E_{-l,-k}), [S^4] \rangle.$$

Linearity and this property imply that there is a constant a such that

$$\langle p_1(E_{k,l}), [S^4] \rangle = a(k - l).$$

To determine a we compute $\langle p_1(E_{0,1}), [S^4]\rangle$. Since, for a fixed element $x \in \mathbb{H}$, the map $y \mapsto y \cdot x$ is $\mathbb{C}$-linear, $E_{0,1}$ is a complex vector bundle. Thus by the formula above:

$$p_1(E_{0,1}) = -2c_2(E_{0,1}) = -2e(E_{0,1})$$

From $\langle e(E_{k,l}), [S^4]\rangle = -k - l$ we conclude

$$\langle p_1(E_{0,1}), [S^4]\rangle = 2$$

and thus we have proved:

Proposition 18.1.

$$\langle p_1(E_{k,l}), [S^4]\rangle = -2(k - l).$$

2. Pontrjagin numbers

To demonstrate the use of characteristic classes we consider the following invariants for closed smooth $4k$-dimensional manifolds M. Let $I := (i_1, i_2, \dots, i_r)$ be a sequence of natural numbers $0 < i_1 \leq \dots \leq i_r$ such that $i_1 + \dots + i_r = k$, i.e. I is a partition of k. Then we define the **Pontrjagin number**

$$p_I(M) := \langle p_{i_1}(TM) \cup \dots \cup p_{i_r}(TM), [M]\rangle \in \mathbb{Z}$$

To compute the Pontrjagin numbers in examples we consider the complex projective spaces and look at their tangent bundle. To determine this bundle we consider the following line bundle over $\mathbb{CP}^n$, the **Hopf bundle**. Its total space H is the quotient of $S^{2n+1} \times \mathbb{C}$ under the equivalence relation $(x, z) \sim (\lambda x, \lambda z)$ for some $\lambda \in S^1$. The projection $p : H \to \mathbb{CP}^n$ maps $[(x, z)]$ to $[x]$. The fibre over $[x]$ is equipped with the structure of a 1-dimensional complex vector space by defining $[(x, z)] + [(x, z')] := [(x, z + z')]$. A local trivialization around $[x]$ is given as follows: Let x_i be non zero and define $U_i := \{[y] \in \mathbb{CP}^n | y_i \neq 0\}$. Then a trivialization over U_i is given by the map $p^{-1}(U_i) \to U_i \times \mathbb{C}$ mapping $[(x, z)]$ to $([x], z/x_i)$.

Proposition 18.2. *There is an isomorphism of complex vector bundles*

$$T\mathbb{CP}^n \oplus \mathbb{CP}^n \times \mathbb{C} \cong \oplus_{n+1} H$$

Proof: We start with the description of $\mathbb{CP}^n$ as $\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^* = \mathbb{C}^{n+1}/\sim$ where $x \sim \lambda x$ for all $\lambda \in \mathbb{C}^*$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{CP}^n$ be the canonical projection. This is a differentiable map. Moreover, if we use complex charts for $\mathbb{C}^n$, it even is a complex differentiable map. Using local coordinates, one checks that for each $x \in \mathbb{C}^{n+1} - \{0\}$ the complex differential $d\pi_x : \mathbb{C}^{n+1} = T_x(\mathbb{C}^{n+1} - \{0\}) \to T_{[x]}\mathbb{CP}^n$ is surjective.

If for some $\lambda \in \mathbb{C}^*$ we consider the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ given by multiplication with λ, its complex differential is for each tangent space $\mathbb{C}^{n+1}$ again multiplication by λ. Thus the differential

$$d\pi : T(\mathbb{C}^{n+1} - \{0\}) \to T\mathbb{CP}^n$$

induces a fibrewise surjective bundle map between two bundles over $\mathbb{CP}^n$

$$[d\pi] : (\mathbb{C}^{n+1} - \{0\}) \times \mathbb{C}^{n+1}/\sim \to T\mathbb{CP}^n$$

where $(x, v) \sim (\lambda x, \lambda v)$. The bundle

$$(\mathbb{C}^{n+1} - \{0\}) \times \mathbb{C}^{n+1}/\sim \to \mathbb{CP}^n$$

given by projection onto the first factor is $\bigoplus_{n+1} H$.

To finish the proof, we have to extend the bundle map $[d\pi]$ to a bundle map

$$(\mathbb{C}^{n+1} - \{0\}) \times \mathbb{C}^{n+1}/_{\sim} \to T\mathbb{CP}^n \oplus \mathbb{CP}^n \times \mathbb{C}$$

which is fibrewise an isomorphism. This map is given by

$$[x, v] \longmapsto ([d\pi]([x, v]), ([x], \langle v/_{||x||}, x/_{||x||}\rangle))$$

where $\langle v, x\rangle$ is the hermitian scalar product $\Sigma v_i \cdot \bar{x}_i$.

Since the kernel of $[d\pi_x]$ consists of all v which are multiples of x, the map is fibrewise injective and thus fibrewise an isomorphism, since both vector spaces have the same dimensions.
q.e.d.

To compute the Pontrjagin classes of the complex projective spaces we have to determine the characteristic classes of H. Since H is a complex line bundle, its first Chern class is equal to $e(H) \in H^2(\mathbb{CP}^n)$. Since $H^2(\mathbb{CP}^n)$ is generated by $e(L)$ we know that $e(H) = k \cdot e(L)$ for some k. To determine k it is enough to consider $p : H \to \mathbb{CP}^1$ and to compute $\langle e(H), [\mathbb{CP}^1]\rangle$. For this consider the section $[x] \to [x, x_0]$ which has just one zero at $[x] = [0, 1]$ where it is transversal. One checks that the local orientation at this point is 1. We conclude:

$$\langle e(H), [\mathbb{CP}^1]\rangle = 1$$

and thus

$$c_1(H) = e(H) = -e(L).$$

Now we use the relation between the Pontrjagin and Chern classes of a complex bundle above and see that

$$p_1(H) = c_1(H)^2 - 2c_2(H) = e(L)^2,$$

since $c_2(H) = 0$. Thus

$$p(H) = 1 + e(L)^2$$

With the product formula for Pontrjagin classes and the fact that the cohomology of $\mathbb{CP}^n$ is torsion free and finitely generated, we conclude from $T\mathbb{CP}^n \oplus \mathbb{CP}^n \times \mathbb{C} = \bigoplus_{n+1} H$ that $p(T\mathbb{CP}^n) = p(\bigoplus_{n+1} H)$ and using the product formula again:

THEOREM 18.3. *The total Pontrjagin class of the complex projective space $\mathbb{CP}^n$ is:*

$$p(T\mathbb{CP}^n) = 1 + p_1(T\mathbb{CP}^n) + \cdots + p_{[n/2]}(T\mathbb{CP}^n) = p(H)^{n+1} = (1 + e(L)^2)^{n+1}$$

or

$$p_k(T\mathbb{CP}^n) = \binom{n+1}{k} \cdot e(L)^{2k}.$$

We use this to compute the following Pontrjagin numbers. We recall that as a consequence of Proposition 11.3 we saw that $e(L) = -x$ and from §11 that $\langle x^n, [\mathbb{CP}^n]\rangle = 1$. Thus $\langle e(L)^{2n}, [\mathbb{CP}^{2n}]\rangle = 1$ and we obtain:

$$\begin{aligned} p_{(1)}(\mathbb{CP}^2) &= 3, \\ p_{(1,1)}(\mathbb{CP}^4) &= 25, \\ p_{(2)}(\mathbb{CP}^4) &= 10, \end{aligned}$$

3. Applications of Pontrjagin numbers to bordism

One of the reasons why Pontrjagin numbers are interesting, is the fact that they are bordism invariants for oriented manifolds. We first note that they are additive under disjoint union and change sign if we pass from M to $-M$ (note that the Pontrjagin classes do not depend on the orientation of a bundle, but the fundamental class does). To see that Pontrjagin numbers are bordism invariants, let W be a compact oriented $(4k+1)$-dimensional smooth manifold. Using our collar we identify an open neighbourhood of ∂W in W with $\partial W \times [0,1)$. Then $T\overset{\circ}{W}|_{\partial W \times (0,1)} = T\partial W \times ((0,1) \times \mathbb{R})$. Thus from the product formula we conclude: $j^*(p_{i_1}(TW) \cup \cdots \cup p_{i_r}(TW)) = p_i(T\partial W) \cup \cdots \cup p_{i_r}(T\partial W)$, where j is the inclusion from ∂W to W. From this we see by naturality:

$$\begin{aligned} p_I(\partial W) &= \langle p_{i_1}(T\partial W) \cup \cdots \cup p_{i_r}(T\partial W), [\partial W]\rangle \\ &= \langle p_{i_1}(TW) \cup \cdots \cup p_{i_r}(TW), j_*[\partial W]\rangle = 0, \end{aligned}$$

the latter following since $j_*[\partial W] = 0$ (W is a zero bordism!). We summarize:

THEOREM 18.4. *The Pontrjagin numbers induce homomorphisms from the oriented bordism group Ω to $\mathbb{Z}$:*

$$p_I : \Omega_{4k} \longrightarrow \mathbb{Z}.$$

Since $p_n(T\mathbb{CP}^{2n}) = \binom{2n+1}{n} \cdot e(L)^{2n}$, the homomorphism $p_{(n)} : \Omega_{4n} \to \mathbb{Z}$ is non-trivial and we have another proof for the fact we have shown using the signature, namely that $\Omega_{4k} \neq 0$ for all $k \geq 0$.

The existence of a homomorphism $\Omega_{4k} \to \mathbb{Z}$ for each partition I of k naturally raises the question whether the corresponding elements in $\mathrm{Hom}(\Omega_{4k}, \mathbb{Z})$ are all linearly independent. This is in fact the case and can be proved with the methods known to the reader of this book [**Mi-St**]. In low dimensions one can easily check this by hand. In dimension 4 there is nothing to show. In dimension 8 we consider $\mathbb{CP}^2 \times \mathbb{CP}^2$. The tangent bundle is $T\mathbb{CP}^2 \times T\mathbb{CP}^2$ or $\pi_1^*T\mathbb{CP}^2 \oplus \pi_2^*T\mathbb{CP}^2$. Thus by the product formula for the Pontrjagin classes $p_1(T(\mathbb{CP}^2 \times \mathbb{CP}^2)) = \pi_1^*3e(L)^2 + \pi_2^*3e(L)^2$ and $p_2(T(\mathbb{CP}^2 \times \mathbb{CP}^2)) = \pi_1^*3e(L)^2 \cup \pi_2^*3e(L)^2 = 9 \cdot (\pi_1^*e(L)^2 \cup \pi_2^*e(L)^2)$ or $9(e(L)^2 \times e(L)^2)$. By definition of the cross-product

$$\langle e(L)^2 \times e(L)^2, [\mathbb{CP}^2 \times \mathbb{CP}^2]\rangle = \langle e(L)^2, [\mathbb{CP}^2]\rangle \cdot \langle e(L)^2, [\mathbb{CP}^2]\rangle = 1$$

and so

$$p_{(2)}(\mathbb{CP}^2 \times \mathbb{CP}^2) = 9$$

and, using $p_1(T\mathbb{CP}^2) = 3e(L)^2$ we compute:

$$\begin{aligned}(p_1(T(\mathbb{CP}^2\times\mathbb{CP}^2)))^2 &= (\pi_1^*3e(L)^2+\pi_2^*3e(L)^2)^2 = 9\pi_1^*e(L)^4+18(\pi_1^*e(L)^2\cup\pi_2^*e(L)^2)+9\pi_2^*e(L)^4\\ &= 18(\pi_1^*e(L)^2 \cup \pi_2^*e(L)^2) = 18(e(L)^2 \times e(L)^2).\end{aligned}$$

We conclude

$$p_{(1,1)}(\mathbb{CP}^2 \times \mathbb{CP}^2) = 18.$$

With this information one checks that the matrix

$$\begin{pmatrix} p_{(1,1)}(\mathbb{CP}^4) & p_{(1,1)}(\mathbb{CP}^2 \times \mathbb{CP}^2) \\ p_{(2)}(\mathbb{CP}^4) & p_{(2)}(\mathbb{CP}^2 \times \mathbb{CP}^2) \end{pmatrix} = \begin{pmatrix} 25 & 18 \\ 10 & 9 \end{pmatrix}$$

is invertible and the two homomorphisms on Ω_8 are linearly independent. We summarize:

THEOREM 18.5. *rank* $\Omega_4 \geq 1$ *and rank* $\Omega_8 \geq 2$.

For the first information we had already another argument using the signature (Corollary 15.3).

4. Classification of some Milnor manifolds

For a final application of characteristic classes in this section, we return to the Milnor manifolds $M_{k,l}$. For dimensional reasons, there is just one Pontrjagin class which might be of some use, namely $p_1(TM_{k,l}) \in H^4(M_{k,l};\mathbb{Z})$. Since this group is zero except for $k+l=0$ (Proposition 11.4), we only look at $M_{k,-k}$. Since $\mathbf{H}_4(M_{k,-k}) \cong \mathbb{Z}$, there is up to sign a unique generator $[V,g] \in \mathbf{H}_4(M_{k,-k})$. Thus we can obtain a numerical invariant by evaluating $p_1(TM_{k,-k})$ on $[V,g]$ and taking its absolute value:

$$M_{k,-k} \longmapsto |\langle p_1(TM_{k,-k}), [V,g]\rangle|.$$

This is an invariant of the diffeomorphism type of $M_{k,-k}$.

To compute this number, recall that $M_{k,l}$ is the sphere bundle of $E_{k,l}$. Thus $TM_{k,l} \oplus M_{k,l} \times \mathbb{R} = TD(E_{k,l})|_{M_{k,l}} = TE_{k,l}|_{M_{k,l}}$ (for the first identity use a collar of $SE_{k,l} = M_{k,l}$ in $DE_{k,l}$). Let $j : M_{k,l} \to E_{k,l}$ be the inclusion. Then our invariant is

$$|\langle p_1(TM_{k,-k}), [V,g]\rangle| = |\langle j^*p_1(TE_{k,-k}), [V,g]\rangle|$$

$$= \quad |\langle p_1(TE_{k,-k}), j_*[V,g]\rangle|$$

$$= \quad |\langle p_1(i^*TE_{k,-k}), [S^4]\rangle|.$$

The last equality comes from two facts, namely that the map $j_* : H_4(M_{k,-k}) \to H_4(E_{k,-k})$ is an isomorphism (this follows from a computation of the homology of $E_{k,-k}$ using the Mayer-Vietoris sequence as for $M_{k,-k}$ and comparing these exact sequences) and that the inclusion $i : S^4 \to E_{k,-k}$ given by the 0-section induces an isomorphism $\mathbf{H}_4(S^4) \to \mathbf{H}_4(E_{k,-k})$. To compute $p_1(i^*TE_{k,-k}) = p_1(TE_{k,-k}|_{S^4})$, we note that $TE_{k,l}|_{S^4} \cong TS^4 \oplus E_{k,l}$. The isomorphism is induced by the differential of i from TS^4 to $TE_{k,l}$ and by the differential of the inclusion of a fibre $(E_{k,l})_x$ to $E_{k,l}$ giving a homomorphism $T((E_{k,l})_x) = E_{k,l} \to TE_{k,l}$. With the help of a local trivialization one checks that this bundle map $TS^4 \oplus E_{k,l} \to TE_{k,l}|_{S^4}$ is fibrewise an isomorphism and thus a bundle isomorphism.

Returning to the Milnor manifolds, since $TS^4 \oplus S^4 \times \mathbb{R} = T\mathbb{R}^5|_{S^4} = S^4 \times \mathbb{R}^5$, we note that:

$$|\langle p_1(i^*TE_{k,-k}), [S^4]\rangle| = |\langle p_1(E_{k,-k}), [S^4]\rangle| = 4|k|.$$

Thus $|k|$ is a diffeomorphism invariant of $M_{k,-k}$ showing an analogy to L_k, where also $|k|$ was a diffeomorphism invariant. But there is a big difference between the two cases since for L_k we have detected $|k|$ as the order of $\mathbf{H}_1(L_k)$, whereas all $M_{k,-k}$ have the same homology and we have used a more subtle invariant to distinguish them.

Finally, we construct an (orientation reversing) diffeomorphism from $M_{k,l}$ to $M_{-k,-l}$ by mapping $D^4 \times S^3$ to $D^4 \times S^3$ via $(x,y) \mapsto (\bar{x}, y)$ and $-D^4 \times S^3$ to $-D^4 \times S^3$ via $(x,y) \mapsto (\bar{x}, y)$. Thus we conclude:

THEOREM 18.6. *Two Milnor manifolds $M_{k,-k}$ and $M_{r,-r}$ are diffeomorphic if and only if $|k| = |r|$.*

CHAPTER 19

Exotic 7-spheres

1. The signature theorem and exotic 7-spheres

At the end of the last section we classified those Milnor manifolds where $\mathbf{H}_4(M) \cong \mathbb{Z}$. In this section we want to look at the other extremal case, namely where all homology groups of $M_{k,l}$ except in dimension 0 and 7 are trivial. By Proposition 11.4 this is equivalent to $k + l = \pm 1$. Then homologically $M_{k,\pm 1-k}$ looks like S^7. We are going to prove that it is actually homeomorphic to S^7, a remarkable result by Milnor [**Mi 1**]:

Theorem 19.1. **(Milnor):** *The Milnor manifolds $M_{k,\pm 1-k}$ are homeomorphic to S^7.*

Although the proof of this result is not related to the main theme of this book we will give it at the end of this chapter for completeness.

This result raises the question whether all manifolds $M_{k,\pm 1-k}$ are diffeomorphic to S^7. We will show that in general this is not the case. We prepare the argument by some considerations concerning bordism groups and the signature.

In §18 we have introduced Pontrjagin numbers, which turned out to be bordism invariants for oriented smooth manifolds. We used them to show that the rank of Ω_4 is at least one and of Ω_8 is at least two. Moreover, we stated that the Pontrjagin numbers can be used to show that for all k the products of complex projective spaces $\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}$ for $i_1 + \cdots + i_r = k$ are linear independent, implying rank $\Omega_{4k} \geq \pi(k)$, the number of partitions of k. In his celebrated paper [**Th 1**] Thom has proved that dim $\Omega_{4k} \otimes \mathbb{Q} = \pi(k)$.

Theorem 19.2. **(Thom):** *The dimension of $\Omega_{4k} \otimes \mathbb{Q}$ is $\pi(k)$ and the products*

$$[\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}]$$

for $i_1 + \cdots i_r = k$ form a basis of $\Omega_{4k} \otimes \mathbb{Q}$.

The original proof of this result consists of three steps. The first is a translation of bordism groups into homotopy groups of the so called Thom space of a certain bundle, the universal bundle over the classifying space for oriented vector bundles. The main ingredient for this so called Pontrjagin-Thom construction is transversality. The second is a computation of the rational cohomology ring. Both steps are explained in the book [**Mi-St**]. The final third step is a computation of the rational homotopy groups of this Thom space. Details for this are not given in Milnor-Stasheff, where the reader is referred to the original paper of Serre. An elementary proof based on [**K-K**] is sketched in [**K-L**],

p. 14 ff.

Now we will apply Thom's result to give a formula for the signature in low dimensions. The key observation here is the bordism invariance of the signature (Theorem 11.6). We recall that the signature induces a homomorphism

$$\tau : \Omega_{4k} \to \mathbb{Z}$$

Combining this fact with Theorem 19.2 we conclude that the signature can be expressed as a linear combination of Pontrjagin numbers. For example in dimension 4, where $\Omega_4 \otimes \mathbb{Q} \cong \mathbb{Q}$, the formula can be obtained by comparing $1 = \tau(\mathbb{CP}^2)$ with $\langle p_1(T\mathbb{CP}^2), [\mathbb{CP}^2]\rangle = 3$ and so, for all closed oriented smooth 4-manifolds, one has the formula:

$$\tau(M) = \frac{1}{3}\langle p_1(TM), [M]\rangle$$

In dimension 8 one knows that $\tau(M) = ap_{(1,1)}(M) + bp_{(2)}(M) = a\langle p_1(TM)^2, [M]\rangle + b\langle p_2(TM), [M]\rangle$. We have computed the Pontrjagin numbers of $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$. We know already that $\tau(\mathbb{CP}^4) = 1$ and one checks that also $\tau(\mathbb{CP}^2 \times \mathbb{CP}^2) = 1$. Comparing the values of the signature and the Pontrjagin numbers for these two manifolds one concludes:

THEOREM 19.3. **(Hirzebruch):** *For a closed oriented smooth 8-dimensional manifold M one has*

$$\tau(M) = \frac{1}{45}(7\langle p_2(TM), [M]\rangle - \langle p_1(TM)^2, [M]\rangle)$$

Proof: We only have to check the formula for $\mathbb{CP}^4$ and for $\mathbb{CP}^2 \times \mathbb{CP}^2$. The values for the Pontrjagin numbers were computed at the end of §18 and with this the reader can verify the formula.
q.e.d.

The two formulas above are special cases of Hirzebruch's famous **signature theorem**, which gives a corresponding formula in all dimensions (see [**Hir**] or [**Mi-St**]).

One of the most spectacular applications of Theorem 19.3 was Milnor's discovery of exotic spheres. Milnor shows that in general $M_{k,1-k}$ is not diffeomorphic to S^7. His argument is the following: Suppose there is a diffeomorphism $f : M_{k,1-k} \to S^7$. Since $M_{k,1-k}$ is the boundary of the disk bundle $DE_{k,1-k}$, we can then form the closed smooth manifold

$$N := DE_{k,1-k} \cup_f D^8$$

We extend the orientation of $DE_{k,1-k}$ to N (which can be done, since the disk has an orientation reversing diffeomorphism) and compute its signature. The inclusion induces an isomorphism $j^* : H^4(N) \cong H^4(\overset{\circ}{DE}_{k,1-k}) \cong H^4(S^4) \cong \mathbb{Z}$. We will show that the signature of N is -1 by constructing a class with negative self intersection number. To do this we consider the cohomology class $j^*([S^4, v]) \in H^4(\overset{\circ}{DE}_{k,1-k})$, where v is the zero section. We also consider $v^*(j^*([S^4, v])) \in H^4(S^4)$. This is equal to the Euler class of $E_{k,1-k}$. By definition the self intersection $S_N([S^4, v], [S^4, v]) = \langle e(E_{k,1-k}), [S^4]\rangle$. We have computed this number in Proposition 16.7 and conclude:

$$S_N([S^4, v], [S^4, v]) = -k - (1 - k) = -1$$

Thus:

$$\tau(N) = -1$$

Now we use the signature theorem to compute $\tau(N)$ in terms of $\langle p_1(TN)^2, [N]\rangle$ and $\langle p_2(TN), [N]\rangle$. Since $v^* : H^4(N) \to H^4(S^4)$ is an isomorphism, we know $p_1(TN) = (v^*)^{-1}(p_1(TN|_{S^4}))$. But $v^*TN \cong TS^4 \oplus E_{k,1-k}$ and so the Whitney formula implies together with Proposition 18.1:

$$\langle v^*(p_1(TN)), [S^4]\rangle = \langle p_1(E_{k,1-k}), [S^4]\rangle = -2(2k - 1)$$

Comparing this information with the Kronecker product $\langle v^*([S^4, v]), [S^4]\rangle = -1$ we conclude:

$$p_1(TN) = 2(2k - 1)[S^4, v].$$

Using

$$S_N([S^4, v], [S^4, v]) = -k - (1 - k) = -1$$

we have:

$$\langle p_1^2(TN), [N]\rangle = -4(2k - 1)^2$$

Now we feed this information into the signature Theorem 19.3:

$$-1 = \tau(N) = \frac{1}{45}(7\langle p_2(TN), [N]\rangle + 4(2k - 1)^2)$$

Since $\langle p_2(TN), [N]\rangle \in \mathbb{Z}$, we obtain the congruence

$$45 + 4(2k - 1)^2 \equiv 0 \mod 7$$

if $M_{k,1-k}$ is diffeomorphic to S^7. Taking $k = 2$ we obtain a contradiction and so have proved

Theorem 19.4. **(Milnor):** *$M_{2,-1}$ is homeomorphic to S^7 but not diffeomorphic.*

This was the first example of a so called exotic smooth structure on a manifold, i.e. a second smooth structure which is not diffeomorphic to the given one.

We give another application of the signature formula. Given a topological manifold M of dimension $2k$ one can ask whether there is a complex structure on M, i.e. an atlas of charts in $\mathbb{C}^k$ whose coordinate changes are holomorphic functions. We suppose now that M is closed and connected. A necessary condition is that M admits a non-trivial class in $\mathbf{H}_{2k}(M)$. One can introduce the concept of orientation for topological manifolds and show that a connected closed n-dimensional manifold is orientable if and only if a non-trivial class in $\mathbf{H}_n(M)$ exists. Thus the necessary condition above amounts to a topological version of orientability. It $k = 1$ it is a classical fact that all orientable surfaces admit a complex structure. As another application of the signature formula we show

THEOREM 19.5. *S^4 admits no complex structure.*

Proof: If S^4 is equipped with a complex structure, the tangent bundle is a complex vector bundle. For a complex vector bundle E we can compute the first Pontrjagin class using the formula from §18:

$$p_1(E) = -2c_2(E)$$

Thus

$$p_1(TS^4) = -2c_2(TS^4) = -2e(TS^4),$$

since $c_1(TS^4) = 0$. Now we use the fact from the Remark after Corollary 12.2 that $\langle e(TM), [M]\rangle = e(M)$ (following from the Poincaré-Hopf Theorem for vector fields) and conclude:

$$\langle e(TS^4), [S^4]\rangle = e(S^4) = 2$$

Next we note that $\tau(S^4) = 0$ and so we obtain a contradiction from the signature formula:

$$0 = \tau(S^4) = 1/3\langle p_1(TS^4), [S^4]\rangle = -4/3$$

q.e.d.

One actually can show that S^{2k} has no complex structure for $k \neq 1, 3$. It is a famous open problem whether S^6 has a complex structure.

2. The Milnor spheres are homeomorphic to the 7-sphere

We finish this chapter with the proof of Theorem 19.1. It is based on an elementary but fundamental argument in Morse theory.

LEMMA 19.6. *Let W be a compact smooth manifold with $\partial W = M_0 + M_1$. If there is a smooth function*

$$f : W \to [0, 1]$$

without critical points and $f(M_0) = 0$ and $f(M_1) = 1$, then W is diffeomorphic to $M_0 \times [0, 1]$.

Proof: Choose a smooth Riemannian metric g on TW (for example embed W smoothly into a Euclidean space and restrict the Euclidean metric to each fibre of the tangent bundle). Consider the so called **normed gradient vector field** of f which is defined by mapping $x \in M$ to the vector $s(x) \in T_xM$ such that

i) $df_x s(x) = 1 \in \mathbb{R} = T_{f(x)}\mathbb{R}$

ii) $\langle s(x), v\rangle_{g(x)} = 0$ for all v with $df_x(v) = 0$.

This is a well defined function since the dimension of $\ker df_x$ is $\dim M - 1$ and $df_x|_{\ker\ df_x^\perp}$ is an isomorphism (the orthogonal complement is taken with respect to g_x). Since W, f and g are smooth, this is a smooth vector field on W.

Now, we consider the ordinary differential equation for a given point $x \in W$:

$$\dot\varphi(t) = s(\varphi(t)) \text{ and } \varphi(0) = x,$$

where φ is a smooth function (a path) from an interval to W and as usual we abbreviate the differential of a path φ at the time t by $\dot\varphi(t)$.

The existence and uniqueness result for ordinary differential equations says that locally (using a chart to translate everything into $\mathbb{R}^m$) there is a unique solution called an integral curve. Furthermore, the solution depends smoothly on x if we vary the initial point x and on t.

Now, for each $x \in M_0$ we consider a maximal interval for which one has a solution φ_x with initial value x. Then

$$df(\dot\varphi_x(t)) = df(s(\varphi_x(t))) = 1$$

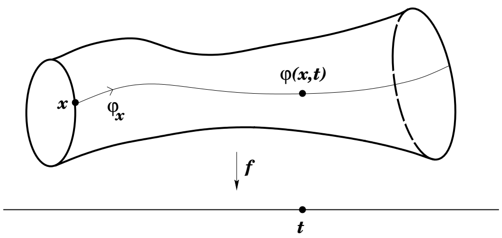
Thus

$$f\varphi_x(t) = t + c$$

for some $c \in \mathbb{R}$. Since $\varphi_x(0) = x$, we conclude $c = 0$ and so $f\varphi_x(t) = t$.

Since W is compact, the interval is maximal and since $f\varphi_x(t) = t$, the interval has to be $[0,1]$. As φ_x depends smoothly on x and t we obtain a smooth function

$$\begin{array}{rlcl} \psi : & M_0 \times [0,1] & \to & W \\ & (x,t) & \mapsto & \varphi_x(t) \end{array}$$



This function is a diffeomorphism since we have an inverse. For this consider for $y \in W$ the integral curve of the differential equation:

$$\dot\eta_y(t) = -s(\eta_y(t)) \text{ and } \eta_y(0) = y$$

(we use the negative gradient field to "travel" backwards). As above, we see that

$$f\eta_y(t) = f(y) - t$$

The integral curve $\eta_{\varphi_x(t)}$ joins $\varphi_x(t)$ with x and is the time inverse of the integral curve φ_x. With this information, we can write down the inverse:

$$\psi^{-1}(y) = (\eta_y(f(y)), f(y))$$

q.e.d.

Proof of Theorem 19.1 after Milnor: For simplicity we only consider the case $M_{k,1-k}$, the other case follows similarly. With Lemma 19.6 we will give the proof by constructing two disjoint embeddings D^7_+ and D^7_- in $M_{k,1-k}$ and constructing a smooth function

$$f : M_{k,1-k} - (\mathring{D}^7_+ + \mathring{D}^7_-) \to [0,1]$$

without critical points. Then by Lemma 19.6 there is a diffeomorphism

$$\varphi : S^6_+ \times [0,1] \longrightarrow M_{k,1-k} - (\mathring{D}^7_+ + \mathring{D}^7_-)$$

with $\varphi(x,0) = x$ for all $x \in S^6_+$.

From this we construct a homeomorphism from $M_{k,1-k}$ to $S^7 = D^7_+ \cup D^7_-$ as follows. We map

$$x \in D^7_+ \text{ to } x \in D^7_+ \subset S^7,$$
$$\varphi(x,t) \text{ to } (1 - t/2) \cdot x \in D^7_- \text{ for } x \in S^6_+ \text{ and } t \in [0,1]$$
$$x \in D^7_- \text{ to } x/2 \in D^7_- \subset S^7.$$

The reader should check that this map is well defined, continuous and bijective. Thus it is a homeomorphism.

Continuing in the proof, we note that

$$M_{k,l} = \mathbb{H} \times S^3 \cup_{f_{k,l}} -\mathbb{H} \times S^3$$

where $f_{k,l} : \mathbb{H} - \{0\} \times S^3 \to -\mathbb{H} - \{0\} \times S^3$ maps

$$(x,y) \mapsto (x/_{||x||^2}, x^k y x^l/_{||x||^{(k+l)}})$$

We have used this description since it gives $M_{k,l}$ as a smooth manifold. Now we consider the smooth functions

$$\begin{array}{rccc} g: & \mathbb{H} \times S^3 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \frac{y_1}{\sqrt{1+||x||^2}} \end{array}$$

and

$$\begin{array}{rccc} h: & -\mathbb{H} \times S^3 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \frac{(x \cdot y^{-1})_1}{\sqrt{1+||x \cdot y^{-1}||^2}} \end{array}$$

where $(\ )_1$ denotes the first component.

If $l = 1 - k$, the two functions are compatible with the gluing function $f_{k,1-k}$ and thus

$$g \cup h : M_{k,1-k} \longrightarrow \mathbb{R}$$

is a smooth function.

What are the singular points of g and h? The function h has no singular points but g has singular points $(0,1)$ and $(0,-1)$, where $1 = (1,0,0,0) \in S^3$. Thus, 1 and -1 are the only singular values of $g \cup h$.

Since $\pm 1/2$ are regular values, we can decompose $M_{k,1-k}$ as $(g \cup h)^{-1}(-\infty, -\frac{1}{2}] \cup (g \cup h)^{-1}[-\frac{1}{2}, \frac{1}{2}]$ and $(g \cup h)^{-1}[\frac{1}{2}, \infty) =: D_+ \cup W \cup D_-$. We identify $D_\pm$ with D^7 as follows:

$D_+ = (g \cup h)^{-1}(-\infty, -\frac{1}{2}] = \{(x,y) \in \mathbb{H} \times S^3 | y_1 \leq -\frac{1}{2}\sqrt{1 + ||x||^2}\}$ using the fact that $y \in S^3$ and so $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1$. From this we conclude that

$$D_+ = \{(x, (y_2, y_3, y_4)) | \, 4\,(y_2^2 + y_3^2 + y_4^2) + ||x||^2 \leq 3\}$$

and thus D_+ is diffeomorphic to D^7. Similarly one shows that D_- is diffeomorphic to D^7. Since $g \cup h|_W$ has no critical points, we are in the situation discussed above and the proof is finished.

q.e.d.

CHAPTER 20

Relation to ordinary singular (co)homology

1. $\mathbf{H}_k(X)$ is isomorphic to ordinary homology for CW-complexes

This chapter has a different character since we use several concepts and results which are not covered in this book. In particular we assume familiarity with ordinary singular homology and cohomology.

Eilenberg and Steenrod showed that if a functor fulfils their homology axioms, then there is a unique natural isomorphism between this homology and ordinary singular homology $H_k(X)$ for finite CW-complexes X, which for a point is the identity [**E-S**]. Their axioms are equivalent to our axioms, if in addition the homology groups of a point are $\mathbb{Z}$ in degree 0 and 0 else. Thus for finite CW-complexes X there is a unique natural isomorphism (which for a point is the identity)

$$\sigma : \mathbf{H}_k(X) \to H_k(X)$$

Since $\mathbf{H}_k(X)$ is compactly supported one can extend σ to a natural transformation for arbitrary CW-complexes. Namely, if X is a CW-complex and $[\mathbf{S}, g]$ is an element of $\mathbf{H}_k(X)$, the image of $\mathbf{S}$ under g is compact. Thus there is a finite subcomplex Y in X such that $g(\mathbf{S}) \subset Y$. Let $i : Y \to X$ be the inclusion, then we consider $i_*(\sigma([\mathbf{S}, g]) \in H_k(X)$, where we consider $[\mathbf{S}, g]$ as element of $\mathbf{H}_k(Y)$. It is easy to see that this gives a well defined natural transformation

$$\sigma : \mathbf{H}_k(X) \to H_k(X)$$

for arbitrary CW-complexes X. We use the fact that if $(\mathbf{T}, h)$ is a bordism, then $g(\mathbf{T})$ is contained in some other finite subcomplex Z with $Y \subset Z$.

Theorem 20.1. *The natural transformation*

$$\sigma : \mathbf{H}_k(X) \to H_k(X)$$

is an isomorphism for all CW-complexes X and all k.

This natural transformation commutes with the $\times$-product.

More generally it is enough to require that X is homotopy equivalent to a CW-complex. All smooth manifolds are homotopy equivalent to CW-complexes [**Mi 3**] and so theorem 20.1 holds for all smooth manifolds.

Proof: We know this already for finite CW-complexes. The argument for arbitrary CW-complexes uses the same idea as the construction of the generalization of σ. Namely if X is an arbitrary CW-complex and $x \in H_k(X)$ is a homology class then there exists a finite subcomplex Y such that $x \in \operatorname{im}(H_k(Y) \to H_k(X))$. From this we conclude using the result for finite CW-complexes that x is in the image of $\sigma : \mathbf{H}_k(X) \to H_k(X)$. Similarly if $x \in \mathbf{H}_k(X)$ maps to zero under σ, we find a finite CW-complex $Z \subset X$ such that $x \in \operatorname{im}(\mathbf{H}_k(Z) \to \mathbf{H}_k(X))$ since $\mathbf{H}_k(X)$ has compact support. Thus we can assume that $x \in \mathbf{H}_k(Z)$. Since $H_k(X)$ has compact support there is a finite CW complex $T \subset X$ such that $Z \subset T$ and $\sigma(x)$ maps to zero in $H_k(T)$. From the result for finite CW-complexes we conclude $\sigma(x) = 0$ in $H_k(T)$ and so $x = 0$.

The argument that the natural transformation commutes with the $\times$-product is based on a description of ordinary singular homology using bordism of stratifolds with additional structure, a so called parametrization. This is defined in [**K**], where we construct a natural isomorphism between homology based on parametrized stratifolds and ordinary homology. This natural isomorphism preserves the $\times$-product. The forgetful map (forgetting the parametrization) gives another natural transformation from homology based on parametrized stratifolds to $\mathbf{H}_k(X)$ which preserves the $\times$-product. Since the natural transformations commute for CW-complexes (by the fact that for a point they are the identity) this shows that the natural transformation above commutes with the $\times$-product. **q. e. d.**

Remark: *A similar argument gives a natural isomorphism*

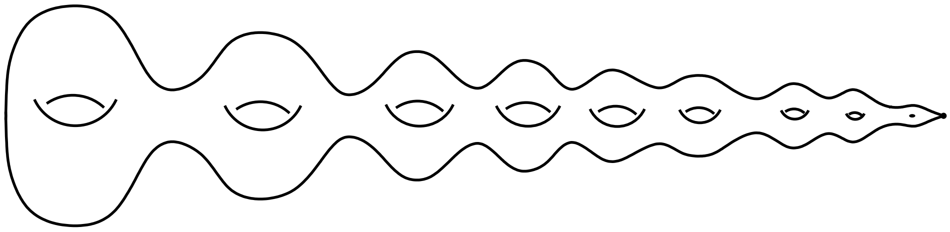
$$\sigma_n : \mathbf{H}_k(X; \mathbb{Z}/2) \to H_k(X; \mathbb{Z}/2).$$

for all CW-complexes X.

2. An example where $\mathbf{H}_k(X)$ and $H_k(X)$ are different

We denote the oriented surface of genus g by F_g. For $g = 1$ we obtain the torus $F_1 = T$ and F_g is the connected sum of g copies of the torus.

We consider the following subspace of $\mathbb{R}^3$ given by an infinite connected sum of tori as in the following picture, where the point on the right side is removed. We call this an infinite sum of tori. This is a non-compact smooth submanifold of $\mathbb{R}^3$ denoted by F_∞. The space on the picture is the one point compactification of F_∞. This is a compact subspace of $\mathbb{R}^3$.



As in example 2.) in §2 we make F_∞^+ a 2-dimensional stratifold denoted $\mathbf{S}$ by the algebra $\mathbf{C}$ consisting of continuous functions which are constant near the additional point and smooth on F_∞. Obviously this stratifold is regular and oriented. Thus we can consider the fundamental class

$$[\mathbf{S}] = [\mathbf{S}, \mathrm{id}] \in \mathbf{H}_2(\mathbf{S})$$

This class has the following property. Let $p_g : \mathbf{S} \to F_g$ be the projection onto F_g (we map all tori added to F_g to obtain F_∞ to a point). Then

$$(p_g)_*([\mathbf{S}]) = [F_g]$$

(why ?). In particular $(p_g)_*([\mathbf{S}])$ is non-trivial for all g.

But there is no class α in $H_2(\mathbf{S})$ such that $p_{g*}(\alpha)$ is non-trivial for all g. The reason is that for each topological space X and each class α in $H_2(X)$ there is a map $f : F \to X$, where F is a closed oriented surface, such that $\alpha = f_*([F])$. This follows from [**C-F**] using the Atiyah-Hirzebruch spectral sequence. Now we suppose that we can find $f : F \to \mathbf{S}$ such that $(p_g)_*(\alpha) \neq 0$ in $H_2(F_g)$ for all g. But this is impossible since the degree of fp_g is non-zero and there is no map $F \to F_g$ with degree non-zero if the genus of F is smaller than g. The reason is that if the degree is non-trivial then the induced map $H_1(F_g) \to H_1(F)$ is injective (as follows from the unimodularity of the intersection form).

We summarize these considerations:

THEOREM 20.2. *The homology theories $\mathbf{H}_k(X)$ and $H_k(X)$ are not equivalent.*

Remark: *For compact metric spaces (where the maps between these spaces are the ordinary continuous maps) there are other homology theories which for finite CW-complexes agree with ordinary homology, for example Steenrod homology. It is natural to ask if $\mathbf{H}_k(X)$ on compact metric spaces agrees with Steenrod homology. The answer is no. We will discuss this relation in* [**K**].

In [**K**] *we will introduce stratifolds with an additional structure, called a parametrization. We will use parametrized stratifolds to define homology groups as we did it here with arbitrary stratifolds. It turns out that the resulting homology theory is ordinary homology for* **all** *spaces X. This will be shown in* [**K**].

3. $H^k(M)$ is isomorphic to ordinary singular cohomology

We also want to identify our cohomology groups $H^k(M)$ constructed via stratifolds with the singular cohomology groups $H^k(M)$. To distinguish the notation we denote the cohomology groups constructed via stratifolds by $\mathbf{H}^k(M)$.

So far we only have defined integral cohomology groups for oriented manifolds. We will now use a trick to extend the definition to arbitrary manifolds. The idea is to associate in a natural way to each non-oriented manifold an oriented manifold whose cohomology will be defined as the cohomology of the original manifold.

For this we consider for a smooth m-dimensional manifold M the determinant line bundle $\Lambda^m TM$. This is the bundle whose fibre at x is $\Lambda^m T_x M$, the 1-dimensional space of alternating m-forms on $T_x M$. The total space of $\Lambda^m TM$ has a canonical orientation. Namely if $p : E \to M$ is a smooth vector bundle, then p^*E, the tangent vectors along the fibre, is a subbundle of TE, and the differential of p induces an isomorphism $TE/_{p^*E} \to p^*TM$. In the situation above we choose for $x \in \Lambda^m TM$ vectors $v_1, \dots, v_m \in T_x(\Lambda^m TM)$ such that $dp(v_i)$ are a basis of $T_{p(x)}M$ and $0 \neq \omega \in \Lambda^m T_{p(x)}M$ and say that $(v_1, \dots, v_m, \omega)$ are positively oriented if $\omega(dp_x(\nu_1), \dots, dp_m(\nu_m)) > 0$. It is easy to check that this determines an orientation of the total space of $\Lambda^m TM$.

After we have oriented $\Lambda^m TM$, we can define $\mathbf{H}^k(M) := H^k(\Lambda^m TM)$. If $g : M' \to M$ is a smooth map, we define $\hat{g} : \Lambda^m TM' \to \Lambda^m TM$ as sgp', where $p' : \Lambda^m TM' \to M'$ is the projection and $s : M \to \Lambda^m TM$ is the 0-section. If : $M'' \to M'$ is a smooth map, then $\hat{g}\hat{h} = sgp'hp'' = sghp'' = \hat{(gh)}$. We further note that since sp' is homotopic to $id : \Lambda^m TM \to \Lambda^m TM$, the maps $\hat{id}$ and $id : \Lambda^m TM \to \Lambda^m TM$ are homotopic.

Now for $g : M' \to M$ we define

$$g^* := (\hat{g})^* : \mathbf{H}^k(M) \to \mathbf{H}^k(M')$$

and the projections above imply that

$$id^* = id$$

$$(gh)^* = h^* g^*$$

If U and V are given subsets of M, then $p^{-1}(U)$ and $p^{-1}(V)$ are open subsets of $\Lambda^m TM$ with $p^{-1}(U) \cap p^{-1}(V) = p^{-1}(U \cap V)$. Thus we obtain a Mayer-Vietoris sequence from the Mayer-Vietoris sequence of oriented manifolds:

$$\cdots \to \mathbf{H}^k(U \cup V) \to \mathbf{H}^k(U) \oplus \mathbf{H}^k(V) \to \mathbf{H}^k(U \cap V) \to \mathbf{H}^{k+1}(U \cap V) \to \dots.$$

By construction it is natural, i.e. commutes with induced maps.

Thus we have defined a cohomology theory for arbitrary smooth manifolds. It remains to show that, if M is oriented, there is a canonical natural isomorphism between $H^k(M)$ as defined previously and $H^k(\Lambda^m TM)$. Such an isomorphism is easily described, namely:

$$p^* : H^k(M) \to H^k(\Lambda^m TM)$$

is an isomorphism since p is a homotopy equivalence. Actually an orientation of M gives an orientation preserving isomorphism between $\Lambda^m TM$ and $M \times \mathbb{R}$. Using this, one sees that p^* is a natural isomorphism.

This finishes our definition of integral cohomology for non-oriented manifolds extending the previous definition if the manifold is oriented. Now we proceed with the comparison with ordinary integral cohomology.

We use a characterization of singular cohomology on smooth manifolds from [**K-S**]. The main result from this paper says that we only have to check the following conditions

for such a cohomology theory h for which the cohomology groups of a point are $\mathbb{Z}$ in degree 0 and 0 else.
i) Let M_i for $i = 1, 2...$ be a sequence of smooth manifolds. Then

$$h^k(+M_i) \cong \prod_i h^k(M_i),$$

where the isomorphism from $h^k(M)$ to the direct product is given by the inclusions.
ii) The $\times$-product gives an isomorphism

$$\times : h^k(S^k) \otimes h^n(S^n) \cong h^{k+n}(S^k \times S^n)$$

Since these conditions hold for our cohomology theory there is a unique natural isomorphism θ from $\mathbf{H}^k(M)$ to $H^k(M)$ commuting with the $\times$ products and inducing the identity on $\mathbf{H}^0(pt)$ [**K-S**]:

THEOREM 20.3. *There is a unique natural isomorphism θ from the cohomology groups constructed in this book via stratifolds to ordinary singular cohomology, commuting with the $\times$ products and inducing the identity on cohomology in degree* 0.

Since the natural transformation θ respects the cup product we obtain a geometric interpretation of the intersection form on ordinary singular cohomology. Let M be a closed smooth oriented manifold of dimension m. Since θ respects the cup-products we conclude:

COROLLARY 20.4. *For a closed smooth oriented m-dimensional manifold M and cohomology classes $x \in H^k(M)$ and $y \in H^{m-k}(M)$ we have the identity:*

$$\langle x \cup y, [M] \rangle = [\mathbf{S}_x, g_x] \pitchfork [\mathbf{S}_y, g_y],$$

where $[S_x, g_x] := \theta(x)$ and $[S_y, g_y] := \theta(y)$ are cohomology classes in $\mathbf{H}^k(M)$ and $\mathbf{H}^{m-k}(M)$ corresponding to x and y via θ and $\pitchfork$ means the transversal intersection.

Thus the traditional geometric interpretation of the intersection form for those cohomology classes on a closed oriented smooth manifold, where the Poincaré duals are represented by a map from a closed oriented smooth manifold to M, as transversal intersection makes sense for arbitrary cohomology classes.

The natural isomorphism between the (co)homology groups defined in this book and ordinary singular cohomology allow - for CW-complexes - to translate results from one of the worlds to the other. Above we have made use of this by interpreting the intersection form on singular cohomology geometrically. The geometric feature is one of the strengths of our approach to (co)homology. There are other aspects of (co)homology which are easier and more natural in ordinary singular (co)homology, in particular those which allow an application of homological algebra. This is demonstrated by the general Künneth Theorem or by the various universal coefficient theorems. It is useful to have both interpretations of (co)homology available so that one can choose in which world one wants to work depending on the questions one is interested in.

APPENDIX A

Constructions of stratifolds

1. The product of two stratifolds

Now we show that $(\mathbf{S} \times \mathbf{S}', \mathbf{C}(\mathbf{S} \times \mathbf{S}'))$ as defined in chapter 2 is a stratifold. It is clear that $\mathbf{S} \times \mathbf{S}'$ is a locally compact Hausdorff space with countable basis. We have to show that $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$ is an algebra. Let f and g be in $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$ and $x \in \mathbf{S}^i$ and $y \in (\mathbf{S}')^j$. Using local retracts one sees that $f + g$ and fg are in $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$. Obviously the constant maps are in $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$. Since we characterize $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$ by local conditions it is locally detectable. Also condition 2.) of a differential space is obvious.

Next we show that restriction gives an isomorphism of germs near $(x, y) \in \mathbf{S}^i \times (\mathbf{S}')^j$:

$$\mathbf{C}(\mathbf{S} \times \mathbf{S}')_{(x,y)} \xrightarrow{\cong} C^\infty(\mathbf{S}^i \times (\mathbf{S}')^j)_{(x,y)}$$

To see that this map is surjective, we consider $f \in C^\infty(\mathbf{S}^i \times (\mathbf{S}')^j)$ and choose for x a local retract $r : U \to V$ near x of $\mathbf{S}$ and for y a local retract $r' : U' \to V'$ near y of $\mathbf{S}'$. Let ρ be a smooth function on $\mathbf{S}$ with support $\rho \subset U$ which is constant 1 near x and ρ' a corresponding smooth function on $\mathbf{S}'$ with support $\rho' \subset U'$ which is constant 1 near y. Then $\rho(z)\rho'(z')f(r(z), r'(z'))$ (which we extend by 0 to the complement of $U \times U'$) is in $\mathbf{C}(\mathbf{S} \times \mathbf{S}')$. (To see this we only have to check for $(z, z') \in U \times U'$ that there are local retracts q near z and q' near z' such that $f(rq(t), r'q'(t')) = f(r(t), r'(t'))$. But since r is a morphism, we can choose q such that $rq(t) = r(t)$ and similarly $r'q'(t') = r'(t')$ implying the statement.) Thus we have found a germ near (x, y) which maps to f under restriction.

To see that the map is injective, we note that if $f \in \mathbf{C}(\mathbf{S} \times \mathbf{S}')$ maps to zero in $C^\infty(\mathbf{S}^i \times (\mathbf{S}')^j)_{(x,y)}$ it vanishes in an open neighborhood of (x, y) in $\mathbf{S}^i \times (\mathbf{S}')^j$ and since there are retracts near x and y such that f commutes with them, it is zero in some open neighborhood of (x, y) in $\mathbf{S} \times (\mathbf{S}')$.

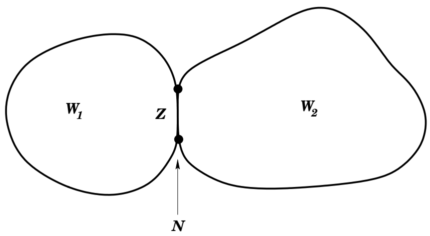
After we have shown that $\mathbf{C}(\mathbf{S} \times \mathbf{S}')_{(x,y)} \xrightarrow{\cong} C^\infty(\mathbf{S}^i \times (\mathbf{S}')^j)_{(x,y)}$ is an isomorphism, we conclude that $T_{(x,y)}(\mathbf{S} \times \mathbf{S}') \cong T_{(x,y)}(\mathbf{S}^i \times (\mathbf{S}')^j)$, and so the induced stratification on $\mathbf{S} \times \mathbf{S}'$ is given by $+_{i+j=k}\mathbf{S}^i \times (\mathbf{S}')^j$. Now condition 1 also follows from the isomorphism of germs, condition 2 is obvious and condition 3 follows form the product $\rho\rho'$ of appropriate bump functions of $\mathbf{S}$ and $\mathbf{S}'$.

Thus $(\mathbf{S} \times \mathbf{S}', \mathbf{C}(\mathbf{S} \times \mathbf{S}'))$ is $(k + l)$-dimensional stratifold.

2. Gluing along part of the boundary

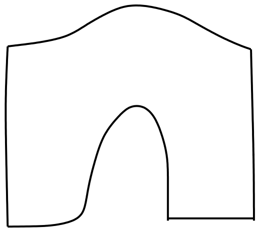
In the proof of the Mayer-Vietoris sequence we will also need gluing along part of the boundary. If one glues naively then corners or cusps occur (see picture below). The corners or cusps can in a natural way be removed or better smoothed. The central tool for this smoothing is given by collars. The constructions will depend on the choice of a collar, not just on the corresponding germ. But up to bordism, these choices are irrelevant.

Now we return to gluing along part of the boundary. Consider two c-stratifolds W_1 and W_2 and suppose that ∂W_1 is obtained by gluing two c-stratifolds Z and Y_1 over the common boundary $\partial Z = \partial Y_1 = N$ (assuming that Z and Y_1 have collars φ_Z and φ_{Y_1}): $\partial W_1 = Z \cup_N Y_1$. Similarly, we assume that $\partial W_2 = Z \cup_N Y_2$ (using collars φ_Z and φ_{Y_2}) and that W_1 and W_2 have collars η_1 and η_2. Then we want to make $W_1 \cup_Z W_2$ a c-stratifold with boundary $Y_1 \cup_N Y_2$. We define $W_1 \overset{\circ}{\cup}_Z W_2$ as $W_1 \cup_Z W_2 - Y_1 \cup_N Y_2$. But this space is equal to $W_1 - Y_1 \cup_{\overset{\circ}{Z}} W_2 - Y_2$, gluing of two c-stratifolds along the full boundary $\overset{\circ}{Z}$, which is a stratifold by the considerations above. If we add the boundary $Y_1 \cup_N Y_2$ naively and use the given collars, we obtain "cusps" along N.

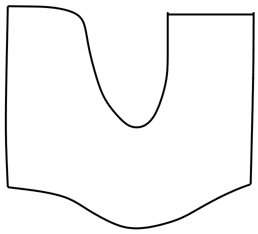


To smooth along N we first combine φ_Z and φ_{Y_1} to an isomorphism $\varphi_1 : N \times (-1,1) \to \partial W_1$ onto its image, where $\varphi_1(x,t) := \varphi_Z(x,t)$ for $t \geq 0$ and $\varphi_1(x,t) := \varphi_{Y_1}(x,-t)$ for $t \leq 0$. $\varphi_1|_{N\times\{0\}}$ is the identity map. Similarly, we combine φ_Z and φ_{Y_2} to $\varphi_2 : N \times (-1,1) \to \partial W_1$ and note that $\varphi_2|_{N\times[0,1)} = \varphi_1|_{N\times[0,1)}$. We denote by $\alpha_1 : N \times (-1,1) \times [0,1) \to W_1$ the map $(x,s,t) \mapsto \eta_1(\varphi_1(x,s),t)$. We denote the image by U_1. This map is an isomorphism away from the boundary. Similarly, we define $\alpha_2 : N \times (-1,1) \times [0,1) \to U_2$. The union $U_1 \cup U_2 := U_N$ is an open neighbourhood of N in $W_1 \cup_Z W_2$.

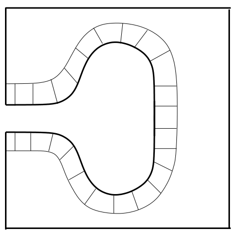
Now we pass in $\mathbb{R}^2$ to polar coordinates (r,φ) and choose a smooth monotone map $\rho : \mathbb{R}_{\geq 0} \to (0,1]$, which is for $r \leq \frac{1}{3}$ equal to $\frac{1}{2}$ and for $r \geq \frac{2}{3}$ equal to 1 (it is important to fix this map for the future and use the same map to make the constructions unique). Then consider the map β_1 from $(-1,1) \times [0,1) \subset \{(r,\varphi) | r \geq 0,\ 0 \leq \varphi \leq \pi\}$ to $\mathbb{R}^2$ mapping (r,φ) to $(r, \rho(r)\cdot\varphi)$ and similarly β_2 mapping (r,φ) to $(r, -\rho(r)\cdot\varphi)$. The images of $(-1,1) \times [0,1)$ in cartesian coordinates look roughly as



and



Identifying $\beta_1([0,1)\times\{0\})$ with $\beta_2([0,1)\times\{0\})$ gives a smooth c-manifold G looking as



where the collar is indicated in the picture. We obtain a homeomorphism $\Phi : U_N \longrightarrow N\times G$ mapping $\alpha_1(x,s,t)$ to $(x,\beta_1(s,t))$ and $\alpha_2(x,s,t)$ to $(x,\beta_2(s,t))$. Φ is an isomorphism of stratifolds outside N. By construction the collar induced from $N\times G$ via Φ and the collars of W_1 along $\partial W_1 - \mathrm{im}\varphi_{Y_1}$ and of W_2 along $\partial W_2 - \mathrm{im}\varphi_{Y_2}$ fit together to give a collar on $W_1 \cup_Z W_2$ finishing the proof of:

Proposition A.1. *Let (for $i = 1,2$) W_i be c-stratifolds such that ∂W_i is obtained by gluing two c-stratifolds Z and Y_i over the common boundary $\partial Z = \partial Y_i = N$:*

$$\partial W_i = Z \cup_N Y_i$$

Choose representatives of the germs of collars for Y_i and Z.

Then there is a c-stratifold $W_1 \cup_Z W_2$ extending the stratifold structures on $W_i - (Z \cup im\varphi_{Y_i})$. The boundary of $W_1 \cup_Z W_2$ is $Y_1 \cup_N Y_2$.

It should be noted that the construction of the collar of $W_1 \cup_Z W_2$ depends on the choice of representatives of the collars of W_i, Y_i and Z. For our application in the proof of the Mayer-Vietoris sequence it is important to observe, that the collar was constructed in such a way, that away from the neighbourhood of the union of the collars of N in Y_i and Z is the original collar of W_1 and W_2.

3. Proof of Proposition 4.1

At the end of this chapter we prove that for a space X the isomorphism classes of pairs $(\mathbf{S}, g)$, where $\mathbf{S}$ is an m-dimensional stratifold, and $g : \mathbf{S} \to X$ is a continuous map, is a set.

Proof of Proposition 4.1: For this we first note that the diffeomorphism classes of manifolds form a set. This follows since a manifold is diffeomorphic to one obtained by taking a countable union of open subsets of $\mathbb{R}^m$ (the domains of a countable atlas) and identifying them according to an appropriate equivalence relation. Since the countable sum of copies of $\mathbb{R}^m$ is a set, the set of subsets of a set is a set, and the possible equivalence relations on these sets form a set, the diffeomorphism classes of m-dimensional manifolds are a subset of the set of all sets obtained from a countable disjoint union of subsets of $\mathbb{R}^m$ by some equivalence relation.

Next we note that a stratifold is obtained from a disjoint union of manifolds, the strata, by introducing a topology (a collection of certain subsets) and a certain algebra. The possible topologies as well as the possible algebras are a set. Thus the isomorphism classes of stratifolds are a set. Finally for a fixed stratifold $\mathbf{S}$ and space X the maps from $\mathbf{S}$ to X are a set, and so we conclude that the isomorphism classes of pairs $(\mathbf{S}, g)$, where $\mathbf{S}$ is an m-dimensional stratifold, and $g : \mathbf{S} \to X$ is a continuous map, is a set.
q.e.d.

APPENDIX B

The detailed proof of the Mayer-Vietoris sequence

The following lemma is the main tool for completing the proof of the Mayer-Vietoris sequence along the lines explained in §5. It is also useful in other contexts. Roughly it says that up to bordism we can separate a regular stratifold $\mathbf{S}$ by an open cylinder over some regular stratifold $\mathbf{P}$. Such an embedding is called a **bicollar**, i.e. an isomorphism $g : \mathbf{P} \times (-\epsilon, \epsilon) \to V$, where V is an open subset of $\mathbf{S}$. The most naive idea would be to "replace" $\mathbf{P}$ by $\mathbf{P} \times (-\epsilon, \epsilon)$, so that as a set we change $\mathbf{S}$ into $(\mathbf{S} - \mathbf{P}) \cup (\mathbf{P} \times (-\epsilon, \epsilon))$. The proof of the following lemma makes this rigorous.

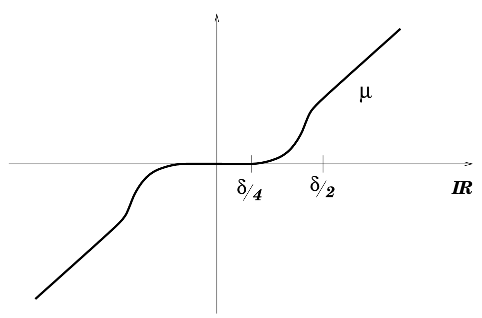
Lemma B.1. *Let $\mathbf{T}$ be a regular c-stratifold. Let $\rho : \mathbf{T} \to \mathbb{R}$ be a continuous function such that $\rho|_{\overset{\circ}{\mathbf{T}}}$ is smooth. Let 0 be a regular value of $\rho|_{\overset{\circ}{\mathbf{T}}}$ and suppose that $\rho^{-1}(0) \subset \overset{\circ}{\mathbf{T}}$ and that there is an open neighbourhood of 0 in $\mathbb{R}$ consisting only of regular values of $\rho|_{\overset{\circ}{\mathbf{T}}}$.*

Then there exists a regular c-stratifold $\mathbf{T}'$ and a continuous map $f : \mathbf{T}' \to \mathbf{T}$ with $\partial \mathbf{T}' = \partial \mathbf{T}$, $f|_{\partial \mathbf{T}'} = id$ such that f commutes with appropriate representatives of the collars of $\mathbf{T}'$ and $\mathbf{T}$. Furthermore there is an $\epsilon > 0$ such that $\rho^{-1}(0) \times (-\epsilon, \epsilon)$ is contained in $\mathbf{T}'$ as open subset and a continuous map $\rho' : \mathbf{T}' \to \mathbb{R}$ whose restriction to the interior is smooth and whose restriction to $\rho^{-1}(0) \times (-\epsilon, \epsilon)$ is the projection to $(-\epsilon, \epsilon)$. The restriction of f to $\rho^{-1}(0) \times (-\epsilon, \epsilon)$ is the projection onto $\rho^{-1}(0)$. In addition $(\rho')^{-1}(-\infty, -\epsilon) \subset \rho^{-1}(-\infty, 0)$ and $(\rho')^{-1}(\epsilon, \infty) \subset \rho^{-1}(0, \infty)$.

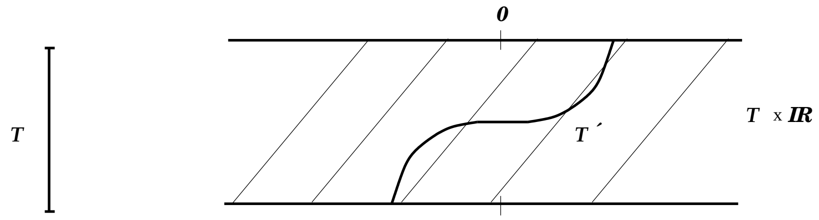
If $\partial \mathbf{T} = \emptyset$ then $(\mathbf{T}, id)$ and $(\mathbf{T}', f)$ are bordant.

Proof: Choose δ such that $(-\delta, \delta)$ consists only of regular values of ρ.

Consider a monotone smooth map $\mu : \mathbb{R} \to \mathbb{R}$ which is the identity for $|t| > \delta/2$ and 0 for $|t| < \delta/4$.



Then $\eta : \mathbf{T} \times \mathbb{R} \to \mathbb{R}$ mapping $(x,t) \mapsto \rho(x) - \mu(t)$ has 0 as regular value. Namely, for those (x,t) mapping to 0 with $|t| < \delta$ we have $|\rho(x)| < \delta$ and thus (x,t) is a regular point of η, and for those (x,t) mapping to 0 with $|t| > \delta/2$ we have $\mu(t) = t$ and again (x,t) is a regular point. Thus $\mathbf{T}' := \eta^{-1}(0)$ is by Proposition 4.2 a regular c-stratifold (the collar is discussed below) containing $V := \rho^{-1}(0) \times (-\delta/4, \delta/4)$. Setting $\epsilon = \delta/4$ we have constructed the desired subset in $\mathbf{T}'$.



To relate $\mathbf{T}'$ to $\mathbf{T}$, consider the map $f : \mathbf{T}' \to \mathbf{T}$ given by the restriction of the projection onto $\mathbf{T}$ in $\mathbf{T} \times \mathbb{R}$. This is an isomorphism outside $\rho^{-1}(0) \times (-\delta/2, \delta/2)$. In particular we can identify the boundaries via this isomorphism: $\partial\mathbf{T}' = \partial\mathbf{T}$. Similarly we use this isomorphism to induce a collar on $\mathbf{T}'$ from a small collar of $\mathbf{T}$ and so the c-structure on $\mathbf{T}$ makes $\mathbf{T}'$ a regular c-stratifold. Finally we define ρ' by the projection onto $\mathbb{R}$. The desired properties are obvious and this finishes the proof of the first statement.

If $\partial\mathbf{T} = \emptyset$, we want to construct a bordism between $(\mathbf{T}, \mathrm{id})$ and $(\mathbf{T}', f)$. For this, choose a smooth map $\zeta : I \to \mathbb{R}$ which is 0 near 0 and 1 near 1. Then consider the smooth map $\mathbf{T} \times \mathbb{R} \times I \to \mathbb{R}$ mapping $(x,t,s) \to \rho(x) - (\zeta(s)\mu(t) + (1-\zeta(s))t)$. This map has again 0 as regular value and the preimage of 0 is the required bordism $\mathbf{Q}$. By construction and Proposition 4.2 $\mathbf{Q}$ is a regular c-stratifold. The projection from $\mathbf{Q}$ to $\mathbf{T}$ is a map $r : \mathbf{Q} \to \mathbf{T}$, whose restriction to $\mathbf{T}$ is the identity on $\mathbf{T}$ and whose restriction to $\mathbf{T}'$ is f. Thus $(\mathbf{Q}, r)$ is a bordism between $(\mathbf{T}, \mathrm{id})$ and $(\mathbf{T}', f)$.
q.e.d.

Now we apply this lemma to the proof of Proposition 5.1 and the detailed proof of Theorem 5.2, the Mayer-Vietoris sequence.

Proofs of Proposition 5.1 and Theorem 5.2: We begin with the proof of Proposition 5.1. For $[\mathbf{S}, g] \in \mathbf{H}_m(X)$ we consider as before Proposition 5.1 the closed subsets $A := g^{-1}(X - V)$ and $B := g^{-1}(X - U)$. By partition of unity we construct a smooth function $\rho : \mathbf{S} \to \mathbb{R}$ and choose a regular value s such that $\rho^{-1}(s) \subset \mathbf{S} - (A \cup B)$ and $A \subset \rho^{-1}(s, \infty)$ and $B \subset \rho^{-1}(-\infty, s)$. After composition with an appropriate translation we can assume $s = 0$. Since $\mathbf{S}$ is compact, by Proposition 4.3 the regular values of ρ are an open set in $\mathbb{R}$.

Thus we can apply Lemma B.1 and we consider $\mathbf{S}'$, f and ρ'. Then $(\mathbf{S}, g)$ is bordant to $(\mathbf{S}', gf)$ (since $(\mathbf{S}', f)$ is bordant to $(\mathbf{S}, \mathrm{id})$) and 0 is a regular value of ρ'. By construction $\rho^{-1}(0) \times (-\epsilon, \epsilon)$ is contained in $\mathbf{S}'$ as open neighbourhood of $\mathbf{P} := (\rho')^{-1}(0) = \rho^{-1}(0)$, with other words we have a bicollar of $\mathbf{P}$. Furthermore by construction gf is on $\mathbf{P} = \rho^{-1}(0)$ equal to g, in particular $gf(\mathbf{P})$ is contained in $U \cap V$. In Proposition 5.1 we had defined

$d([\mathbf{S}, g])$ as $[\rho^{-1}(0), g|_{\rho^{-1}(0)}]$ and the considerations so far implied that this definition is the same if we pass from $(\mathbf{S}, g)$ to the bordant pair $(\mathbf{S}', gf)$ and define $d([\mathbf{S}', gf])$ as $[\mathbf{P}, gf|_{\mathbf{P}}]$, which situation has the advantage that $\mathbf{P}$ has a bicollar.

To show that d is well defined it is enough to show that if $(\mathbf{S}', gf)$ is the boundary of $(\mathbf{T}, F)$, then $[\mathbf{P}, g|_{\mathbf{P}}]$ is zero in $\mathbf{H}_{k-1}(U \cap V)$. Here $\mathbf{T}$ is a c-stratifold with boundary $\mathbf{S}'$. In particular we can take as $\mathbf{T}$ the cylinder over $\mathbf{S}$ and see that d does not depend on the choice of the separating function or the regular value. We choose a representative of the germ of collars $\mathbf{c}$ of $\mathbf{T}$. Define $A_{\mathbf{T}} := F^{-1}(A)$ and $B_{\mathbf{T}} := F^{-1}(B)$ and construct a smooth function $\eta : \mathbf{T} \to \mathbb{R}$ with the following properties:
1.) There is a $\mu > 0$ such that the restriction of η to $\mathbf{P} \times (-\mu, \mu)$ is the projection to $(-\mu, \mu)$,
2.) $\eta(\mathbf{c}(x, t)) = \eta(x)$,
3.) there is an $\delta > 0$ such that $F(\eta^{-1}(-\delta, \delta)) \subset U \cap V$.
The construction of such a map η is easy by partition of unity since $\mathbf{P}$ has a bicollar in $\mathbf{S}'$.

By Sard's theorem there is a t with $|t| < \min\{\delta, \mu\}$ which is a regular value of η. Since the restriction of η to $\mathbf{P} \times (-\mu, \mu)$ is the projection to $(-\mu, \mu)$ we conclude that t is also a regular value of $\eta|_{\mathbf{S}'}$. By condition 2.) we guarantee that $\mathbf{Q} := \eta^{-1}(t)$ is a c-stratifold with boundary $\mathbf{P} \times \{t\}$. By condition 3.) we know that $F(\mathbf{Q}) \subset U \cap V$ and so we see that $[\mathbf{P} \times \{t\}, F|_{\mathbf{P} \times \{t\}}]$ is zero in $\mathbf{H}_{k-1}(U \cap V)$. On the other hand obviously $[\mathbf{P} \times \{t\}, F|_{\mathbf{P} \times \{t\}}]$ is bordant to $[\mathbf{P}, g|_{\mathbf{P}}]$. This finishes the proof of Proposition 5.1.

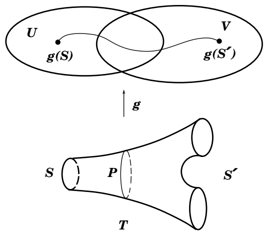
Now we pass to the proof of Theorem 5.2. We first show that d commutes with induced maps. The reason is the following. Let X' be a space with decomposition $X' = U' \cup V'$ and $h : X \to X'$ a continuous map respecting the decomposition. Then if we consider $(\mathbf{S}, hf)$ instead of $(\mathbf{S}, f)$ one can take the same separating function ρ in the definition of d and so $d'([\mathbf{S}, hf]) = [\rho^{-1}(s), hf|_{\rho^{-1}(s)}] = h_*([\rho^{-1}(s), f|_{\rho^{-1}(s)}]) = h_*(d([\mathbf{S}, f]))$.

Now we begin with the proof of the exactness by looking at

$$\mathbf{H}_n(U \cap V; \mathbb{Z}/2) \to \mathbf{H}_n(U; \mathbb{Z}/2) \oplus \mathbf{H}_n(V; \mathbb{Z}/2) \to \mathbf{H}_n(U \cup V; \mathbb{Z}/2)$$

Since $j_U i_U = i : U \cap V \to U \cup V$, the inclusion map, and also $j_V i_V = i$, the difference of the composition of the two maps is zero. To show the other inclusion, consider $[\mathbf{S}, f] \in \mathbf{H}_n(U; \mathbb{Z}/2)$ and $[\mathbf{S}', f'] \in H(V; \mathbb{Z}/2)$ with $(j_U)_*([\mathbf{S}, f]) = (j_V)_*([\mathbf{S}', f'])$. Let $(\mathbf{T}, g)$ be a bordism between $[\mathbf{S}, f]$ and $[\mathbf{S}', f']$, where $g : \mathbf{T} \to U \cup V$. Similarly as in the proof that d is well defined, we consider the closed disjoint subsets $A_{\mathbf{T}} := \mathbf{S} \cup g^{-1}(X - V)$ and $B_{\mathbf{T}} := \mathbf{S}' \cup g^{-1}(X - U)$. By partition of unity we construct a smooth function $\rho : \mathbf{T} \to \mathbb{R}$ with $\rho(A) = -1$ and $\rho(B) = 1$ and choose a regular value s such that $\rho^{-1}(s) \subset \mathring{\mathbf{T}} - (A_{\mathbf{T}} \cup B_{\mathbf{T}})$. After composition with an appropriate translation we can assume $s = 0$. Since $\mathbf{T}$ is compact, by Proposition 4.3 the regular values of ρ are an open set in $\mathbb{R}$. Applying Lemma B.1 we can assume after replacing $\mathbf{T}$ by $\mathbf{T}'$ that $\rho^{-1}(s)$ has a bicollar φ. Then $[\rho^{-1}(s), g|_{\rho^{-1}(s)}] \in \mathbf{H}_n(U \cap V)$ and— as explained in §3—$(\rho^{-1}[s, \infty), g|_{\rho^{-1}[s,\infty)})$

is a bordism between $(\mathbf{S}, f)$ and $(\rho^{-1}(s), g|_{\rho^{-1}(s)})$ in U.



Similarly $(\rho^{-1}(-\infty, s]), g|_{\rho^{-1}(-\infty,s])})$ is a bordism between $(\mathbf{S}', f')$ and $(\rho^{-1}(s), g|_{\rho^{-1}(s)})$ in V. Thus $((i_U)_*([\rho^{-1}(s), g|_{\rho^{-1}(s)}]), (i_V)_*([\rho^{-1}(s), g|_{\rho^{-1}(s)}])) = ([\mathbf{S}, f], [\mathbf{S}', f'])$.

Next we consider the exactness of

$$\mathbf{H}_n(U \cup V; \mathbb{Z}/2) \xrightarrow{d} \mathbf{H}_{n-1}(U \cap V; \mathbb{Z}/2) \to \mathbf{H}_{n-1}(U; \mathbb{Z}/2) \oplus \mathbf{H}_{n-1}(V; \mathbb{Z}/2).$$

By construction of the boundary operator the composition of the two maps is zero. Namely $(\rho^{-1}[s, \infty), f|_{\rho^{-1}[s,\infty)})$ is a zero-bordism of $d([\mathbf{S}, f])$ in U and $(\rho^{-1}(-\infty, s], f|_{\rho^{-1}(-\infty,s]})$ is a zero-bordism of $d([\mathbf{S}, f])$ in V. Here we again apply Lemma B.1 and assume that $\rho^{-1}(s)$ has a bicollar.

To show the other inclusion, start with $[\mathbf{P}, r] \in \mathbf{H}_{n-1}(U \cap V; \mathbb{Z}/2)$ and suppose $(i_U)_*([\mathbf{P}, r]) = 0$ and $(i_V)_*([\mathbf{P}, r]) = 0$. Let $(\mathbf{T}_1, g_1)$ be a zero bordism of $(i_U)_*([\mathbf{P}, r])$ and $(\mathbf{T}_2, g_2)$ be a zero bordism of $(i_V)_*([\mathbf{P}, r])$. Then we consider $\mathbf{T}_1 \cup_{\mathbf{P}} \mathbf{T}_2$. Since $g_1|_{\mathbf{P}} = g_2|_{\mathbf{P}} = r$, we can - as in the proof of the transitivity of the bordism relation - extend r to $\mathbf{T}_1 \cup_{\mathbf{P}} \mathbf{T}_2$ using g_1 and g_2 and denote this map by $g_1 \cup g_2$. Thus $[\mathbf{T}_1 \cup_{\mathbf{P}} \mathbf{T}_2, g_1 \cup g_2] \in \mathbf{H}_n(U \cup V; \mathbb{Z}/2)$. By construction of the boundary operator we have $d([\mathbf{T}_1 \cup_{\mathbf{P}} \mathbf{T}_2, g_1 \cup g_2]) = [\mathbf{P}, r]$. Here one constructs using the bicollar a separating function which near $\mathbf{P}$ is the projection from $\mathbf{P} \times (-\epsilon, \epsilon)$ to the second factor.

Finally, we prove exactness of

$$\mathbf{H}_n(U; \mathbb{Z}/2) \oplus \mathbf{H}_n(V; \mathbb{Z}/2) \to \mathbf{H}_n(U \cup V; \mathbb{Z}/2) \xrightarrow{d} \mathbf{H}_{n-1}(U \cap V; \mathbb{Z}/2).$$

The composition of the two maps is obviously zero. Now, consider $[\mathbf{S}, f] \in \mathbf{H}_n(U \cup V; \mathbb{Z}/2)$ with $d([\mathbf{S}, f]) = 0$. Consider ρ, s and $\mathbf{P}$ as in the definition of the boundary map d and assume by Lemma B.1 that $\rho^{-1}(s)$ has a bicollar. We denote $\mathbf{S}_+ := \rho^{-1}[s, \infty)$ and $\mathbf{S}_- := \rho^{-1}(-\infty, s]$. Then $\mathbf{S} = \mathbf{S}_+ \cup_{\mathbf{P}} \mathbf{S}_-$. If $d([\mathbf{S}, f]) = [\mathbf{P}, f|_{\mathbf{P}}] = 0$ in $\mathbf{H}_{n-1}(U \cap V; \mathbb{Z}/2)$ there is $\mathbf{Z}$ with $\partial \mathbf{Z} = \mathbf{P}$ and an extension of $f|_{\mathbf{P}}$ to $r : \mathbf{Z} \to U \cap V$. We glue to obtain $\mathbf{S}_+ \cup_{\mathbf{P}} \mathbf{Z}$ and $\mathbf{S}_- \cup_{\mathbf{P}} \mathbf{Z}$. Since $f|_{\mathbf{P}} = r|_{\mathbf{P}}$ the maps $f|_{\mathbf{S}_+}$ and r give a continuous map $f_+ : \mathbf{S}_+ \cup_{\mathbf{P}} \mathbf{Z} \to U$ and similarly we obtain $f_- : \mathbf{S}_- \cup_{\mathbf{P}} \mathbf{Z} \to V$. We are finished if $(j_U)_*([\mathbf{S}_+ \cup_{\mathbf{P}} \mathbf{Z}, f_+]) - (j_V)_*([\mathbf{S}_- \cup_{\mathbf{P}} \mathbf{Z}, f_-]) = [\mathbf{S}, f]$. For this we have to find a bordism $(\mathbf{T}, g)$ such that $\partial \mathbf{T} = \mathbf{S}_+ \cup \mathbf{Z} + \mathbf{S}_- \cup \mathbf{Z} + \mathbf{S}$ (recall that $-[\mathbf{P}, r] = [\mathbf{P}, r]$ for all elements in $\mathbf{H}_n(Y; \mathbb{Z}/2)$) and g extends the given three maps on the pieces.

This bordism is given as $\mathbf{T} := ((\mathbf{S}_+ \cup_{\mathbf{P}} \mathbf{Z}) \times [0, 1]) \cup_{\mathbf{Z}} ((\mathbf{S}_- \cup_{\mathbf{P}} \mathbf{Z}) \times [1, 2])$ with $\partial \mathbf{T} = (\mathbf{S}_+ \cup_{\mathbf{P}} \mathbf{Z}) \times \{0\} + (\mathbf{S}_- \cup_{\mathbf{P}} \mathbf{Z}) \times \{2\} + \mathbf{S}_+ \cup_{\mathbf{P}} \mathbf{S}_-$. Here we apply Lemma A.1

to smooth the corners or cusps. This finishes the proof of Theorem 5.2.
q.e.d.

Finally we discuss the modification needed to prove the Mayer-Vietoris sequence in cohomology. Everything works with appropriate obvious modifications as for homology except where we argue that the regular values of the separating map ρ are an open set. This used that the stratifold on which ρ is defined is compact, which is not the case for regular stratifolds representing cohomology classes. We are free in the choice of the separating function and we show now that we can always find a separating function ρ and a regular value, which is an inner point of the set of regular values.

Let $g : \mathbf{S} \to M$ be a proper smooth map and C and D be disjoint closed subsets of M. We choose a smooth map $\rho : M \to \mathbb{R}$ which on C is 1 and on D is -1. We select a regular value s of ρg. The set of singular points of ρg is closed by Proposition 4.3, and since a proper map on a locally compact space is closed ([**Sch**], p. 72), the image of the singular points of ρg under g is a closed subset F of M.

Now we consider a bicollar $\varphi : U \to M - F$ of $\rho^{-1}(s)$, where $U = \{(x,t) \in \rho^{-1}(s) \times \mathbb{R} \,|\, |t| < \delta(x)$ for some continuous map $\delta : \rho^{-1}(s) \to \mathbb{R}_{>0}$. We can choose φ in such a way, that $\rho\varphi(x,t) = t$. Now we "expand" this bicollar by choosing a diffeomorphism from U to $\rho^{-1}(s) \times (-1/2, 1/2)$ mapping (x,t) to $(x, \eta(x,t))$, where $\eta(x,..)$ is a diffeomorphism for each $x \in \rho^{-1}(s)$. Using this it is easy to find a new separating function ρ', such that $\rho'\varphi(x,t) = t$ and $\rho'^{-1}(-1/2, 1/2) = U$. By construction the interval $(-1/2, 1/2)$ consists only of regular values of $\rho' f$.

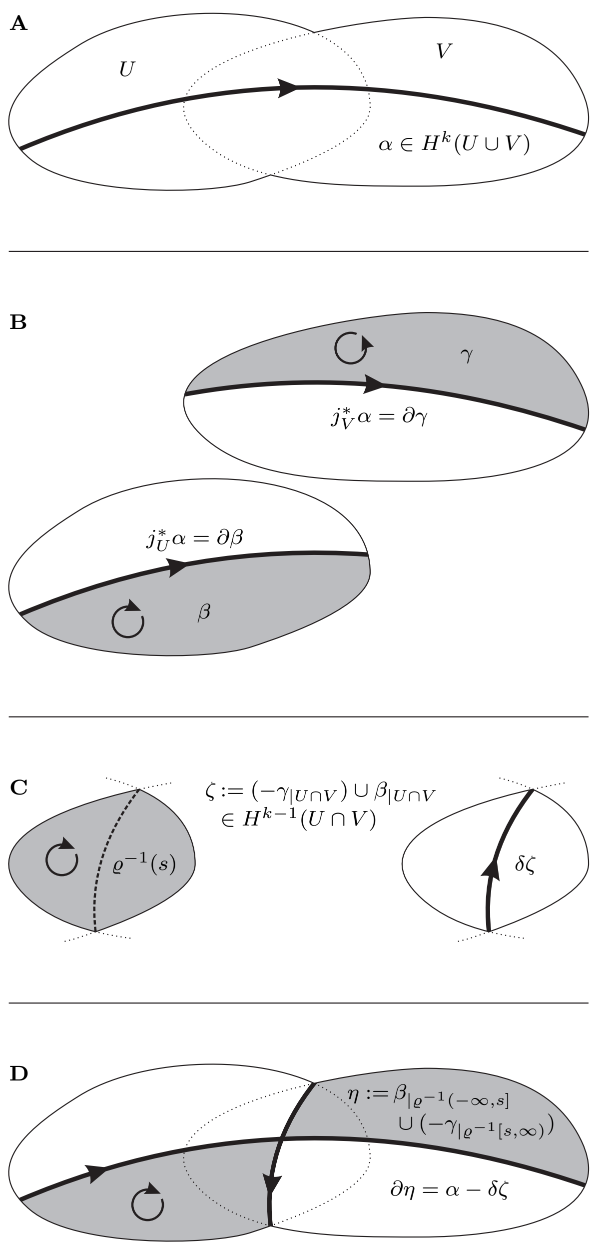
We apply this in the proof of the Mayer-Vietoris sequence for cohomology as follows. Let U and V be open subsets of $M = U \cup V$. We consider the closed subsets $C := M - U$ and $D := M - V$. Then we construct ρ' as above and note that $\rho' g$ is a separating function of $A := g^{-1}(C)$ and $B := g^{-1}(D)$, and s is a regular value which is an inner point in the set of regular values. With this the definition of the boundary operator works as explained in chapter 12.

Now we explain why the Mayer-Vietoris sequence is exact. We do this separately at the three places and only explain the non-obvious steps. We explain the arguments with pictures.

We begin with the exactness of

$$H^{k-1}(U \cap V) \to H^k(U \cup V) \to H^k(U) \oplus H^k(V)$$

Let $\alpha \in H^k(U \cup V)$ (picture A) such that it maps to zero. That is there are stratifolds with boundary and proper maps extending the map representing α after restricting to U and V respectively. We abbreviate these extension by β and γ and write $\partial\beta = j_U^*(\alpha)$ and $\partial\gamma = j_V^*(\alpha)$ (picture B). Now we restrict β and γ to the intersection $U \cap V$ and glue them (respecting the orientation) along the common boundary to obtain $\zeta := (-\gamma|_{U\cap V}) \cup \beta|_{U\cap V} \in H^{k-1}(U \cap V)$ (picture C). Using a separating function ρ we determine the image of ζ under the boundary operator: $\delta(\zeta)$. Finally we have to show that $\delta(\zeta)$ is bordant to α. For this we consider $\eta := \beta|_{\rho^{-1}(-\infty,s]} \cup (-\gamma|\rho^{-1}[s,\infty))$, which gives such a bordism (picture D).

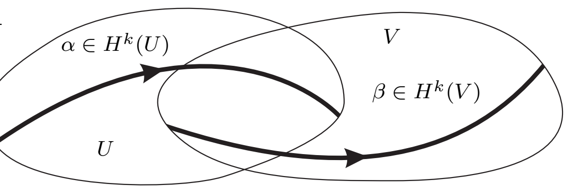

A
U
V
$\alpha \in H^k(U \cup V)$
B
γ
$j_V^* \alpha = \partial \gamma$
$j_U^* \alpha = \partial \beta$
β
C
$\zeta := (-\gamma_{|U\cap V}) \cup \beta_{|U\cap V}$
$\in H^{k-1}(U \cap V)$
$\varrho^{-1}(s)$
$\delta\zeta$
D
$\eta := \beta_{|\varrho^{-1}(-\infty,s]}$
$\cup (-\gamma_{|\varrho^{-1}[s,\infty)})$
$\partial\eta = \alpha - \delta\zeta$

Now we consider the exactness of

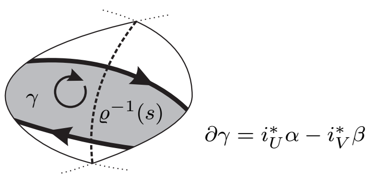
$$H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V)$$

For this we consider $\alpha \in H^k(U)$ and $\beta \in H^k(V)$ (picture A) such that (α, β) maps to zero in $H^k(U \cap V)$. This means there is γ, a stratifold with boundary together with a proper map to $U \cap V$, such that $\partial\gamma = i_U^*(\alpha) - i_V^*(\beta)$ (picture B). Next we choose a separating function ρ as indicated in picture B. Using ρ we consider $\zeta := \alpha_{|\varrho^{-1}(-\infty,s]} \cup (-\delta\gamma) \cup \beta_{|\varrho^{-1}[s,\infty)}$ $\in H^k(U \cup V)$ (picture C). Finally we have to construct a bordism between $j_U^*(\zeta)$ and α resp. $j_V^*(\zeta)$ and β. This is given by the equations $j_U^*\zeta + \partial(\gamma_{|\varrho^{-1}[s,\infty)}) = \alpha$ and $j_V^*\zeta - \partial(\gamma_{|\varrho^{-1}(-\infty,s]}) = \beta$ (picture D).

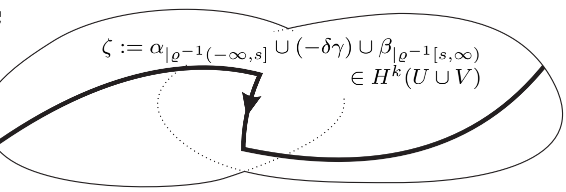
A



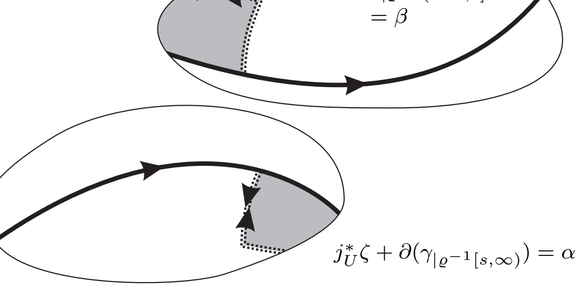
B



C



D

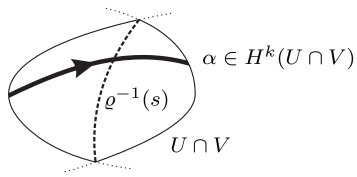


Finally we consider the exactness of

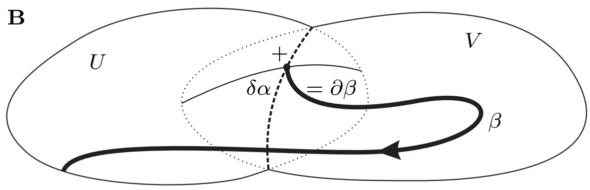
$$H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(U \cup V)$$

Let α be in $H^k(U \cap V)$ (and ρ a separating function) such that $\delta\alpha = 0$ (picture A). This means that there is a stratifold β with boundary $\delta(\alpha)$ and a proper map extending the given map (picture B). From this we construct the classes $\zeta_1 := \alpha_{|\varrho^{-1}(-\infty,s]} \cup \beta_{|U} \in H^k(U)$ and $\zeta_2 := (-\alpha_{|\varrho^{-1}[s,\infty)}) \cup (-\beta_{|V}) \in H^k(V)$ (picture C). Finally we note that $i_U^*(\zeta_1) - i_V^*(\zeta_2) = \alpha$ (picture D).

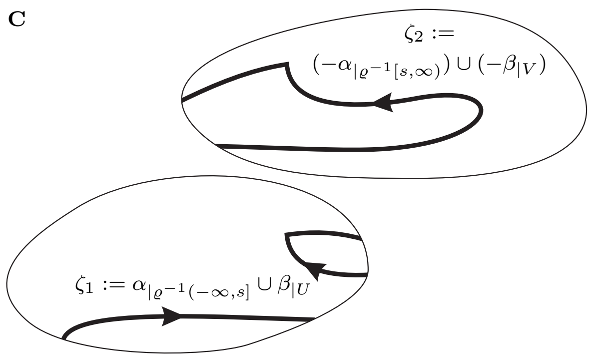
A



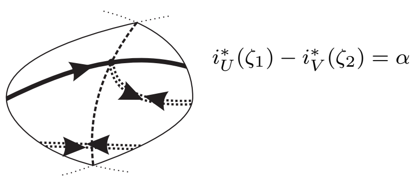
B



C



D



APPENDIX C

The tensor product

We want to describe an important construction in linear algebra, the tensor product. This assigns to two R-modules another R-module. The slogan is: Bilinearity is transferred to linearity. More precisely, let R be a commutative ring with unit, for example $\mathbb{Z}$ or a field. Consider a bilinear map $f : V \times W \to P$ between R-modules. Then we will construct another R-module denoted $V \otimes_R W$ together with a canonical map $V \times W \to V \otimes_R W$ such that f induces a map from $V \otimes_R W \to P$ commuting with the canonical map and f.

Since we are particularly interested in the case of $R = \mathbb{Z}$ we note that a $\mathbb{Z}$-module is the same as an abelian group. If A is an abelian group we make it a $\mathbb{Z}$-module by defining (for $n \geq 0$) $n \cdot a := a + ... + a$, where the sum is taken over n summands, and for $n < 0$ we define $n \cdot a := -(-n \cdot a)$.

We begin with the definition of $V \otimes_R W$. This is an R-module generated by all pairs (v, w) with $v \in V$ and $w \in W$. One denotes the corresponding generators by $v \otimes w$ and calls them **pure tensors**. The fact that these will be the generators means that we will get a surjective map

$$\bigoplus_{(v,w) \in V \times W} (v, w) \cdot R \longrightarrow V \otimes_R W$$

mapping (v, w) to $v \otimes w$. In order to finish the definition of $V \otimes_R W$ we only need to define the kernel K of this map. We describe the generators of the kernel, these are :

$$\begin{array}{l} (rv, w) - (v, rw) \text{ for all } v, w \in V, r \in R \text{ and} \\ (rv, w) - (v, w)r \text{ for all } v, w \in V, r \in R \text{ and} \\ (v, w) + (v', w) - (v + v', w) \text{ respectively} \\ (v, w) + (v, w') - (v, w + w') \text{ for all } v, v', w, w' \in V \end{array}$$

Let K be the submodule generated by these elements. Then we define the **tensor product**

$$V \otimes_R W := \left(\bigoplus_{(v,w) \in V \times W} (v, w) \cdot R \right) \Big/ K$$

Remark: *The following rules are translations of the relations and very useful for working with tensor products:*

$$r \cdot (v \otimes w) = (r \cdot v) \otimes w = v \otimes (r \cdot w)$$

$$v \otimes w + v' \otimes w = (v + v') \otimes w$$

$$v \otimes w + v \otimes w' = v \otimes (w + w')$$

These rules imply that the following **canonical map** is well defined and bilinear:

$$\begin{array}{ccc} V \times W & \longrightarrow & V \otimes_R W \\ (v,w) & \longmapsto & v \otimes w \end{array}$$

Let $f : V \times W \to P$ be bilinear. Then f induces a linear map

$$\begin{array}{cccc} f: & V \otimes_R W & \longrightarrow & P \\ & v \otimes w & \longmapsto & f(v,w) \end{array}$$

This map is well defined since $(rv) \otimes w - v \otimes (rw) \mapsto f(rv,w) - f(v,rw) = r\,f(v,w) - r\,f(v,w) = 0$ and $v \otimes w + v' \otimes w - (v+v') \otimes w \mapsto f(v,w) + f(v',w) - f(v+v,w) = 0$, respectively $v \otimes w + v \otimes w' - v \otimes (w+w') \mapsto 0$.

In turn, if we have a linear map from $V \otimes_R W$ to P, the composition of the canonical map with this map is a bilinear map from $V \times W$ to P. Thus as indicated above we have seen the fundamental fact:

The linear maps from $V \otimes_R W$ to P correspond isomorphically to the bilinear maps from $V \times W$ to P.

Example: Let $V = W = R$. Then the bilinear map

$$\begin{array}{ccc} R \times R & \to & R \\ (x,y) & \mapsto & x \cdot y \end{array}$$

induces

$$\begin{array}{ccc} R \otimes_R R & \to & R \\ x \otimes y & \mapsto & x \cdot y \end{array}$$

Obviously this map is surjective since $1 \otimes 1 \mapsto 1$. It is also injective, since

$$\begin{array}{c} R \to R \otimes_R R \\ x \mapsto x \otimes 1 \end{array}$$

is the inverse map.

What is $(V \oplus V') \otimes_R W$? The reader should convince himself that the following maps are bilinear

$$\begin{array}{ccc} (V \oplus V') \times W & \longrightarrow & (V \otimes_R W) \oplus (V' \otimes_R W) \\ ((v,v'),w) & \longmapsto & (v \otimes w, v' \otimes w) \end{array}$$

and

$$\begin{array}{lcl} V \times W \longrightarrow (V \oplus V') \otimes_R W & \text{and} & V' \times W \longrightarrow (V \oplus V') \otimes_R W \\ (v,w) \longmapsto (v,0) \otimes w & & (v',w) \longmapsto (0,v') \otimes w \end{array}$$

These maps induce homomorphisms

$$\begin{array}{ccl} (V \oplus V') \otimes_R W & \longrightarrow & V \otimes_R W \oplus V' \otimes_R W \;\text{ and} \\ (v, v') \otimes w & \longmapsto & (v \otimes w, v' \otimes w) \\ V \otimes_R W \oplus V' \otimes_R W & \longrightarrow & (V \oplus V') \otimes_R W \\ ((v \otimes w_1), (v' \otimes w_2)) & \longmapsto & (v, 0) \otimes w_1 + (0, v') \otimes w_2 \end{array}$$

and these are inverse to each other. Thus we have shown:

PROPOSITION C.1. $(V \oplus V') \otimes_R W \xrightarrow{\cong} (V \otimes_R W) \oplus (V' \otimes_R W)$.

It follows:

$$R^n \otimes_R R^m = (R^{n-1} \oplus R) \otimes_R R^m \cong R^{n-1} \otimes_R R^m \oplus R \otimes_R R^m \cong R^{n-1} \otimes_R R^m \oplus (R \otimes_R [R \oplus \cdots \oplus R]) = (R^{n-1} \otimes_R R^m) \oplus R^m.$$

Thus dim $R^n \otimes_R R^m = n \cdot m$ and

$$\begin{array}{c} R^n \otimes_R R^m \cong R^{n \cdot m} \cong M(n, m) \\ e_i \otimes e_j \longmapsto e_{i,j} \end{array}$$

where $e_{i,j}$ denotes the $n \times m$-matrix whose coefficients are 0 except at the place (i, j), where it is 1.

Example:

$$\begin{array}{ccc} R \otimes_R M & \cong & M \\ r \otimes x & \mapsto & r \cdot x \end{array}$$

If $R = \mathbb{Z}$ we note that a $\mathbb{Z}$-module is the same as an abelian group. For abelian groups A and B we write $A \otimes B$ instead of $A \otimes_{\mathbb{Z}} B$.

We want to determine $\mathbb{Z}/n \otimes \mathbb{Z}/m$. We prepare this by some general considerations. Let $f : A \to B$ and $g : C \to D$ be homomorphisms of R-modules. They induce a homomorphism

$$\begin{array}{cccc} f \otimes g : & A \otimes_R C & \to & B \otimes_R D \\ & a \otimes c & \mapsto & f(a) \otimes g(c). \end{array}$$

called the **tensor product** of f and g.

If we have an exact sequence of R-modules

$$... \to A_{k+1} \to A_k \to A_{k-1} \to ...$$

and a fixed R-module P we can tensorize all A_k with P and tensorize all maps in the exact sequence with Id on P, and obtain a new sequence of maps

$$... \to A_{k+1} \otimes_R P \to A_k \otimes_R P \to A_{k-1} \otimes_R P \to ...$$

called the induced sequence and ask if this is again exact. This is in general not the case and this is one of the starting points of homological algebra which systematically investigates the failure of exactness. Here we only study a very special case.

PROPOSITION C.2. *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. Then the induced sequence

$$A \otimes_R P \to B \otimes_R P \to C \otimes_R P \to 0$$

is again exact. In general the map $A \otimes_R P \to B \otimes_R P$ is not injective.

Proof: Denote the map from $A \to B$ by f and the map from B to C by g. Obviously $(g \otimes Id)(f \otimes Id)$ is zero. Thus $g \otimes Id$ induces a homomorphism $B \otimes_R P/_{(f\otimes Id)(A\otimes_R P)} \to C \otimes_R P$. We have to show that this is an isomorphism. We give an inverse by defining a bilinear map $C \times P$ to $B \otimes_R P/_{(f\otimes Id)(A\otimes_R P)}$ by assigning to (c, p) an element $[b \otimes p]$, where $g(b) = c$. The exactness of the original sequence shows that this induces a well defined homomorphism from $C \otimes_R P$ to $B \otimes_R P/_{(f\otimes Id)(A\otimes_R P)}$ and that it is an inverse of $B \otimes_R P/_{(f\otimes Id)(A\otimes_R P)} \to C \otimes_R P$.

The last statement follows from the next example.
q.e.d.

As an application we compute $\mathbb{Z}/n \otimes \mathbb{Z}/m$. For this consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$$

where the first map is multiplication by n, and tensorize it with $\mathbb{Z}/m$ to obtain an exact sequence

$$\mathbb{Z} \otimes \mathbb{Z}/m \to \mathbb{Z} \otimes \mathbb{Z}/m \to \mathbb{Z}/n \otimes \mathbb{Z}/m \to 0$$

where the first map is multiplication by n. This translates by the isomorphism above to

$$\mathbb{Z}/m \to \mathbb{Z}/m \to \mathbb{Z}/n \otimes \mathbb{Z}/m \to 0$$

where again the first map is multiplication by n (if n and m are not coprime, the left map is not injective finishing the proof of Proposition C.2). Thus $\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/_{gcd(m,n)}$ and we have shown:

Corollary C.3.

$$\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/gcd(n, m)$$

If A is a finitely generated abelian group it is isomorphic to $F \oplus T$, where $F \cong \mathbb{Z}^k$ is a free abelian group, and T is the torsion subgroup. The number k is called the **rank** of A. A finitely generated torsion group is isomorphic to a finite sum of cyclic groups $\mathbb{Z}/n_i$ for some $n_i > 0$. Thus Propositions C.1 and C.3 allow to compute the tensor products of arbitrary finitely generated abelian groups.

Now we study the tensor product of an abelian group with the rationals $\mathbb{Q}$. Let A be an abelian group and K be a field. We first introduce the structure of a K-vector space on $A \otimes K$ (where we consider K as abelian group to construct the tensor product) by: $\alpha \cdot (a \otimes \beta) := a \otimes \alpha \cdot \beta$ for a in A and α and β in K. Decompose $A = F \oplus T$ as above. The tensor product $T \otimes \mathbb{Q}$ is zero, since $a \otimes q = n \cdot a \otimes q/n = 0$, if $n \cdot a = 0$. The tensor product $F \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}^k$. Thus $A \otimes \mathbb{Q}$ is - considered as $\mathbb{Q}$-vector space - a vector space of dimension rank A.

Finally we consider an exact sequence of abelian groups

$$... \to A_{k+1} \to A_k \to A_{k-1} \to ...$$

and the tensor product with an abelian group P.

Proposition C.4. *Let*

$$... \to A_{k+1} \to A_k \to A_{k-1} \to ...$$

be an exact sequence of abelian groups and P either be $\mathbb{Q}$ or a finitely generated free abelian group, then the induced sequence

$$... \to A_{k+1} \otimes P \to A_k \otimes P \to A_{k-1} \otimes P \to ...$$

is exact.

Proof: The case of a free finitely generated abelian group P can by Lemma C.1 be reduced to the case $P = \mathbb{Z}$, which is clear.

If $P = \mathbb{Q}$ we return to Lemma C.2 and note that we are finished if we can show the injectivity of $f \otimes Id : A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$. Consider an element of $A \otimes \mathbb{Q}$, a finite sum $\sum_i a_i \otimes q_i$, and suppose $\sum_i f(a_i) \otimes q_i = 0$. Let m be the product of the denominators of the q_i's and consider $m(\sum_i a_i \otimes q_i) = \sum_i a_i \otimes m \cdot q_i$. The latter is an element of $A \otimes \mathbb{Z}$ mapping to zero in $B \otimes \mathbb{Q}$. Thus its image in $B \otimes \mathbb{Z}$ is a torsion element (the kernel of $B \cong B \otimes \mathbb{Z} \to B \otimes \mathbb{Q}$ is the torsion subgroup of B (why?)). Since $f \otimes Id : A \otimes \mathbb{Z} \to B \otimes \mathbb{Z}$ is injective, this implies that $\sum_i a_i \otimes m \cdot q_i$ is a torsion element mapping to zero in $A \otimes \mathbb{Q}$. Since this is a $\mathbb{Q}$-vector space $m(\sum_i a_i \otimes q_i) = 0$ implies $\sum_i a_i \otimes q_i = 0$.
q.e.d.

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