Guide to Integration

Mathematics 101

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Table of integrals

Recognise these from a table of derivatives.

The very basics

1. $$\int 1 \, dx = x + c$$
2. $$\int \frac{1}{x} \, dx = \log|x| + c$$
3. $$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$
4. $$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

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The very basics

1. $\displaystyle\int 1\,\mathrm{d}x = x + c$
2. $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \log|x| + c$ — don't forget the $|.|$.
3. $\displaystyle\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$ — if $n \neq -1$.
4. $\displaystyle\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax} + c$

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Trigonometry

1. $$\int \sin(ax)\,\mathrm{d}x = \frac{-1}{a}\cos(ax) + c$$
2. $$\int \cos(ax)\,\mathrm{d}x = \frac{1}{a}\sin(ax) + c$$
3. $$\int \sec^2(ax)\,\mathrm{d}x = \frac{1}{a}\tan(ax) + c$$
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Inverse trig

1. $$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}(x/a) + c$$
2. $$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}(x/a) + c.$$

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Inverse trig

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Substitution rule

From the chain rule we get

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u \qquad u = g(x)$$

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$$= f(u) + c = f(g(x)) + c$$

Look for a function and its derivative in the integrand.

Substitution example

Example $\int \frac{\sin(3\log x)}{x}\,dx$

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WARNING — you must turn all the x's into the new variable.

Substitution with definite integrals

You have 2 choices of what to do with the integration terminals.

Transform terminals

We make $u = \log x$ — so change the terminals too.

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$$\int_1^2 \frac{\sin(3\log x)}{x}\,\mathrm{d}x = \int_{\log 1}^{\log 2} \sin 3u\,\mathrm{d}u$$

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$$\int_1^2 \frac{\sin(3\log x)}{x}\,dx = \int_{\log 1}^{\log 2} \sin 3u\,du$$

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$$\begin{aligned}
\int_1^2 \frac{\sin(3\log x)}{x}\,dx &= \int_{\log 1}^{\log 2} \sin 3u\,du \\
&= \left[\frac{-1}{3}\cos(3u)\right]_{\log 1 = 0}^{\log 2} \\
&= \frac{-1}{3}\cos(3\log 2) + \frac{1}{3}\cos(0) \\
&= \frac{-1}{3}\cos(3\log 2) + \frac{1}{3}
\end{aligned}$$

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You have 2 choices of what to do with the integration terminals.

Keep terminals, remember to change everything back to x

$$\int_1^2 \frac{\sin(3\log x)}{x}\,dx = \int_{x=1}^{x=2} \sin 3u\,du$$

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Of course the answers are the same.

Trigonometric integrals

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Useful trig identities

$$
\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
\cos^2 x &= \frac{1}{2}(1 + \cos 2x) \\
\sin^2 x &= \frac{1}{2}(1 - \cos 2x)
\end{aligned}
$$

Trigonometric integrals

Example $\int \sin^a x \cos^b x \, dx$

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- If a and b are both even then use

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Trigonometric integrals

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- If a and b are both even then use

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

- If a or b is odd then use

$$\cos^2 x = 1 - \sin^2 x$$
$$\sin^2 x = 1 - \cos^2 x$$

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Trigonometric integrals

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$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

Now set $u = \sec x + \tan x$:

$$\frac{du}{dx} = \sec x \tan x + \sec^2 x = \sec x \left(\tan x + \sec x \right)$$

Trigonometric integrals

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Now set $u = \sec x + \tan x$:

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Hence we have

$$\int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{1}{u} \cdot \frac{du}{dx} \, dx$$

Trigonometric integrals

Example $\int \sec x \, \mathrm{d}x$

Now this is not at all obvious, but you should see it...

$$\int \sec x \, \mathrm{d}x = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x$$

Now set $u = \sec x + \tan x$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \tan x + \sec^2 x = \sec x \left(\tan x + \sec x \right)$$

Hence we have

$$\begin{aligned} \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x &= \int \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x \\ &= \int \frac{1}{u} \, \mathrm{d}u = \log |u| + c \end{aligned}$$

Trigonometric integrals

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Hence we have

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Integration by parts

From the product rule we get

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

- Frequently used when you have the product of 2 different types of functions.

Integration by parts

From the product rule we get

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

- Frequently used when you have the product of 2 different types of functions.
- You have to choose $f(x)$ and $g'(x)$ — there are 2 options.
- Usually one will work and the other will not.

Integration by parts

Example $\int xe^x \, dx$

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$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

$$\int xe^x\,\mathrm{d}x = xe^x - \int e^x \cdot 1\,\mathrm{d}x$$

$$= xe^x - e^x + c$$

Example $\int xe^x \, dx$

- What if we choose f and g' the other way around?

Integration by parts

Example $\int xe^x \, dx$

- What if we choose f and g' the other way around?
  $f = e^x$ and $g' = x$ — so $f' = e^x$ and $g = x^2/2$

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Example $\int xe^x \, dx$

- What if we choose f and g' the other way around?
  $f = e^x$ and $g' = x$ — so $f' = e^x$ and $g = x^2/2$

$$\int xe^x \, dx = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, dx$$

Integration by parts

Example $\int xe^x \,\mathrm{d}x$

- What if we choose f and g' the other way around?
  $f = e^x$ and $g' = x$ — so $f' = e^x$ and $g = x^2/2$

$$\int xe^x \,\mathrm{d}x = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x \,\mathrm{d}x$$

- This is not getting easier, so stop!

Integration by parts

Sometimes one of the parts is "1".

Example $\int \log x \, dx$

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- Choose $f = \log x$ and $g' = 1$

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- Choose $f = \log x$ and $g' = 1$ — so $f' = 1/x$ and $g = x$:

Integration by parts

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Example $\int \log x \, dx$

- Choose $f = \log x$ and $g' = 1$ — so $f' = 1/x$ and $g = x$:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Integration by parts

Sometimes one of the parts is "1".

Example $\int \log x \, \mathrm{d}x$

- Choose $f = \log x$ and $g' = 1$ — so $f' = 1/x$ and $g = x$:

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

$$\int \log x\,\mathrm{d}x = x\log x - \int x/x\,\mathrm{d}x$$

$$= x\log x - x + c$$

Trigonometric substitutions

Based on

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$

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Things to associate

If the integrand contains

$$\sqrt{a^2 - x^2} \longrightarrow \qquad \sin^2\theta = 1 - \cos^2\theta$$
$$a^2 + x^2 \longrightarrow \qquad 1 + \tan^2\theta = \sec^2\theta$$

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\,\mathrm{d}x$

- Contains $\sqrt{a^2 - x^2}$ so put $x = \sqrt{5}\sin\theta$.

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\,\mathrm{d}x$

- Contains $\sqrt{a^2 - x^2}$ so put $x = \sqrt{5}\sin\theta$.
- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{5}\cos\theta$ and

$$\int (5 - x^2)^{-3/2}\,\mathrm{d}x = \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1 - \sin^2\theta)^{3/2}}\,\mathrm{d}\theta$$

Compute $\int (5 - x^2)^{-3/2}\,\mathrm{d}x$

- Contains $\sqrt{a^2 - x^2}$ so put $x = \sqrt{5}\sin\theta$.
- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{5}\cos\theta$ and

$$\begin{aligned}\int (5 - x^2)^{-3/2}\,\mathrm{d}x &= \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1 - \sin^2\theta)^{3/2}}\,\mathrm{d}\theta \\ &= \int \frac{\cos\theta}{5(\cos^2\theta)^{3/2}}\,\mathrm{d}\theta\end{aligned}$$

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\, dx$

- Contains $\sqrt{a^2 - x^2}$ so put $x = \sqrt{5}\sin\theta$.
- Hence $\frac{dx}{d\theta} = \sqrt{5}\cos\theta$ and

$$\int (5 - x^2)^{-3/2}\, dx = \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1 - \sin^2\theta)^{3/2}}\, d\theta$$
$$= \int \frac{\cos\theta}{5(\cos^2\theta)^{3/2}}\, d\theta$$
$$= \int \frac{1}{5\cos^2}\, d\theta = \frac{1}{5}\int \sec^2\theta\, d\theta$$

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\,\mathrm{d}x$

- Contains $\sqrt{a^2 - x^2}$ so put $x = \sqrt{5}\sin\theta$.
- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{5}\cos\theta$ and

$$
\begin{aligned}
\int (5 - x^2)^{-3/2}\,\mathrm{d}x &= \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1 - \sin^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \int \frac{\cos\theta}{5(\cos^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \int \frac{1}{5\cos^2}\,\mathrm{d}\theta = \frac{1}{5}\int \sec^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{5}\tan\theta + c
\end{aligned}
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Compute $\int (5 - x^2)^{-3/2}\,\mathrm{d}x$

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\int (5 - x^2)^{-3/2}\,\mathrm{d}x &= \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1 - \sin^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \int \frac{\cos\theta}{5(\cos^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \int \frac{1}{5\cos^2}\,\mathrm{d}\theta = \frac{1}{5}\int \sec^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{5}\tan\theta + c
\end{aligned}$$

- We aren't done yet — we have to change back to the x variable.

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\,\mathrm{d}x$

- We substituted $x = \sqrt{5}\sin\theta$ and got $\frac{1}{5}\tan\theta + c$

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\, dx$

- We substituted $x = \sqrt{5}\sin\theta$ and got $\frac{1}{5}\tan\theta + c$
- We can express $\tan\theta$ in terms of $\sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}}$$

$$= \frac{x/\sqrt{5}}{\sqrt{1 - x^2/5}} = \frac{x}{\sqrt{5}\sqrt{1 - x^2/5}} = \frac{x}{\sqrt{5 - x^2}}$$

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\, dx$

- We substituted $x = \sqrt{5}\sin\theta$ and got $\frac{1}{5}\tan\theta + c$
- We can express $\tan\theta$ in terms of $\sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$
$$= \frac{x/\sqrt{5}}{\sqrt{1-x^2/5}} = \frac{x}{\sqrt{5}\sqrt{1-x^2/5}} = \frac{x}{\sqrt{5-x^2}}$$

- Hence the integral is

$$\int (5-x^2)^{-3/2}\, dx = \frac{x}{5\sqrt{5-x^2}} + c$$

Trigonometric substitutions

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Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- Contains $a^2 + x^2$, so sub $x = 2\tan\theta$.

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- Contains $a^2 + x^2$, so sub $x = 2\tan\theta$.
- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec^2\theta$ and

$$\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \int \frac{2\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,\mathrm{d}\theta$$

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

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$$\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \int \frac{2\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,\mathrm{d}\theta$$
$$= \int \frac{2\sec^2\theta}{2\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,\mathrm{d}\theta$$

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- Contains $a^2 + x^2$, so sub $x = 2\tan\theta$.
- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec^2\theta$ and

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x &= \int \frac{2\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,\mathrm{d}\theta \\ &= \int \frac{2\sec^2\theta}{2\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,\mathrm{d}\theta \\ &= \int \sec\theta\,\mathrm{d}\theta\end{aligned}$$

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- Contains $a^2 + x^2$, so sub $x = 2\tan\theta$.
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$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x &= \int \frac{2\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,\mathrm{d}\theta \\ &= \int \frac{2\sec^2\theta}{2\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,\mathrm{d}\theta \\ &= \int \sec\theta\,\mathrm{d}\theta\end{aligned}$$

- We have assumed $\sec\theta > 0$. We did similarly in the previous example.

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- We substituted $x = 2\tan\theta$ and got

$$\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \int sec\theta\,\mathrm{d}\theta$$

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- We substituted $x = 2\tan\theta$ and got

$$\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \int sec\theta\,\mathrm{d}\theta$$

$$= \log|\sec\theta + \tan\theta| + c \qquad \text{previous work}$$

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- We substituted $x = 2\tan\theta$ and got

$$\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \int sec\theta\,\mathrm{d}\theta$$

$$= \log|\sec\theta + \tan\theta| + c \qquad \text{previous work}$$

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- The $\tan\theta = x/2$ is easy. But $\sec\theta$ is harder:

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- Hence: $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \log|\sqrt{1+x^2/4} + x/2| + c.$

Trigonometric substitutions

- Sometimes you need to complete the square in order to get started.

Try $\dfrac{dx}{4x^2 + 12x + 13}$

Partial fractions

Based on partial fraction decomposition of rational functions

- There are some very general rules for this technique.
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- Any polynomial with real coefficients can be factored into linear and quadratic factors with real coefficients

$$
\begin{aligned}
Q(x) = {} & k(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j} \\
& \times (x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2}\cdots(x^2+b_lx+c_l)^{n_l}
\end{aligned}
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Partial fractions

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\begin{aligned}
f(x) = {} & \frac{A_{11}}{(x-a_1)^1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1m_1}}{(x-a_1)^{m_1}} \\
& + \text{similar terms for each linear factor} \\
& + \frac{B_{11}x + C_{11}}{(x^2+b_1x+c_1)^1} + \frac{B_{12}x + C_{12}}{(x^2+b_1x+c_1)^2} + \cdots \frac{B_{1n_1}x + C_{1n_1}}{(x^2+b_1x+c_1)^{n_1}} \\
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- Once in this form, we can integrate term-by-term.

Partial fractions

$$\int \frac{dx}{x(x-1)}$$

- Write in partial fraction form

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Now find A and B.

$$= \frac{A(x-1) + Bx}{x(x-1)}$$

Compare numerators

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- Hence we have 2 equations

$$\left.\begin{aligned} A+B &= 0 \\ -A &= 1 \end{aligned}\right\} \Rightarrow A = -1, B = 1$$

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always check this!

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- Now integrate term-by-term

$$\begin{aligned}\int \frac{1}{x(x-1)}\,dx &= -\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx \\ &= -\log|x| + \log|x-1| + c \\ &= \log\left|\frac{x-1}{x}\right| + c\end{aligned}$$

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Try $\int \frac{1}{x^2 - a^2}\,\mathrm{d}x$.

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About 100 Integrals ...

25. $\int \tan^3 x \sec^4 x \, dx$

27. $\int \tan x \sec^5 x \, dx$

29. $\int \cot^3 x \csc^2 x \, dx$

31. $\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x \, dx$

33. $\int \cot x \csc^{-2} x \, dx$

35. $\int \frac{\tan x}{\cos^3 x} dx$

37. $\int \frac{\tan^2 x}{\sec^5 x} dx$

26. $\int_0^{\pi/3} \tan x \sec^{3/2}$

28. $\int \csc^3 x \, dx$

30. $\int \cot^3 s \csc^4 s \, ds$

32. $\int_{\pi/3}^{\pi/4} \cot x \csc^3 x$

34. $\int \frac{\cot t}{\csc^3 t} dt$

36. $\int \tan^3 x \csc^2 x \, d$

38. $\int \sin^5 w \cot^3 w$

39. $\int \frac{\tan x}{\sec^2 x} dx$

40. $\int \frac{\tan^5 x}{\sec^4 x} dx$

41. $\int \tan^2 x \, dx$ (*Hint:* $\tan^2 x = \sec^2 x - 1$.)

42. $\int \tan^3 x \, dx$

43. $\int \tan^4 x \, dx$

44. $\int \cot^5 x \, dx$

45. $\int \sin 2x \cos 3x \, dx$

46. $\int_0^{2\pi/3} \sin x \cos 2x \, dx$

47. $\int \sin(-4x) \cos(-$

48. $\int \sin 3x \cos \frac{1}{2}x \, dx$

49. $\int \sin \frac{1}{2}x \cos \frac{2}{3}x \, dx$

2. $\int x \sec^2 x \, dx$

4. $\int x \ln x^2 \, dx$

6. $\int x^2 \ln x \, dx$

8. $\int x e^{-x} \, dx$

10. $\int x^2 \sin x \, dx$

12. $\int e^x \sin x \, dx$

14. $\int \frac{\sin x}{e^x} dx$

16. $\int t \cdot 3^{-t} \, dt$

18. $\int \log_6 x \, dx$

20. $\int t^2 \cosh t \, dt$

22. $\int \sin^{-1} 3x \, dx$

33. $\int \frac{1}{(9x^2-4)^{5/2}} dx$

35. $\int \sqrt{4+x^2} \, dx$

37. $\int_{3\sqrt{2}}^{6} \frac{1}{x^4 \sqrt{x^2-9}} dx$

34. $\int \frac{\sqrt{4+x^2}}{x^3} dx$

36. $\int \frac{\sqrt{x^2+1}}{x^4} dx$

38. $\int \frac{2x-3}{\sqrt{4x-x^2-3}} dx$

1. $\int \frac{x}{x+1} dx$

3. $\int \frac{x^2}{x^2-1} dx$

2. $\int \frac{x^2}{x^2+1} dx$

4. $\int \frac{t^2-1}{t^2+1} dt$

5. $\int \frac{x^2+4}{x(x-1)^2} dx$

7. $\int_3^4 \frac{5}{(x-2)(x+3)} dx$

9. $\int \frac{3t}{t^2-8t+15} dt$

6. $\int \frac{2x^3+x^2+12}{x^2-4} dx$

8. $\int \frac{5x}{(x-2)(x+3)} dx$

10. $\int \frac{2}{x^2-x-6} dx$

15. $\int_0^{5/4} \frac{1}{\sqrt{25-4x^2}} dx$

17. $\int \frac{1}{\sqrt{4x^2+4x+2}} dx$

19. $\int \frac{1}{(1-2w^2)^{5/2}} dw$

21. $\int_0^1 \frac{1}{2x^2-2x+1} dx$

23. $\int \sqrt{x-x^2} \, dx$

25. $\int \frac{x^2}{\sqrt{9x^2-1}} dx$

27. $\int \frac{1}{x\sqrt{x^2+4}} dx$

29. $\int \frac{1}{x^2\sqrt{4x^2-9}} dx$

31. $\int \frac{x^2}{\sqrt{1+x^2}} dx$

1. $\int_0^{1/2} \sqrt{1-4x^2} \, dx$

3. $\int_{-2}^{2} \sqrt{1-\frac{x^2}{4}} \, dx$

5. $\int \frac{1}{(3-x^2)^{3/2}} dx$

7. $\int \frac{1}{(x^2+1)^{3/2}} dx$

9. $\int_0^1 \frac{1}{(3x^2+2)^{5/2}} dx$

11. $\int_2^{2\sqrt{2}} \frac{\sqrt{x^2-4}}{x} dx$

13. $\int \frac{1}{(9+t^2)^2} dt$

2. $\int_0^4 \sqrt{16-x^2} \, dx$

4. $\int \sqrt{1+(2x-1)^2} \, dx$

6. $\int_{4\sqrt{2}}^{8} \frac{1}{(t^2-16)^{3/2}} dt$

8. $\int \frac{1}{(2x^2+1)^{3/2}} dx$

10. $\int \frac{x^2}{(x^2-2)^{5/2}} dx$

12. $\int \frac{1}{(y^2-4)^{3/2}} dy$

14. $\int_1^{\sqrt{2}} \frac{1}{\sqrt{2x^2-1}} dx$

39. $\int \frac{x}{\sqrt{2x^2+12x+19}} dx$

40. $\int \frac{1}{(w^2+2w+5)^{3/2}} dw$

42. $\int \frac{e^{3w}}{\sqrt{1-e^{2w}}} dw$

*41. $\int \sqrt{x^2+6x+5} \, dx$

43. $\int e^w \sqrt{1+e^{2w}} \, dw$

1. $\int \sqrt{4x-5} \, dx$; $u = 4x - 5$

2. $\int (1-5x^2)^{2/3}(10x) \, dx$; $u = 1 - 5x^2$

3. $\int \cos \pi x \, dx$; $u = \pi x$

4. $\int 3 \sin(-2x) \, dx$; $u = -2x$

5. $\int x \cos x^2 \, dx$; $u = x^2$

6. $\int \sin^3 t \cos t \, dt$; $u = \sin t$

7. $\int \cos^{-4} t \sin t \, dt$; $u = \cos t$

8. $\int \frac{2t-3}{(t^2-3t+1)^2} dt$; $u = t^2 - 3t + 1$

9. $\int \frac{2t-3}{(t^2-3t+1)^{7/2}} dt$; $u = t^2 - 3t + 1$

10. $\int x\sqrt{x-1} \, dx$; $v = x - 1$

11. $\int (x-1)\sqrt{x+1} \, dx$; $v = x + 1$

12. $\int x^2\sqrt{x+3} \, dx$; $v = x + 3$