Contents

Unit 1

Topological Spaces

Unit 1: *Topological Spaces*

§1 Definition and Examples:

Definition 1.1: Let X be any non-empty set. A family $\mathfrak{J}$ of subsets of X is called a topology on X if it satisfies the following conditions:

(i) $\phi \in \mathfrak{J}$ and $X \in \mathfrak{J}$

(ii) $A, B \in \mathfrak{J} \Rightarrow A \cap B \in \mathfrak{J}$

(iii) $A_\lambda \in \mathfrak{J}$, $\forall \lambda \in \Lambda$ (where Λ is any indexing set) $\Rightarrow \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$

If $\mathfrak{J}$ is a topology on X, then the ordered pair $\langle X, \mathfrak{J} \rangle$ is called a topological space (or T-space)

Examples 1.2: Throughout X denotes a non-empty set.

1) $\mathfrak{J} = \{\emptyset, X\}$ is a topology on X. This topology is called **indiscrete topology** on X and the T-space $\langle X, \mathfrak{J} \rangle$ is called indiscrete topological space.

2) $\mathfrak{J} = \wp(X)$, ($\wp(X)$ = power set of X is a topology on X and is called **discrete topology** on X and the T-space $\langle X, \mathfrak{J} \rangle$ is called discrete topological space.

Remark: If $|X| = 1$, then discrete and indiscrete topologies on X coincide, otherwise they are different.

3) Let $X = \{a, b, c\}$ then $\mathfrak{J}_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathfrak{J}_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ are topologies on X whereas $\mathfrak{J}_3 = \{\emptyset, \{a\}, \{b\}, X\}$ is a not a topology on X.

4) Let X be an infinite set. Define $\mathfrak{J} = \{\emptyset\} \cup \{A \subseteq X \mid X - A \text{ is finite}\}$ then $\mathfrak{J}$ is topology on X.

(i) $\emptyset \in \mathfrak{J}$ …… (by definition of $\mathfrak{J}$)

As X – X = $\emptyset$, a finite set, $X \in \mathfrak{J}$

(ii) Let $A, B \in \mathfrak{J}$. If either $A = \emptyset$ or $B = \emptyset$, then $A \cap B \in \mathfrak{J}$. Assume that $A \neq \emptyset$ and $B \neq \emptyset$.

Then $X - A$ is finite and $X - B$ is finite. Hence $X - (A \cap B) = (X - A) \cup (X - B)$ is finite set. Therefore $A \cap B \in \mathfrak{J}$. Thus $A, B \in \mathfrak{J} \implies A \cap B \in \mathfrak{J}$.

(iii) Let $A_\lambda \in \mathfrak{J}$, for each $\lambda \in \Lambda$ (where Λ is any indexing set). If each $A_\lambda = \emptyset$, then

$$\bigcup_{\lambda\in\Lambda} A_\lambda = \emptyset \in \mathfrak{J}.$$

If $\exists\, \lambda_0 \in \Lambda$ such that $A_{\lambda_0} \neq \emptyset$, then $A_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda \implies X - A_{\lambda_0} \supseteq X - \bigcup_{\lambda\in\Lambda} A_\lambda$.

As $X - A_{\lambda_0}$ is a finite set and subset of finite set being finite we get $X - \bigcup_{\lambda\in\Lambda} A_\lambda$ is finite

and hence $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$. Thus in either case,

$$A_\lambda \in \mathfrak{J},\ \ \forall\, \lambda \in \Lambda \implies \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}.$$

From (i), (ii) and (iii) is a topology on X. This topology is called **co-finite topology** on X and the topological space is called co-finite topological space.

Remark: If X is finite set, then co-finite topology on X coincides with the discrete topology on X.

5) Let X be any uncountable set. Define $\mathfrak{J} = \{\emptyset\} \cup \{A \subseteq X \mid X - A \text{ is countable}\}$ Then $\mathfrak{J}$ is a topology on X.

i. $\emptyset \in \mathfrak{J}$ (by definition).

   As X – X = $\emptyset$ and $\emptyset$ is countable (Since $\emptyset$ is finite) we get X $\in \mathfrak{J}$.

ii. Let $A, B \in \mathfrak{J}$. If either $A = \emptyset$ or $B = \emptyset$ we get $A \cap B \in \mathfrak{J}$.

   Let $A \neq \emptyset$ and $B \neq \emptyset$.

   Then by definition of $\mathfrak{J}$, X – A and X – B both are countable sets and hence $X - (A \cap B) = (X - A) \cup (X - B)$ is countable. This shows that $A \cap B \in \mathfrak{J}$. Thus $A, B \in \mathfrak{J}$ implies $A \cap B \in \mathfrak{J}$.

iii. Let $A_\lambda \in \mathfrak{J}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set. If for each $\lambda \in \Lambda$, $A_\lambda = \emptyset$

then $\bigcup_{\lambda\in\Lambda} A_\lambda = \emptyset$ will imply $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$. Let $A_{\lambda_0} \neq \emptyset$ for some $\lambda_0 \in \Lambda$.

Then $A_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda \Rightarrow X - A_{\lambda_0} \supseteq X - \bigcup_{\lambda\in\Lambda} A_\lambda$

$\Rightarrow X - \bigcup_{\lambda\in\Lambda} A_\lambda$ is a subset of a countable set $X - A_{\lambda_0}$ (Since $A_{\lambda_0} \in \mathfrak{I}$ and $A_{\lambda_0} \neq \emptyset$)

$\Rightarrow X - \bigcup_{\lambda\in\Lambda} A_\lambda$ is a countable set. (since subset of countable set is countable)

$\Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$

Thus in either case, $A_\lambda \in \mathfrak{I}, \forall \lambda \in \Lambda \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$

From (i) , (ii) and (iii) we get $\mathfrak{I}$ is a topology on X. This topology on X is called **co-countable topology** on X and the T-space $\langle X, \mathfrak{I}\rangle$ is called co-countable topological space.

Remark: If X is a countable set, the co-countable topology on X coincides with the discrete topology on X.

6) Let $A \subseteq X$. Define $\mathfrak{I} = \{\emptyset\} \cup \{B \subseteq X \mid A \subseteq B\}$. Then $\mathfrak{I}$ is a topology on X.

(i) $\emptyset \in \mathfrak{I}$ by definition. As $A \subseteq X$, $X \in \mathfrak{I}$.

(ii) Let $B, C \in \mathfrak{I}$. If $B = \emptyset$ or $C = \emptyset$, then $B \cap C = \emptyset$ will give $B \cap C \in \mathfrak{I}$. Let $B \neq \emptyset$ or $C \neq \emptyset$. Then $A \subseteq B \cap C$ will imply $B \cap C \in \mathfrak{I}$.

(iii) Let $B_\lambda \in \mathfrak{I}\ \forall \lambda \in \Lambda$, where Λ is any indexing set. If for each $\lambda \in \Lambda$, $B_\lambda = \phi$ then $\bigcup_{\lambda\in\Lambda} B_\lambda = \emptyset$ and in this case $\bigcup_{\lambda\in\Lambda} B_\lambda \in \mathfrak{I}$.

Assume that $B_{\lambda_0} \neq \emptyset$ for some $\lambda_0 \in \Lambda$. Then $A \subseteq B_{\lambda_0}$ and $B_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} B_\lambda$ imply $A \subseteq \bigcup_{\lambda\in\Lambda} B_\lambda$.

Therefore $\bigcup_{\lambda\in\Lambda} B_\lambda \in \mathfrak{I}$.

From (i), (ii) and (iii) $\mathfrak{I}$ is a topology on X.

Remarks: (1) If $A = \emptyset$ then $\mathfrak{J}$ is discrete topology on X.

(2) If $A = X$ then $\mathfrak{J}$ is indiscrete topology on X.

(3) If $A = \{p\}$, then $\mathfrak{J} = \{\emptyset\} \cup \{B \subseteq X \mid p \in B\}$ is called ***p*-inclusive topology on X**.

7) Let $p \in X$. Define $\mathfrak{J} = \{X\} \cup \{A \subseteq X \mid p \notin A\}$. Then $\mathfrak{J}$ is topology on X.

(i) $p \notin \emptyset$ implies $\emptyset \in \mathfrak{J}$. By definition $X \in \mathfrak{J}$.

(ii) Let $A, B \in \mathfrak{J}$. If $A = X$ or $B = X$, then $A \cap B = X$. In this case $A \cap B \in \mathfrak{J}$. Assume that either $A \neq X$ or $B \neq X$. Then $p \notin A$ or $p \notin B$ and hence $p \notin A \cap B$ which proves $A \cap B \in \mathfrak{J}$.

Thus $A, B \in \mathfrak{J}$ implies $A \cap B \in \mathfrak{J}$.

(iii) Let $A_\lambda \in \mathfrak{J}\ \forall \lambda \in \Lambda$, where Λ is any indexing set. If for some $\lambda \in \Lambda$, $A_\lambda = X$ then $\bigcup_{\lambda \in \Lambda} A_\lambda = X$ will give $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

Assume that $A_\lambda \neq X$ for each $\in \Lambda$. Then $p \notin A_\lambda$ for each $\lambda \in \Lambda$ will imply,

$p \notin \bigcup_{\lambda \in \Lambda} A_\lambda$ and hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

Thus in either case, $A_\lambda \in \mathfrak{J}\ \forall \lambda \in \Lambda \implies \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

From (i), (ii) and (iii) $\mathfrak{J}$ is a topology on X.

This topology on X is called ***p*-exclusive topology on X.**

8) Let $\langle X, \mathfrak{J} \rangle$ be topological space and $A \subseteq X$. Define $\mathfrak{J}^* = \{G \cup (A \cap H) \mid G, H \in \mathfrak{J}\}$. Then $\mathfrak{J}^*$ is a topology on X.

(i) Take $G = \emptyset$ and $H = \emptyset$. Then $G \cup (A \cap H) = \emptyset \cup (A \cap \emptyset) = \emptyset \Rightarrow \emptyset \in \mathfrak{J}^*$. Take $G = X$. Then for any $H \in \mathfrak{J}$ we get $X \cup (A \cap H) = X$. Hence $X \in \mathfrak{J}^*$.

(ii) Let $G_1 \cup (A \cap H_1) \in \mathfrak{J}^*$ and $G_2 \cup (A \cap H_2) \in \mathfrak{J}^*$ for $G_1, H_1, G_2, H_2 \in \mathfrak{J}$.

Then $[G_1 \cup (A \cap H_1)] \cap [G_2 \cup (A \cap H_2)]$

$= (G_1 \cap G_2) \cup (G_1 \cap A \cap H_2) \cup (A \cap H_1 \cap G_2) \cup (A \cap H_1 \cap H_2)$

$= (G_1 \cap G_2) \cup [A \cap [(G_1 \cap H_2) \cup (H_1 \cap G_2) \cup (H_1 \cap H_2)]]$

As $(G_1 \cap G_2) \in \mathfrak{J}$ and $[(G_1 \cap H_2) \cup (H_1 \cap G_2) \cup (H_1 \cap H_2)] \in \mathfrak{J}$ we get,

$[G_1 \cup (A \cap H_1)] \cap [G_2 \cup (A \cap H_2)] \in \mathfrak{J}$.

(iii) Let $G_\lambda \cup (A \cap H_\lambda) \in \mathfrak{I}^*$ for $\lambda \in \Lambda$, where Λ is any indexing set. Then $G_\lambda \in \mathfrak{I}$ and $H_\lambda \in \mathfrak{I}, \forall \lambda \in \Lambda$.

$$\bigcup_{\lambda\in\Lambda}[G_\lambda \cup (A \cap H_\lambda)] = \left[\bigcup_{\lambda\in\Lambda} G_\lambda\right] \cup \left[A \cap \left[\bigcup_{\lambda\in\Lambda} H_\lambda\right]\right]$$

As $\bigcup_{\lambda\in\Lambda} G_\lambda \in \mathfrak{I}$ and $\bigcup_{\lambda\in\Lambda} H_\lambda \in \mathfrak{I}$, we get $\bigcup_{\lambda\in\Lambda}[G_\lambda \cup (A \cap H_\lambda)] \in \mathfrak{I}^*$.

From (i), (ii) and (iii) we get $\mathfrak{I}^*$ is a topology on X.

Remark: This example shows that every topology on X induces another topology on X.

9) Let X and Y be any two non-empty sets and let $f : X \longrightarrow Y$ be any function. Let $\mathfrak{I}$ be topology on Y. Define $\mathfrak{I}^* = \{f^{-1}(G) \mid G \in \mathfrak{I}\}$, where $f^{-1}(G) = \{x \in X \mid f(x) \in G\}$. Then $\mathfrak{I}^*$ is topology on X.

(i) $f^{-1}(\emptyset) = \emptyset \quad \Rightarrow \quad \emptyset \in \mathfrak{I}^*$ and $f^{-1}(Y) = X \quad \Rightarrow \quad X \in \mathfrak{I}^*$

(ii) Let $f^{-1}(G) \in \mathfrak{I}^*$ and $f^{-1}(H) \in \mathfrak{I}^*$ for $, H \in \mathfrak{I}$. Then $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$ and $G, H \in \mathfrak{I}$ will imply $f^{-1}(G) \cap f^{-1}(H) \in \mathfrak{I}^*$.

(iii) Let $f^{-1}(G_\lambda) \in \mathfrak{I}^*\ \forall \lambda \in \Lambda$, where Λ any indexing set is. Then

$$f^{-1}\left(\bigcup_{\lambda\in\Lambda} G_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{-1}(G_\lambda). \quad \text{As } \bigcup_{\lambda\in\Lambda} G_\lambda \in \mathfrak{I}, \text{ we get } \bigcup_{\lambda\in\Lambda} f^{-1}(G_\lambda) \in \mathfrak{I}^*.$$

Thus from (i), (ii) and (iii) we get $\mathfrak{I}^*$ is a topology on X.

10) Let X be any uncountable set and let ∞ be a fixed point of X. Let $\mathfrak{I} = \{G \subseteq X \mid \infty \notin G\} \cup \{G \subseteq X \mid \infty \in G \text{ and } X - G \text{ is finite}\}$. Then $\mathfrak{I}$ is a topology on X.

Define $\mathfrak{I}_1 = \{G \subseteq X \mid \infty \notin G\}$ and $\mathfrak{I}_2 = \{G \subseteq X \mid \infty \in G \text{ and } X - G \text{ is finite}\}$ then $\mathfrak{I} = \mathfrak{I}_1 \cup \mathfrak{I}_2$.

(i) $\infty \notin \emptyset \Rightarrow \emptyset \in \mathfrak{I}$. $\infty \in X$ and $X - X = \emptyset$ is a finite set $\Longrightarrow X \in \mathfrak{I}_2 \Rightarrow X \in \mathfrak{I}$.

(ii) Let $A, B \in \mathfrak{I}$.

Case 1: $A, B \in \mathfrak{I}_1$. Then $\infty \notin A$ and $\infty \notin B$. Hence $\infty \notin A \cap B$.

Therefore $A \cap B \in \mathfrak{I}_1 \Longrightarrow A \cap B \in \mathfrak{I}$.

Case 2 : $A, B \in \mathfrak{J}_2$. Then $A \in \mathfrak{J}_2 \implies \infty \in A$ and $X - A$ is finite. $B \in \mathfrak{J}_2 \implies \infty \in B$ and $X - B$ is finite. Then $\infty \in A \cap B$ and $X - (A \cap B) = (X - A) \cup (X - B)$ is finite. Thus $A \cap B \in \mathfrak{J}_2$ which gives $A \cap B \in \mathfrak{J}$.

Case 3 : $A \in \mathfrak{J}_1$ and $B \in \mathfrak{J}_2$. Then $\infty \notin A$ will imply $\infty \notin A \cap B$.

Hence $A \cap B \in \mathfrak{J}_1 \Rightarrow A \cap B \in \mathfrak{J}$.

Case 4 : $A \in \mathfrak{J}_2$ and $B \in \mathfrak{J}_1$. Then $\infty \notin B$ will imply $\infty \notin A \cap B$.

Hence $A \cap B \in \mathfrak{J}_1 \Rightarrow A \cap B \in \mathfrak{J}$.

Thus in all the cases $A, B \in \mathfrak{J} \implies A \cap B \in \mathfrak{J}$.

(iii) $A_\lambda \in \mathfrak{J}\ \forall \lambda \in \Lambda$, where Λ is any indexing set . If $A_\lambda \in \mathfrak{J}_1\ \forall \lambda \in \Lambda$ then $\infty \notin A_\lambda\ \forall \lambda \in \Lambda$ will imply $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}_1$. Hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

If $\exists\ \lambda_0 \in \Lambda$ such that $A_{\lambda_0} \notin \mathfrak{J}_1$ then $A_{\lambda_0} \in \mathfrak{J}_2$. In this case $\infty \in A_{\lambda_0}$ and $X - A_{\lambda_0}$ is a finite set.

$A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$ implies $\infty \in \bigcup_{\lambda \in \Lambda} A_\lambda$ and $X - \bigcup_{\lambda \in \Lambda} A_\lambda \subseteq X - A_{\lambda_0}$.

As $X - A_{\lambda_0}$ is finite we get $X - \bigcup_{\lambda \in \Lambda} A_\lambda$ a is finite set. Thus in this case $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}_2$ and hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

Thus in either case, $A_\lambda \in \mathfrak{J}, \forall \lambda \in \Lambda \implies \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

From (i), (ii) and (iii) $\mathfrak{J}$ is a topology on X .

This topology $\mathfrak{J}$ is called **Fort's topology** on X and the T-space $\langle X, \mathfrak{J} \rangle$ is called **Fort's space**.

Some Special Topologies on Special sets .

Apart from the topologies given in the above examples there exist some special topologies on $\mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{N}$. ($\mathbb{R}$ = the set of all real numbers , $\mathbb{Z}$ = the set of all integers and $\mathbb{N}$ = the set of all natural numbers). We list some of them in the following examples.

(11) Let $\mathfrak{I}_u = \{\emptyset\} \cup \{A \subseteq \mathbb{R} \mid \forall\, a \in A\ \exists\ \ r > 0 \text{ such that } (a - r\,, a + r) \subseteq A\}$. Then $\mathfrak{I}_u$ is a topology on $\mathbb{R}$.

(i) $\emptyset \in \mathfrak{I}_u$ (by definition) and $\mathbb{R} \in \mathfrak{I}_u$ as for any $a \in \mathbb{R}$, $(a - 1\,, a + 1) \subseteq \mathbb{R}$.

(ii) Let $A, B \in \mathfrak{I}_u$. If $A = \emptyset$ or $B = \emptyset$, then $A \cap B \in \mathfrak{I}_u$. Let $A \neq \emptyset$ and $B \neq \emptyset$.

Then $x \in A \cap B \ \Rightarrow\ x \in A$ and $x \in B \ \Rightarrow \exists\ \ r_1 > 0$ such that $(x - r_1\,,\ x + r_1) \subseteq A$ and $\exists\ \ r_2 > 0$ such that $(x - r_2\,,\ x + r_2) \subseteq B$.

Define $r = min(r_1, r_2)$. Then $r > 0$ and $(x - r\,, x + r) \subseteq A \cap B$. But this shows that $A \cap B \in \mathfrak{I}_u$. Thus in either case $A, B \in \mathfrak{I}_u \ \Rightarrow\ A \cap B \in \mathfrak{I}_u$.

(iii) $A_\lambda \in \mathfrak{I}_u\ \ \forall\, \lambda \in \Lambda$, where Λ is any indexing set .

If $\bigcup\limits_{\lambda \in \Lambda} A_\lambda = \emptyset$, then obviously, $\bigcup\limits_{\lambda \in \Lambda} A_\lambda \in \mathfrak{I}_u$.

Hence, assume that $\bigcup\limits_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Let $x \in \bigcup\limits_{\lambda \in \Lambda} A_\lambda$. Then $x \in A_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

As $A_{\lambda_0} \in \mathfrak{I}_u\ \ \exists\ \ r > 0$ such that $(x - r\,,\ x + r) \subseteq A_{\lambda_0}$.

But then $(x - r\,, x + r) \subseteq \bigcup\limits_{\lambda \in \Lambda} A_\lambda$. But this shows that $\bigcup\limits_{\lambda \in \Lambda} A_\lambda \in \mathfrak{I}_u$.

Thus in either case $A_\lambda \in \mathfrak{I}_u$, $\forall\, \lambda \in \Lambda \ \Longrightarrow\ \bigcup\limits_{\lambda \in \Lambda} A_\lambda \in \mathfrak{I}_u$.

From (i), (ii) and (iii) $\mathfrak{I}_u$ is a topology on $\mathbb{R}$.

This topology is called **usual topology** on $\mathbb{R}$.

Remarks: (1) The usual topology on $\mathbb{R}$ is also called standard topology or Euclidean topology .

(2) Any open interval in $\mathbb{R}$ is a member of $\mathfrak{I}_u$. Consider the open interval (a, b) and $x \in (a, b)$. Take $r = min(x - a\,, b - x)$. Then $(x - r\,, x + r) \subseteq (a\,, b)$. This shows that $(a\,, b) \in \mathfrak{I}_u$.

(12) Let $\mathfrak{I}_r = \{\emptyset\} \cup \{A \subseteq \mathbb{R} \mid \forall\, p \in A\ \exists\, a, b \in \mathbb{R} \text{ such that } p \in [a\,,\ b) \subseteq A\}$. Then $\mathfrak{I}_r$ is a topology on $\mathbb{R}$.

(i) $\emptyset \in \mathfrak{I}_r$ (by definition). $\mathbb{R} \in \mathfrak{I}_r$ as for any $p \in \mathbb{R}\ \exists\, a, b \in \mathbb{R}$ such that $p \in [p\,,\ p + 1) \subseteq \mathbb{R}$.

(ii) Let $A, B \in \mathfrak{I}_r$. If $A \cap B = \emptyset$, then $A \cap B \in \mathfrak{I}_r$. If $A \cap B \neq \emptyset$ then for $x \in A \cap B$ there exist half open intervals H_1 and H_2 in $\mathbb{R}$ such that $x \in H_1 \subseteq A$ and

$x \in H_2 \subseteq B$. But then $H_1 \cap H_2$ will be an half open interval in $\mathbb{R}$ with $x \in H_1 \cap H_2 \subseteq A \cap B$. This shows that $A \cap B \in \mathfrak{I}_r$. Thus $A, B \in \mathfrak{I}_r \Rightarrow A \cap B \in \mathfrak{I}_r$.

(iii) Let $A_\lambda \in \mathfrak{I}_r \ \forall \lambda \in \Lambda$, where Λ is any indexing set.

If $\bigcup_{\lambda \in \Lambda} A_\lambda = \emptyset$, then obviously $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{I}_r$.

Let $\bigcup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Let $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$.

Then $x \in A_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. As $A_{\lambda_0} \in \mathfrak{I}_r \ \exists \ [a, b)$ such that,

$x \in [a, b) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$

But this shows that $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{I}_r$.

Thus in either case $A_\lambda \in \mathfrak{I}_r , \ \forall \lambda \in \Lambda \implies \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{I}_r$.

From (i), (ii) and (iii) $\mathfrak{I}_r$ is a topology on $\mathbb{R}$.

This topology is called **lower limit topology** or **right half open topology** on $\mathbb{R}$ **.**

(13) Let $\mathfrak{I}_l = \{\emptyset\} \cup \{A \subseteq \mathbb{R} \mid \forall p \in A \ \exists a, b \in \mathbb{R}$ such that $p \in (a, b] \subseteq A\}$. Then $\mathfrak{I}_l$ is a topology on $\mathbb{R}$.

This topology is called **upper limit topology** or **left half open topology** on $\mathbb{R}$ **.** (Proof as in Ex.12)

(14) For each $a \in \mathbb{R}$ define $L_a = \{x \in \mathbb{R} \mid x < a\}$. Define $\mathfrak{I} = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{L_a \mid a \in \mathbb{R}\}$. Then $\mathfrak{I}$ is a topology on $\mathbb{R}$. [Note that $L_a = (-\infty, a)$].

(i) $\emptyset \in \mathfrak{I}$ and $\mathbb{R} \in \mathfrak{I}$ (by definition)

(ii) Let $A, B \in \mathfrak{I}$.

Case (1) : $A = \emptyset$ or $B = \emptyset$ in this case $A \cap B = \emptyset \in \mathfrak{I}$.

Case (2) : $A = \mathbb{R}$ or $B = \mathbb{R}$ in this case $A \cap B = A$ or $A \cap B = B$. Hence $A \cap B \in \mathfrak{I}$.

Case (3) : $A = L_a$ and $B = L_b$. Then $a, b \in \mathbb{R}$. Define $c = \min(a, b)$. Hence $A \cap B = L_a \cap L_b = L_c \in \mathfrak{I}$. Thus in all cases $A, B \in \mathfrak{I} \implies A \cap B \in \mathfrak{I}$.

(iii) Let $A_\lambda \in \mathfrak{J}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set.

Case (1): $A_\lambda = \emptyset\ ,\ \ \forall\, \lambda \in \Lambda$. Then $\bigcup_{\lambda\in\Lambda} A_\lambda = \emptyset \ \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

Case (2): $A_\lambda = \mathbb{R}$ for some $\lambda \in \Lambda$. Then $\bigcup_{\lambda\in\Lambda} A_\lambda = \mathbb{R} \ \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

Case (3): $A_\lambda = L_\lambda\ ,\ \ \forall\, \lambda \in \Lambda$. Then $\Lambda \subseteq \mathbb{R}$. $\bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\mathbb{R}} A_\lambda = \mathbb{R}\ ,\ if\ \ \Lambda = \mathbb{R}$ or

$\bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} L_\lambda = L_u$ if $\Lambda \subset \mathbb{R}$.

And $u = l.u.b.\{\, \lambda \mid \lambda \in \Lambda\}$.

Thus in all cases $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

From (i), (ii) and (iii) is a topology on $\mathbb{R}$. This topology is called the **left ray topology** on $\mathbb{R}$.

(15) Define $\mathfrak{J}^* = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{R_a \mid a \in \mathbb{R}\}$ where $R_a = \{\, x \in \mathbb{R} \mid x > a\}$.

[Note that $R_a = (a\,,\infty)$]. Then $\mathfrak{J}^*$ is a topology on $\mathbb{R}$.

This topology is called **right ray topology** on $\mathbb{R}$.

[Proof is similar to example 4].

(16) Let $\mathfrak{J} = \{\emptyset\} \cup \{A_n \mid n = 1,2, \dots\}$ where $A_n = \{n, n+1, n+2, \dots\}$. Then $\mathfrak{J}$ is a topology on N .

(i) $\emptyset \in \mathfrak{J}$ (by definition). As $\mathbb{N} = \{1,2,3, \dots\} = A_1$ we get $\mathbb{N} \in \mathfrak{J}$.

(ii) Let $A\,, B \in \mathfrak{J}$. If either $A = \emptyset$ or $B = \emptyset$ we get $A \cap B = \emptyset \in \mathfrak{J}$. Hence assume that $A \neq \emptyset$ and $B \neq \emptyset$. Then $A = A_m$ or $B = A_n$ for some $\ ,n \in \mathbb{N}$. But then $A \cap B = A_m \cap A_n = A_m$ if $m \geq n$ or $A \cap B = A_m \cap A_n = A_n$ if $n \geq m$. Thus in either case $A \cap B \in \mathfrak{J}$ for $A, B \in \mathfrak{J}$.

(iii) Let $A_\lambda \in \mathfrak{J}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set . As $A_\lambda \in \mathfrak{J}\ \forall\, \lambda \in \Lambda$, by definition of $\mathfrak{J}$, $\Lambda = \mathbb{N}$. Hence Λ is well ordered set.

Define $m = l.u.b\ \{\lambda \mid \lambda \in \Lambda\}$. This m exists.

Hence $\bigcup_{\lambda\in\Lambda} A_\lambda = A_m$. This shows that $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

From (i), (ii) and (iii), $\mathfrak{I}$ is a topology on.

(17) Let $\mathfrak{I} = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{A_n \mid n = 1,2, \dots\}$ where $A_n = \{1,2,3, \dots, n\}$.

Then $\mathfrak{I}$ is a topology on $\mathbb{N}$.

(i) $\emptyset \in \mathfrak{I}$ and $\mathbb{N} \in \mathfrak{I}$ (by definition).

(ii) Let $A, B \in \mathfrak{I}$. If $A, B \in \{\emptyset\} \cup \{\mathbb{N}\}$, then $A \cap B \in \mathfrak{I}$. Let $A, B \in \{A_n \mid n = 1,2, \dots\}$ then $A = A_n$ and $B = A_m$. As $m, n \in \mathbb{N}$, either $m \leq n$ or $n \leq m$. Hence $A \cap B = A_m$ if $m \leq n$ or $A \cap B = A_n$ if $n \leq m$.

(iii) $A_\lambda \in \mathfrak{I}$, for each $\lambda \in \Lambda$ (where Λ is any indexing set).

$Case$ (1): $A_\lambda = \emptyset$, $\forall\, \lambda \in \Lambda$. Then $\bigcup\limits_{\lambda\in\Lambda} A_\lambda = \emptyset \implies \bigcup\limits_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$.

$Case$ (2): $A_\lambda = \mathbb{N}$, for some $\lambda \in \Lambda$, then $\bigcup\limits_{\lambda\in\Lambda} A_\lambda = \mathbb{N}$. Hence $\bigcup\limits_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$.

$Case$ (3): Let $A_\lambda \neq \emptyset$ and $A_\lambda \neq \mathbb{N}$, $\forall\, \lambda \in \Lambda$. Then $\Lambda \subseteq \mathbb{N}$. $\therefore$ $\bigcup\limits_{\lambda\in\Lambda} A_\lambda = \mathbb{N}$.

if $\Lambda = \mathbb{N}$ and $\bigcup\limits_{\lambda\in\Lambda} A_\lambda = A_m$ $where\ m = sup\{\lambda \mid \lambda \in \Lambda\}$ if $\Lambda \neq \mathbb{N}$.

Hence in all the cases, $\bigcup\limits_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$ whenever $A_\lambda \in \mathfrak{I}$, $\forall\, \lambda \in \Lambda$.

From (i), (ii) and (iii) $\mathfrak{I}$ is a topology on $\mathbb{N}$.

(18) $\mathfrak{I} = \{\emptyset\} \cup \{G_z \mid z \in \mathbb{Z}\}$, where $G_z = \{z + 2n \mid n \in \mathbb{Z}\}$. Then $\mathfrak{I}$ is a topology on $\mathbb{Z}$.

(19) Let $X = \mathbb{R}$ and define $\mathfrak{I} = \{\emptyset\} \cup \{A \subseteq \mathbb{R} \mid x \in A$ implies $-x \in A\}$.

Then $\mathfrak{I}$ is a topology on $\mathbb{R}$.

(i) $\emptyset \in \mathfrak{I}$ (by definition). $\mathbb{R} \in \mathfrak{I}$ as $x \in \mathbb{R}$ implies $-x \in \mathbb{R}$.

(ii) Let $A, B \in \mathfrak{I}$. If $A \cap B = \emptyset$, then $A \cap B \in \mathfrak{I}$. Let $A \cap B \neq \emptyset$.

Then $x \in A \cap B \Rightarrow x \in A$ and $x \in B$. As $A, B \in \mathfrak{I}$ we get $-x \in A$ and $-x \in B$. Thus

$x \in A \cap B \Rightarrow -x \in A \cap B$. Hence $A \cap B \in \mathfrak{I}$.

(iii) Let $A_\lambda \in \mathfrak{I}$, for each $\lambda \in \Lambda$ (where Λ is any indexing set).

If $\bigcup_{\lambda\in\Lambda} A_\lambda = \emptyset \;\Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$. Let $\bigcup_{\lambda\in\Lambda} A_\lambda \neq \emptyset$ and let $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$.

Then $x \in A_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

As $A_{\lambda_0} \in \mathfrak{J}$ we get $-x \in A_{\lambda_0}$ and hence $-x \in \bigcup_{\lambda\in\Lambda} A_\lambda$.

Thus $A_\lambda \in \mathfrak{J}$, $\forall\, \lambda \in \Lambda \;\Rightarrow\; \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

From (i), (ii) and (iii) $\mathfrak{J}$ is a topology on $\mathbb{R}$.

Note that for this topology $\mathfrak{J}$ on $\mathbb{R}$, $A \in \mathfrak{J} \Leftrightarrow \mathbb{R} - A \in \mathfrak{J}$. Let $A \in \mathfrak{J}$. If $A = \emptyset$ or $\mathbb{R}$ then obviously $\mathbb{R} - A \in \mathfrak{J}$. Hence, let $\emptyset \subset A \subset \mathbb{R}$.

$x \in \mathbb{R} - A \Rightarrow x \notin A \Rightarrow -x \notin A$ [since $-x \in A \Rightarrow -(-x) = x \in A$]

$\Rightarrow -x \in \mathbb{R} - A$.

Thus $A \in \mathfrak{J} \Rightarrow \mathbb{R} - A \in \mathfrak{J}$. Similarly $\mathbb{R} - A \in \mathfrak{J} \Rightarrow \mathbb{R} - (\mathbb{R} - A) = A \in \mathfrak{J}$.

Hence $A \in \mathfrak{J} \Leftrightarrow \mathbb{R} - A \in \mathfrak{J}$.

Remarks :

1) In a T-space $\langle X, \mathfrak{J}\rangle$, each member of $\mathfrak{J}$ is a subset of X but not conversely.

For this consider the T-space $\langle X, \mathfrak{J}\rangle$ where $X = \{a, b, c\}$ and $\mathfrak{J} = \{\emptyset, \{a\}, \{a, b\}, X\}$.

Then $\{c\} \subset X$ but $\{c\} \notin \mathfrak{J}$.

2) Intersection of finite number of members of $\mathfrak{J}$ is a member of $\mathfrak{J}$ but intersection of any number members of $\mathfrak{J}$ need not be member of $\mathfrak{J}$.

For this consider the T-space $\langle \mathbb{R}, \mathfrak{J}_u\rangle$. $(-n, n) \in \mathfrak{J}_u$ for each $n \in \mathbb{N}$. But

$$\bigcap_{n\in\mathbf{N}} (-n, n) = \{0\} \notin \mathfrak{J}_u.$$

3) Let $X \neq \emptyset$. Every subset of the power set of X need not be a topology on X.

For this consider the following examples:

(i) Let $X = \mathbb{R}$ and $\mathcal{K} = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}$. Define $A_n = \left[\frac{1}{n}, \infty\right)\ \forall\, n \in \mathbb{N}$.

Then $A_n \in \mathcal{K}$ for each $n \in \mathbb{N}$ but $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}, \infty\right) = (0, \infty) \notin \mathcal{K}$.

Hence $\mathcal{K}$ is not a topology on $\mathbb{R}$.

(ii) Let $X = \{a, b, c\}$. Define $\mathcal{K} = \{\emptyset, \{a\}, \{c\}, X\}$. Then $\mathcal{K}$ is not a topology on X as $\{a\} \cup \{c\} = \{a, c\} \notin \mathcal{K}$.

4) For any two topologies $\mathfrak{J}_1$ and $\mathfrak{J}_2$ on X, $\mathfrak{J}_1 \cup \mathfrak{J}_2$ need not be a topology on X.
For this, consider $X = \{a, b, c\}$. Let $\mathfrak{J}_1 = \{\emptyset, \{a\}, X\}$ and $\mathfrak{J}_2 = \{\emptyset, \{b\}, X\}$ be two topologies on X, but $\mathfrak{J}_1 \cup \mathfrak{J}_2 = \{\emptyset, \{a\}, \{b\}, X\}$ is not a topology on X.

Definition 1.3: Let $\langle X, \mathfrak{J} \rangle$ be a topological space. Members of $\mathfrak{J}$ are called open sets in X with respect to the topology $\mathfrak{J}$.

Obviously, we have,

(i) $\emptyset$ and X are open sets in X w.r.t. any topology $\mathfrak{J}$ on X.

(ii) Intersection of finite number of open sets in a T-space is an open set.

(iii) Union of arbitrary number of open sets in a T-space is an open set.

(iv) Every subset of X is open in X w.r.t. the topology $\mathfrak{J}$ if and only if the $\mathfrak{J}$ is a discrete topology on X.

§2 The set of all topologies on X ($\neq \emptyset$)

Given X ($\neq \emptyset$), there always exists a topology on X viz. the discrete topology or the indiscrete topology. Hence, every non-empty set can be considered as a T-space.

The collection $\mathcal{K}$ of all topologies defined on a non-empty set X is surely non-empty and is partially ordered set (poset in short) under the partial ordering relation $\leq$ defined by $\mathfrak{J}_1 \leq \mathfrak{J}_2$ if and only if $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$, for $\mathfrak{J}_1, \mathfrak{J}_2 \in \mathcal{K}$. The poset $\langle \mathcal{K}, \leq \rangle$ is a bounded poset with indiscrete topology as the smallest element and discrete topology as the greatest element.

If $\mathfrak{J}_1, \mathfrak{J}_2 \in \mathcal{K}$, then $\mathfrak{J}_1 \cap \mathfrak{J}_2 \in \mathcal{K}$. Actually, $\cap \{\mathfrak{J}_\alpha \mid \mathfrak{J}_\alpha \in \mathcal{K}\}$ for any $\mathfrak{J}_\alpha \in \mathcal{K}$.
Thus $\mathcal{K}$ is closed for arbitrary intersection and contains the greatest element. Hence $\mathcal{K}$ forms a Moore family of subsets of X (A family of subsets of a non empty set X is said to form a Moore

family of subsets of X if it is closed for arbitrary intersection and contains X). But $\mathfrak{J}_1 \cup \mathfrak{J}_2$ need not be a topology on X for $\mathfrak{J}_1, \mathfrak{J}_2 \in \mathcal{K}$. For $\mathfrak{J}_1, \mathfrak{J}_2 \in \mathcal{K}$, define

$$\mathfrak{J} = \bigcap_{\alpha} \{\mathfrak{J}_\alpha \mid \mathfrak{J}_\alpha \text{ is a topology on } X \text{ such that } \mathfrak{J}_1 \subseteq \mathfrak{J}_\alpha \text{and } \mathfrak{J}_2 \subseteq \mathfrak{J}_\alpha\}.$$

Then $\mathfrak{J}$ is the smallest topology on X containing both $\mathfrak{J}_1$ and $\mathfrak{J}_2$.

This topology $\mathfrak{J}$ is called the topology generated by $\mathfrak{J}_1$ and $\mathfrak{J}_2$.

The set $\langle \mathcal{K}, \wedge, \vee \rangle$ is a complete lattice with $\mathfrak{J}_1 \wedge \mathfrak{J}_2 = \mathfrak{J}_1 \cap \mathfrak{J}_2$ and $\mathfrak{J}_1 \vee \mathfrak{J}_2 =$ topology generated by $\mathfrak{J}_1$and $\mathfrak{J}_2$. Note that $\mathfrak{J}_1 \cup \mathfrak{J}_2$ is a topology on X if $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$or $\mathfrak{J}_2 \subseteq \mathfrak{J}_1$.

§3 Topological spaces and metric spaces

Theorem 3.1 :-Let $\langle X, d\rangle$ be a metric space.

For $x \in X$ and $r > 0$, $S(x,r) = \{y \in X \mid d(x,y) < r\}$.

Define $\mathfrak{J}_d = \{A \subseteq X \mid \forall x \in A \; \exists \; r > 0$ such that $S(x,r) \subseteq A\} \cup \{\emptyset\}$.

Then $\mathfrak{J}_d$ is a topology on X.

Proof :- (i) $\emptyset \in \mathfrak{J}_d$ (by definition). For $x \in X$, $S(x,1) \subseteq X$. Hence $X \in \mathfrak{J}_d$.

(ii) Let $A, B \in \mathfrak{J}_d$. If $A \cap B = \emptyset$, then $A \cap B \in \mathfrak{J}_d$. Let $A \cap B \neq \emptyset$. Then for $x \in A \cap B$ we get $x \in A$ and $x \in B$.

$x \in A \Longrightarrow \exists \; r_1 > 0$ such that $S(x,r_1) \subseteq A$. $x \in B \Longrightarrow \exists \; r_2 > 0$ such that $S(x,r_2) \subseteq B$. Select $r = \min(r_1, r_2)$. Then $S(x,r) \subseteq S(x,r_1)$ and $S(x,r) \subseteq S(x,r_2)$. Hence $S(x,r) \subseteq A \cap B$. Thus given $x \in A \cap B$, $\exists \; r > 0$ such that $S(x,r) \subseteq A \cap B$. Hence $A \cap B \in \mathfrak{J}_d$. Thus $A, B \in \mathfrak{J}_d$implies $A \cap B \in \mathfrak{J}_d$.

(iii) Let $A_\lambda \in \mathfrak{J}_d$, for each $\lambda \in \Lambda$ (where Λ is any indexing set).

If $\bigcup_{\lambda \in \Lambda} A_\lambda = \emptyset$, then by definition of $\mathfrak{J}_d$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}_d$.

Let $\bigcup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Then $x \in \bigcup_{\lambda \in \Lambda} A_\lambda \Longrightarrow x \in A_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

As $A_{\lambda_0} \in \mathfrak{J}_d \; \exists \; r > 0$ such that $S(x,r) \subseteq A_{\lambda_0}$ and hence $S(x,r) \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$.

Thus given $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$, $\exists \; r > 0$ such that $S(x,r) \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$.

Hence $A_\lambda \in \Im_d$, for each $\lambda \in \Lambda \Longrightarrow \bigcup_{\lambda \in \Lambda} A_\lambda \in \Im_d$.

From (i), (ii) and (iii) $\Im_d$ is a topology on X.

Hence $\langle X, \Im_d \rangle$ is a topological space.

This topology $\Im_d$ is called the topology induced by the metric d on X.

Remarks:

(1) Every open set in a metric space $\langle X, d \rangle$ is an open set in T-space $\langle X, \Im_d \rangle$.
Obviously for any $x \in X$ and $r > 0$, $S(x, r) \in \Im_d$.

(2) Every metric d on X $(\neq \emptyset)$ induces a topology $\Im$ on X.

Example : Let d be a discrete metric on $X(\neq \emptyset)$ i.e.

$$d(x, y) = \begin{cases} 1 & if \quad x \neq y \\ 0 & if \quad x = y \end{cases}$$

As $S(x, 1) = \{x\}$, we get $\{x\} \in \Im_d$ for each $x \in X$. Hence $\Im_d$ is the discrete topology on X. Thus discrete metric on X induces the discrete topology on X.

Example : Let d denote usual metric on $\mathbb{R}$ i.e. $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Then $S(x, r) = (x - r, x + r)$ for each $x \in \mathbb{R}$ and $r > 0$, Hence $\Im_d = \Im_u$ (by definition of $\Im_d$ and $\Im_u$) This shows that the usual topology on $\mathbb{R}$ is same as the topology induced by induced usual metric on $\mathbb{R}$.

Definition 3.2: A T-space $\langle X, \Im \rangle$ is said to be metrizable if there exists a metric d on X such that $\Im_d = \Im$.

Examples :

(1) $\langle \mathbb{R}, \Im_u \rangle$ is a metrizable space.

(2) Discrete topological space is a metrizable space.

Remarks: Every topological space need not be a metrizable space.

For this, consider the following topological spaces.

(1) The topological space $\langle X, \mathfrak{I} \rangle$ where $X = \{a, b\}$ and $\mathfrak{I} = \{\emptyset, \{a\}, X\}$. This topological space $\langle X, \mathfrak{I} \rangle$ is not metrizable. Let there exist a metric d on X such that $\mathfrak{I}_d = \mathfrak{I}$. As $a \neq b$, $d(a, b) > 0$. For $r = d(a, b)$, $S(b, r) = \{b\}$. Thus $\{b\} \in \mathfrak{I}_d$ but $\{b\} \notin \mathfrak{I}_d$.
Hence a contradiction. This shows that $\langle X, \mathfrak{I} \rangle$ is not a metrizable space.

(2) Co-finite topological space $\langle \mathbb{N}, \mathfrak{I} \rangle$ is not a metrizable. $\langle \mathbb{N}, \mathfrak{I} \rangle$ be a co-finite topological space. Assume that there exists a metric d on $\mathbb{N}$ such that $\mathfrak{I}_d = \mathfrak{I}$.
Then $S(x, 1) = \{x\} \in \mathfrak{I}_d = \mathfrak{I}$, which is a contradiction, as $\mathbb{N} - \{x\}$ is not finite.

(3) Indiscrete topological space $\langle X, \mathfrak{I} \rangle$ with $|X| > 1$, is not a metrizable space.
Let $|X| > 1$ and let $\mathfrak{I}$ be indiscrete topology on X. If possible assume that there exists a metric d on X such that $\mathfrak{I}_d = \mathfrak{I}$. Select $x, y \in X$ such that $x \neq y$ (this possible as $|X| > 1$). Hence $d(x, y) = r > 0$. Then $S(x, r) \neq \emptyset$ as $x \in S(x, r)$. Thus $S(x, r) \in \mathfrak{I}_d$ but $S(x, r) \notin \mathfrak{I} = \{\emptyset, X\}$. This contradicts $\mathfrak{I} = \mathfrak{I}_d$. Thus there does not exist any metric d on X such that $\mathfrak{I} = \mathfrak{I}_d$. ($\mathfrak{I}$ = indiscrete topology).

Exercises

(1) List four distinct topologies on
(i) $X = \{a, b, c, d\}$ (ii) $X = \{1,2,3\}$

(2) Show that in a co-finite (co-countable) topological space $\langle X, \mathfrak{I} \rangle$,

$$\bigcap \{G \in \mathfrak{I} \mid x \in G\} = \{x\} \text{ for any } x \in X.$$

(3) Show that no two (non-empty) open sets in a co-finite topological space are disjoint.

(4) Define a metrizable space. Show that every metric d defined on X induces a topology on X.

(5) Prove or disprove:
(i) Union of two topologies defined on the same non-empty set X is a topology on X.
(ii) Every topological space is metrizable.
(iii)The set of all topologies defined on a non-empty set X is a complete lattice.

(6) Give four different topologies on $\mathbb{R}$.

(7) Show that the co-finite topology on a finite set is the discrete topology.

(8) Show that the co-countable topology on a countable set is the discrete topology.

(9) Let $\langle X, \mathfrak{J} \rangle$ be a topological space and $A \subseteq X$. Show that $\{U \cup (V \cap A) \mid U, V \in \mathfrak{J}\}$ is a topology on X .

(10) Let $X = \{1,2,3\}$. $\mathfrak{J}_1 = \{\emptyset, X, \{2\}, \{2,3\}\}$ and $\mathfrak{J}_2 = \{\emptyset, X, \{2\}, \{1,3\}\}$. Find the smallest topology on X containing $\mathfrak{J}_1$ and $\mathfrak{J}_2$ and the largest topology contained in $\mathfrak{J}_1$ and $\mathfrak{J}_2$.

(11) Prove or disprove: $\mathfrak{J} = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}$ is a topology on $\mathbb{R}$.

(12) Find the mutually non-comparable topologies on $X = \{p, q, r\}$.

(13) Let $X \neq \emptyset$ and $A \subseteq X$. Show that the family of all subsets of X which contain A together with the empty set $\emptyset$ is a topology on X. Discuss the special cases

(i) $A = \emptyset$ (ii) $A = X$

(14) Prove or disprove:

(1) Every topological space is metrizable.

(2) Any metric defined on X $(\neq \emptyset)$ induces a topology on X.

(15) Show that the usual metric on $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ induces the discrete topology on X.

(16) Prove that usual metric on $\mathbb{R}$ induces usual topology on $\mathbb{R}$.

(17) Let $X = \{a, b, c, d. e\}$ and $\mathcal{K} = \{\{a\}, \{c, d\}, \{a, b, c\}\}$. Find the topology $\mathfrak{J}$ on X generated by the family $\mathcal{K}$.

Unit 2

Bases and Subspaces

Unit 2: Bases and Subspaces

§1 Base for a topology

Definition 1.1: Let $\langle X, \mathfrak{I} \rangle$ be a topological space and let $\mathfrak{B} \subseteq \mathfrak{I}$. $\mathfrak{B}$ is a base for $\mathfrak{I}$ if members of $\mathfrak{I}$ can be can be expressed as a union of members of $\mathfrak{B}$ or equivalently for each and each $x \in G$ there exists $B \in \mathfrak{B}$ such that $x \in B \subseteq G$.

The members of the base $\mathfrak{B}$ are called basic open sets.

Examples 1.2:

(1) Let $\langle X, \mathfrak{I} \rangle$ be a discrete topological space. $\mathfrak{B} = \{\{x\} \mid x \in X\}$ is base for $\mathfrak{I}$.

(2) Let $X = \{a, b, c, d\}$ and $\mathfrak{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\langle X, \mathfrak{I} \rangle$ is a T-space and $\mathfrak{B} = \{\{a\}, \{b\}, \{c, d\}\}$ is a base for $\mathfrak{I}$.

(3) For the T-space $\langle \mathbb{R}, \mathfrak{I}_u \rangle$, $\mathfrak{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$ is a base for $\mathfrak{I}_u$. Obviously $\mathfrak{B} \subseteq \mathfrak{I}_u$. Select $G \in \mathfrak{I}_u$, and $x \in G$. As $G \in \mathfrak{I}_u$, for $x \in G$, $\exists\, r > 0$ such that $(x - r, x + r) \subseteq G$ (by definition of $\mathfrak{I}_u$). As $(x - r, x + r) \in \mathfrak{B}$ and $x \in (x - r, x + r) \subseteq G$ we get $\mathfrak{B}$ is a base for $\mathfrak{I}_u$.

§2 Characterizations of bases

Theorem 2.1: Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two topologies on a set X having bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Then $\mathfrak{I}_1 \leq \mathfrak{I}_2$ if and only if every member $\mathfrak{B}_1$ of can be expressed as a union of some members of $\mathfrak{B}_2$.

Proof : Only if part.

Let $\mathfrak{I}_1 \leq \mathfrak{I}_2$. As $\mathfrak{B}_1 \subseteq \mathfrak{I}_1$ we get $\mathfrak{B}_1 \subseteq \mathfrak{I}_2$. As $\mathfrak{B}_2$ is base for topology $\mathfrak{I}_2$, each member of $\mathfrak{B}_1$ being member of $\mathfrak{I}_2$, can be expressed as union of some members of $\mathfrak{B}_2$.

If part.

By the given condition, each member of $\mathfrak{B}_1$can be expressed as union of some members of $\mathfrak{B}_2$. As $\mathfrak{B}_1$ is a base for $\mathfrak{I}_1$, each member of $\mathfrak{I}_1$ is expressed as union some members of $\mathfrak{B}_1$ and hence each member $\mathfrak{I}_1$ of is expressed as union of some members of $\mathfrak{B}_2$. As $\mathfrak{B}_2 \subseteq \mathfrak{I}_2$ we get each member of $\mathfrak{I}_1$ is a member of $\mathfrak{I}_2$ also. Hence $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ i.e. $\mathfrak{I}_1 \leq \mathfrak{I}_2$.

Note that not every family of subsets of X will form a base for some topology on X.

The necessary and sufficient condition for $\mathfrak{B} \subseteq \wp(X)$ to be a base for some topology $\mathfrak{J}$ on X is given in following theorem.

Theorem 2.2: Let X be non-empty set and $\mathfrak{B} \subseteq \wp(X)$. $\mathfrak{B}$ is a base for some topology on X if and only if it satisfies the following conditions :

(i) $X = \bigcup\{B \mid B \in \mathfrak{B}\}$ and (ii) for $B_1, B_2 \in \mathfrak{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathfrak{B}$ such that $x \in B \subseteq B_1 \cap B_2$ (i.e. $B_1 \cap B_2$ is expressed as union of members of $\mathfrak{B}$)

Proof : Only if part

Let $\mathfrak{B}$ be base for some topology $\mathfrak{J}$ on X. Then $X \in \mathfrak{J} \implies X = \bigcup\{B \mid B \in \mathfrak{B}\}$. Let $B_1, B_2 \in \mathfrak{B}$. Then as $\mathfrak{B} \subseteq \mathfrak{J}$ we get $B_1, B_2 \in \mathfrak{J}$ and hence $B_1 \cap B_2 \in \mathfrak{J}$. $\mathfrak{B}$ being a base for topology for each $x \in B_1 \cap B_2$, $\exists\, B \in \mathfrak{B}$ such that $x \in B \subseteq B_1 \cap B_2$. Thus both conditions are satisfied.

If part

Let $\mathfrak{B} \subseteq \wp(X)$ and let $\mathfrak{B}$ satisfy the given two conditions (i) and (ii). To prove that $\mathfrak{B}$ is a base for some topology $\mathfrak{J}$ on X. Define

$$\mathfrak{J} = \{\emptyset\} \cup \{A \subseteq X \mid A \text{ is union of some members of } \mathfrak{B}\}$$

(1) $\emptyset \in \mathfrak{J}$ and $X \in \mathfrak{J}$ (by condition (i)).

(2) Let $G, H \in \mathfrak{J}$. If $G \cap H = \emptyset$, then $G \cap H \in \mathfrak{J}$. Let $G \cap H \neq \emptyset$.

$x \in G \cap H \implies x \in G$ and $x \in H$. As $G, H \in \mathfrak{J}$ $\exists\, B_1, B_2 \in \mathfrak{B}$ such that $x \in B_1 \subseteq G$ and $x \in B_2 \subseteq H$ (by definition of $\mathfrak{J}$). Thus $x \in B_1 \cap B_2$ and $B_1, B_2 \in \mathfrak{B}$. Hence by condition (ii), there exists $B \in \mathfrak{B}$ such that $x \in B \subseteq B_1 \cap B_2$. This shows that for any $x \in G \cap H$, $\exists\, B \in \mathfrak{B}$ such that $x \in B \subseteq G \cap H$. Hence

$$G \cap H = \bigcup_{x \in G \cap H} \{x\} \subseteq \bigcup_{B \in \mathfrak{B}} B \subseteq G \cap H$$

i.e. $G \cap H$ is union of some members of $\mathfrak{B}$. Hence $G \cap H \in \mathfrak{J}$.

(3) Let $A_\lambda \in \mathfrak{J}$, $\forall \lambda \in \Lambda$, where Λ is any indexing set. Then obviously $\bigcup_{\lambda \in \Lambda} A_\lambda$ is union of some members of . Hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

From (1), (2) and (3), $\mathfrak{J}$ is a topology on X. Now as $\mathfrak{B} \subseteq \mathfrak{J}$ and each member of is expressed as union of members of $\mathfrak{B}$, we get $\mathfrak{B}$ is a base for this topology $\mathfrak{J}$ on X.

Corollary 2.3: If $\mathfrak{B}$ is a family of subsets of X ($\neq \emptyset$) such that

(i) $X = \bigcup\{B \mid B \in \mathfrak{B}\}$ and

(ii) $B_1, B_2 \in \mathfrak{B} \implies B_1 \cap B_2 \in \mathfrak{B}$.

Then $\mathfrak{B}$ is a base for topology $\mathfrak{J}$ on X.

§3 Solved problems

Problem 1: Show that $\{[a, b] \mid a < b,\ \ a, b \in \mathbb{R}\}$ will not form base for any topology $\mathfrak{J}$ on $\mathbb{R}$.

Solution : Let $\mathfrak{J}$ be a topology on $\mathbb{R}$ for which $\mathfrak{B} = \{[a, b] \mid a < b,\ \ a, b \in \mathbb{R}\}$ is a base for $\mathfrak{J}$. As $\mathfrak{B} \subseteq \mathfrak{J}$, $[1\,,2] \cap [2\,,3] \in \mathfrak{J} \implies \{2\} \in \mathfrak{J}$. But $\{2\}$ cannot be expressed a union of members of $\mathfrak{B}$; a contradiction. Hence there does not exists any topology $\mathfrak{J}$ on $\mathbb{R}$ for which $\mathfrak{B}$ is a base.

Problem 2 : Let $\langle X, \mathfrak{J}\rangle$ be a topological space and $\mathfrak{B}$ be a base for $\mathfrak{J}$. If $\mathfrak{J}\,'$ is a topology on X with same base $\mathfrak{B}$ then $\mathfrak{J} = \mathfrak{J}\,'$.

Solution : Let $G \in \mathfrak{J}$. Then by definition of base $G = \bigcup\{B \mid B \in \mathfrak{B}, B \subseteq G\}$. As $\mathfrak{B}$ is a base for $\mathfrak{J}\,'$ also, $\mathfrak{B} \subseteq \mathfrak{J}\,'$ and hence $G \in \mathfrak{J}\,'$. Thus $\mathfrak{J} \subseteq \mathfrak{J}\,'$. Similarly we can prove $\mathfrak{J}\,' \subseteq \mathfrak{J}$. Hence $\mathfrak{J} = \mathfrak{J}\,'$.

Problem 3 : Let $\langle X, \mathfrak{J}\rangle$ be a discrete topological space. Let $\mathfrak{B} = \big\{\{x\} \mid x \in X\big\}$. Show that any family $\mathfrak{B}^*$ (of subsets of X) is a base for $\mathfrak{J}$ if and only if $\mathfrak{B} \subseteq \mathfrak{B}^*$.

Solution : Only if part –

Let $\mathfrak{B}^*$ be a base for $\mathfrak{J}$. $\{x\} \in \mathfrak{J}$ and $x \in \{x\}$. Hence by definition of base $\exists\, B^* \in \mathfrak{B}^*$ such that $x \in B^* \subseteq \{x\}$. But then $B^* = \{x\}$. As this is true for each $x \in X$ we get $\mathfrak{B} \subseteq \mathfrak{B}^*$.

If part – Let $\mathfrak{B} \subseteq \mathfrak{B}^*$. As $\mathfrak{J} = \wp(X)$, $\mathfrak{B}^* \subseteq \mathfrak{J}$. ____________________ (1)

As $G \in \mathfrak{J}$. Then $= \bigcup_{x \in G} \{x\}$.

As $\{x\} \in \mathfrak{B}^*$, $\forall\, x \in G$ we get, G is union of members of $\mathfrak{B}^*$__________ (2)

Hence from (1) and (2) $\mathfrak{B}^*$ is base for $\mathfrak{J}$.

Problem 4 : Let $\mathfrak{J}$ and $\mathfrak{J}^*$ be any two topologies on X $(\neq \emptyset)$ with $\mathfrak{B}$ and $\mathfrak{B}^*$ as bases. If each $G \in \mathfrak{J}$ is union of members of $\mathfrak{B}^*$, then show that $\mathfrak{J} \leq \mathfrak{J}^*$.

Solution : As $\mathfrak{B}^* \subseteq \mathfrak{J}^*$, each $G \in \mathfrak{J}$ is union of members of $\mathfrak{J}^*$ and $\mathfrak{J}^*$ being a topology on X, $G \in \mathfrak{J}^*$. This shows that $\mathfrak{J} \leq \mathfrak{J}^*$.

Problem 4: Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two T-spaces.

Let $\mathfrak{B} = \{G_1 \times G_2 |\, G_1 \in \mathfrak{J}_1 \text{ and } G_2 \in \mathfrak{J}_2\}$

Then show that $\mathfrak{B}$ is a base for some topology on $X \times Y$.

Proof : Obviously $\mathfrak{B}$ is a family of subsets of $X \times Y$. As $X \in \mathfrak{J}_1$ and $Y \in \mathfrak{J}_2$, we get

$X \times Y = \bigcup\{B |\, B \in \mathfrak{B}\}$ _______ (1)

Further let $G_1 \times G_2 \in \mathfrak{B}$, $H_1 \times H_2 \in \mathfrak{B}$ and $(x, y) \in (G_1 \times G_2) \cap (H_1 \times H_2)$.

Then $(x, y) \in (G_1 \cap G_2) \times (H_1 \cap H_2)$. As $G_1 \cap G_2 \in \mathfrak{J}_1$ and $H_1 \times H_2 \in \mathfrak{J}_2$ we get

$(G_1 \cap G_2) \times (H_1 \cap H_2) \in \mathfrak{B}$. Thus $(x, y) \in (G_1 \cap G_2) \times (H_1 \cap H_2) = (G_1 \times H_1) \cap (G_2 \times H_2)$.

This shows that both the conditions (i) and (ii) of the theorem 1.4.3 are satisfied. Hence $\mathfrak{B}$ is base for some topology $\mathfrak{J}$ on $X \times Y$.

Definition : The topology $\mathfrak{J}$ defined on $X \times Y$ for which $\mathfrak{B} = \{G \times H \mid G \in \mathfrak{J}_1 \text{ and } H \in \mathfrak{J}_2\}$ is called the product topology on $X \times Y$ and the T-space $\langle X \times Y, \mathfrak{J} \rangle$ is called product space, where $\mathfrak{J}$ is product topology on $X \times Y$..

§4 Sub-base – Definition and Examples.

Definition 4.1: A family S of subsets of X is said to be a sub-base for the topology $\mathfrak{J}$ on X if the family of all finite intersections of members of $\mathfrak{J}$ is base for $\mathfrak{J}$.

Examples 4.2:

(1) Every base for topology $\mathfrak{I}$ is obviously a sub-base for $\mathfrak{I}$.

(2) $\{A \subseteq \mathbb{R} \mid A = (a, \infty)\ or\ A = (-\infty, b)\ for\ a, b\ \in \mathbb{R}\}$ is a sub-base for the usual topology $\mathfrak{I}_u$ on $\mathbb{R}$.

§5 Subspace of a topological space

Theorem 5.1: Let $\langle X, \mathfrak{I}\rangle$ be a T-space and let Y be any non-empty subset of X. Define $\mathfrak{I}^* = \{G \cap Y \mid G \in \mathfrak{I}\}$. Then $\mathfrak{I}^*$ is a topology on Y.

Proof : (i) $\emptyset \in \mathfrak{I} \implies \emptyset \cap Y = \emptyset \in \mathfrak{I}^*$.

$X \in \mathfrak{I} \implies X \cap Y = Y \in \mathfrak{I}^*$.

(ii) Let $G^*, H^* \in \mathfrak{I}^*$. Then $G^* = G \cap Y$ and $H^* = H \cap Y$ for some $G, H \in \mathfrak{I}$. Hence $G^* \cap H^* = (G \cap H) \cap Y$. As $(G \cap H) \in \mathfrak{I}$ we get $G^* \cap H^* \in \mathfrak{I}^*$.

(iii) Let ${G_\lambda}^* \in \mathfrak{I}^*$, $\forall\, \lambda \in \Lambda$, where Λ is any indexing set. Then ${G_\lambda}^* = G_\lambda \cap Y$ for some $G_\lambda \in \mathfrak{I}$.

$$\bigcup_{\lambda \in \Lambda} {G_\lambda}^* = \bigcup_{\lambda \in \Lambda} (G_\lambda \cap Y) = \left[\bigcup_{\lambda \in \Lambda} G_\lambda\right] \cap Y$$

As $\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{I}$, we get $\bigcup_{\lambda \in \Lambda} {G_\lambda}^* \in \mathfrak{I}^*$.

From (i), (ii) and (iii) we get $\mathfrak{I}^*$ is a topology on Y. Hence $\langle Y, \mathfrak{I}^*\rangle$ is a T-space.

Definition 5.2: This topology $\mathfrak{I}^*$ on Y is called relative topology on Y and the T-space $\langle Y, \mathfrak{I}^*\rangle$ is called the subspace of T- space $\langle X, \mathfrak{I}\rangle$.

Note that a subset$A \subseteq Y$ is open in $\langle Y, \mathfrak{I}^*\rangle$ if and only if $A = G \cap Y$ for some open set G in $\langle X, \mathfrak{I}\rangle$.

Examples 5.3:

(1) Let $\langle X, \mathfrak{I}\rangle$ be a T-space where $X = \{a, b, c, d\}$ and $\mathfrak{I} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. If $Y = \{b, c, d\}$ then the relative topology $\mathfrak{I}^*$ on Y is given by $\mathfrak{I}^* = \{\emptyset, \{b, c\}, Y\}$

(2) Let $\langle X, \mathfrak{I}\rangle$ be any T-space. Let $Y = \{a\}$ for some $a \in X$. Then the relative topology $\mathfrak{I}^*$ on Y is the indiscrete topology on Y as $\mathfrak{I}^* = \big\{\emptyset, \{a\}\big\}$.

(3) $\mathbb{N} \subseteq \mathbb{R}$. $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a T-space. The relative topology $\mathfrak{J}^*$ on $\mathbb{N}$ is a discrete topology on $\mathbb{N}$ as for any $n \in \mathbb{N}$, $\{n\} = \left(n - \frac{1}{2}, n + \frac{1}{2}\right) \cap \mathbb{N} \in \mathfrak{J}^*$. Similarly we can prove that the relative topology of $\mathfrak{J}_u$ to $\mathbb{Z}$ is the discrete topology.

(4) Any subspace of a discrete (indiscrete) T-space is a discrete (indiscrete) T-space.

Theorem 5.4: Let $\langle X, \mathfrak{J} \rangle$ be a T-space and $Z \subset Y \subset X$. Denote $\mathfrak{J}/_Y$, the relative topology on Y induced by $\mathfrak{J}$. Show that $(\mathfrak{J}/Y)/_Z = \mathfrak{J}/_Z$.

Proof: We have $\mathfrak{J}/_Z = \{G \cap Z \mid G \in \mathfrak{J}\}$ and $\mathfrak{J}/_Y = \{G \cap Y \mid G \in \mathfrak{J}\}$.

Then
$$
\begin{aligned}
(\mathfrak{J}/Y)/_Z &= \left\{G^* \cap Z \mid G^* \in \mathfrak{J}/_Y\right\} \\
&= \{(G \cap Y) \cap Z \mid G^* = (G \cap Y),\ G \in \mathfrak{J}\} \\
&= \{G \cap (Y \cap Z) \mid\ G \in \mathfrak{J}\} \\
&= \{G \cap Z \mid\ G \in \mathfrak{J}\} \qquad \ldots\ldots\ldots (\because Z \subset Y) \\
&= \mathfrak{J}/_Z
\end{aligned}
$$

Remark: Let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of $\langle X, \mathfrak{J} \rangle$. For each subset open in the subspace $\langle Y, \mathfrak{J}^* \rangle$ to be open in $\langle X, \mathfrak{J} \rangle$, it is necessary and sufficient that Y is open in X.

For this consider the T-space $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ and $Y = [0,1]$. Then Y is not open in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$. $\left[0, \frac{1}{2}\right)$ is open in Y as $\left[0, \frac{1}{2}\right) = \left(\frac{-1}{2}, \frac{1}{2}\right) \cap Y = \left(\frac{-1}{2}, \frac{1}{2}\right) \cap [0,1]$ and $\left(\frac{-1}{2}, \frac{1}{2}\right) \in \mathfrak{J}_u$. But $\left[0, \frac{1}{2}\right)$ is not open in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$.

Theorem 5.5: Let $\langle X, \mathfrak{J} \rangle$ be a T-space and let $\mathfrak{B}$ be a base for $\mathfrak{J}$. If $\langle Y, \mathfrak{J}^* \rangle$ is a subspace of $\langle X, \mathfrak{J} \rangle$, then $\mathfrak{B}^* = \{B \cap Y \mid B \in \mathfrak{B}\}$ is a base for $\mathfrak{J}^*$.

Proof : $\mathfrak{B}$ is a base for $\mathfrak{J} \Rightarrow \mathfrak{B} \subseteq \mathfrak{J} \Rightarrow \mathfrak{B}^* \subseteq \mathfrak{J}^*$ ________________________(i)

Let $G^* \in \mathfrak{J}^*$ and $y \in G^*$. As $G^* \in \mathfrak{J}^*$, $G^* = G \cap Y$ for some $G \in \mathfrak{J}$. As $y \in G^*$, we get $y \in G \cap Y$. As $\mathfrak{B}$ is a base for $\mathfrak{J}$, $y \in G$ and $G \in \mathfrak{J}$ will imply $y \in B \subseteq G$ for some $B \in \mathfrak{B}$. But then $y \in B \cap Y \subseteq G \cap Y = G^*$. Define $B^* = B \cap Y$. Then $B^* \in \mathfrak{B}^*$.

Thus for $G^* \in \mathfrak{J}^*$ and $y \in G^*$ $\exists$ $B^* \in \mathfrak{B}^*$ such that $y \in B^* \subseteq G^*$________________(ii)

Hence from (i) and (ii) $\mathfrak{B}^*$ is a base for $\mathfrak{J}^*$.

Definition 5.6: A property of a topological space is said to be hereditary if every subspace of the space has that property.

Examples 5.7:

(1) The property of a topological space being a discrete space is a hereditary property.

(2) A property of a topological space being a indiscrete space is a hereditary property.

(3) Metrisability is a hereditary property i.e. subspace of a metrizable space is metrizable space.

Proof: Let a T-space $\langle X, \mathfrak{J} \rangle$ be a metrizable. Hence $\exists$ a metric d defined on X such that the induced topology $\mathfrak{J}_d$ by the metric d coincides with $\mathfrak{J}$. $d: X \times X \longrightarrow \mathbb{R}$ and $Y \subseteq X$. Restrict d to $Y \times Y$ and denote it by d_1. Then $d_1: Y \times Y \longrightarrow \mathbb{R}$.

For any $y \in Y$, $S(y, r)$ in Y = ($S(y, r)$ in X) $\cap$ Y __________________ (I)

The base for topology $\mathfrak{J}_d = \mathfrak{J}$ is given by $\{S(x, r) \mid x \in X \text{ and } r > 0\}$. From theorem 5.5, $\{S(x, r) \cap Y \mid x \in X \text{ and } r > 0\}$ will be a base for $\mathfrak{J}^*$, where $\mathfrak{J}^*$ denotes the relative topology on Y. Thus by (I) $\{S(y, r) \mid y \in Y \text{ and } r > 0\}$ is a base for $\mathfrak{J}^*$. As the base for the topology $\mathfrak{J}^*$ and the topology $\mathfrak{J}_{d_1}$ are the same we get $\mathfrak{J}^* = \mathfrak{J}_{d_1}$. This shows that for the relative topology $\mathfrak{J}^*$ on $Y \subseteq X$, $\exists$ a metric d_1 on Y such that $\mathfrak{J}_{d_1} = \mathfrak{J}^*$. Hence the subspace $\langle Y, \mathfrak{J}^* \rangle$ is metrizable. Thus subspace of a metrizable space is metrizable.

Remark : There are some properties of a topological $\langle X, \mathfrak{J} \rangle$ which are not hereditary, e.g. compactness or connected which we will study in unit 6 and 7.

Exercises

1) Let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of $\langle X, \mathfrak{J} \rangle$. Consider the following statements :

(i) $\langle X, \mathfrak{J} \rangle$ is discrete topology $\implies$ $\langle Y, \mathfrak{J}^* \rangle$ is discrete topology.

(ii) $\langle Y, \mathfrak{J}^* \rangle$ is discrete topology $\implies$ $\langle X, \mathfrak{J} \rangle$ is discrete topology.

Which of the statements (i) and (ii) is true? Justify your answer.

Unit 3

Special Subsets

Unit 3: Special Subsets

§1 Derived set of a set

Definition 1.1:- Let $\langle X, \mathfrak{J} \rangle$ be a topological space. Let $A \subseteq X$ and $x \in X$. Then x is a limit point or accumulation point of A if each open set containing x contains a point of A other than x.
i.e. for each open set G containing x, $G \cap A - \{x\} \neq \emptyset$.

Remarks:

(1) $x \in X$ is not a limit point of $A \subseteq X$ if $G \cap A = \emptyset$ or $G \cap A = \{x\}$ for some open set G containing x.

(2) The set of all limit points of A is denoted by $d(A)$ and is called derived set of A.

Examples 1.2:

1) Let $\langle X, \mathfrak{J} \rangle$ be a discrete topological space and let $A \subseteq X$. For any $x \in X$ we get $x \in \{x\}$ and $\{x\} \in \mathfrak{J}$. Hence $\{x\} \cap A = \emptyset$ if $x \notin A$ or $\{x\} \cap A = \{x\}$ if $x \in A$.
Hence $\{x\} \cap A - \{x\} = \emptyset$.Hence, x is not a limit point of A.
Thus no point of X will be a limit point of A. Hence $d(A) = \emptyset$ for each $A \subseteq X$ in a discrete topological space.
2) Let $\langle X, \mathfrak{J} \rangle$ be indiscrete topological space, $A \subseteq X$ and $x \in X$. The only open set containing x is X. Hence $X \cap A - \{x\} = A - \{x\}$.
If $A = \emptyset$, then no point of X will be a limit point of A. Hence $d(A) = \emptyset$.
If $A = \{x\}$, then each point of $X - \{x\}$ will be a limit point of A.
Hence $d(\{x\}) = X - \{x\}$.
If $|A| > 1$, then each point of X will be a limit point of A. Hence $d(A) = X$.
3) Consider the topological space $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ and $\mathbb{N} \subseteq \mathbb{R}$. For $x \in \mathbb{R}$,
$(x - r\ ,\ x + r) \in \mathfrak{J}_u$.
For $r < 1$ we get $(x - r\ ,\ x + r) \cap \mathbb{N} = \emptyset$. Hence x is not a limit point of $\mathbb{N}$.
As this is true for any $x \in \mathbb{R}$, $d(\mathbb{N}) = \emptyset$.
4) Let $X = \{a, b, c\}$ and let $\mathfrak{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Take $A = \{a\}$.

(i) $x = a$. For the open set $\{a\}$ containing a, we get,

$\{a\} \cap A - \{a\} = \{a\} \cap \{a\} - \{a\} = \{a\} - \{a\} = \emptyset$. Hence a is not a limit point of A.

(ii) $x = b$. For the open set $\{b\}$ containing b, we get,

$\{b\} \cap A - \{b\} = \emptyset$. Hence b is not a limit point of A .

(iii) $x = c$. The only open set containing c is X and $X \cap A - \{c\} = \{a\} - \{c\} = \{a\} \neq \emptyset$.

This shows that c is a limit point of A

Hence $d(A) = \{c\}$.

Theorem 1.3: In any topological space $\langle X, \mathfrak{J} \rangle$ we have,

1) $d(\emptyset) = \emptyset$.
2) $A \subseteq B \implies d(A) \subseteq d(B)$, $\forall A, B \subseteq X$.
3) $x \in d(A) \implies x \in d(A - \{x\})$,$\forall A \subseteq X$.
4) $d(A \cup B) = d(A) \cup d(B)$, $\forall A, B \subseteq X$.
5) $d(A \cap B) \subseteq d(A) \cap d(B)$ $\forall A, B \subseteq X$.

Proof: -

(1)Let $x \in X$ and G be any open set containing x. Then $G \cap \emptyset - \{x\} = \emptyset - \{x\} = \emptyset$, shows that no $x \in X$ will be a limit point of $\emptyset$.

Hence $d(\emptyset) = \emptyset$.

(1) Let $x \in d(A)$. If $x \notin d(B)$, then $\exists$ an open set G containing x such that $G \cap B - \{x\} = \emptyset$. As $A \subseteq B$ we get $G \cap A - \{x\} = \emptyset$. Hence $x \notin d(A)$; a contradiction. Thus $x \in d(A) \implies x \in d(B)$, if $A \subseteq B$.

Hence, $A \subseteq B \implies d(A) \subseteq d(B)$.

(2) Let $x \in d(A)$. To prove that $x \in d(A - \{x\})$.

Assume that $x \notin d(A - \{x\})$. Then $\exists$ an open set G containing x such that $G \cap (A - \{x\}) - \{x\} = \emptyset$. But this implies $G \cap (A \cap \{x\}') \cap \{x\}' = \emptyset$ $(\{x\}' = X - \{x\})$ i.e. $G \cap A \cap \{x\}' = \emptyset$ i.e. $G \cap A - \{x\} = \emptyset$. Hence $x \notin d(A)$; a contradiction.

Hence $x \in d(A) \implies x \in d(A - \{x\})$.

(3) To prove that $d(A \cup B) = d(A) \cup d(B)$, $\forall A, B \subseteq X$.

By (2) we get,

$A \subseteq A \cup B \implies d(A) \subseteq d(A \cup B)$ and

$B \subseteq A \cup B \implies d(B) \subseteq d(A \cup B)$.

Hence $d(A) \cup d(B) \subseteq d(A \cup B)$ ____________________________ (I)

Let $x \in d(A \cup B)$. To prove $x \in d(A) \cup d(B)$.

Assume that $x \notin d(A) \cup d(B)$. Therefore $x \notin d(A)$ and $x \notin d(B)$.

$x \notin d(A) \implies \exists\, G \in \mathfrak{J}$ such that $x \in G$ and $G \cap A - \{x\} = \emptyset$.

$x \notin d(B) \implies \exists\, H \in \mathfrak{J}$ such that $x \in H$ and $H \cap B - \{x\} = \emptyset$.

$G, H \in \mathfrak{J} \implies G \cap H \in \mathfrak{J}$ and we get $(G \cap H) \cap A - \{x\} = \emptyset$ and $(G \cap H) \cap B - \{x\} = \emptyset$.

Combining both we get, $(G \cap H) \cap (A \cap B) - \{x\} = \emptyset$.

As $G \cap H \in \mathfrak{J}$ and $x \in G \cap H$, we get $x \notin d(A \cup B)$; a contradiction.

Thus $x \in d(A \cup B) \implies x \in d(A) \cup d(B)$

Hence $d(A \cup B) \subseteq d(A) \cup d(B)$ ____________________________ (II)

From (I) and (II) we get,

$$d(A \cup B) = d(A) \cup d(B)$$

(4) $A \cap B \subseteq A \implies d(A \cap B) \subseteq d(A)$ *and* $A \cap B \subseteq B \implies d(A \cap B) \subseteq d(B)$ Hence we get $d(A \cap B) \subseteq d(A) \cap d(B)$.

§2 Closed sets

Definition 2.1: Let $\langle X, \mathfrak{J} \rangle$ be a topological space and $A \subseteq X$. A is said to be closed if A contains all its limit points i.e. if $d(A) \subseteq A$.

Examples 2.2:

1) $d(\emptyset) = \emptyset \subseteq \emptyset \implies \emptyset$ is closed in $\langle X, \mathfrak{J} \rangle$.
2) $d(X) \subseteq X$ always $\implies X$ is closed in $\langle X, \mathfrak{J} \rangle$.
3) In $\langle \mathbb{R}, \mathfrak{J}_u \rangle$, (a, b) [where $a < b$] is not a closed set as it does not contain its limit point a.
4) In a discrete topological space $\langle X, \mathfrak{J} \rangle$, $(|X| > 1)$, $d(A) = \emptyset$, $\forall\, A \subseteq X$.
   Hence $d(A) \subseteq A$, $\forall\, A \subseteq X$. Hence each subset of X is closed.
5) In an indiscrete topological space the only closed sets are $\emptyset$ and X.

Theorem 2.3: Let F be a closed set in a topological space $\langle X, \mathfrak{J} \rangle$ and let $x \notin F$. Then $\exists$ an open set G such that $x \in G \subseteq X - F$.

Proof: F is closed set $\Longrightarrow d(F) \subseteq F$.

$x \notin F \;\Longrightarrow x \notin d(F)$

$\Longrightarrow x$ is not a limit point of F.

$\Longrightarrow\; \exists$ an open set G containing x such that $G \cap F - \{x\} = \emptyset$.

$\Longrightarrow\; \exists\, G \in \mathfrak{J}$ such that $x \in G$ and $G \cap F = \emptyset \; or \; G \cap F = \{x\}$.

$\Longrightarrow\; \exists\, G \in \mathfrak{J}$ such that $x \in G$ and $G \cap F = \emptyset$ (as $x \notin F$, $G \cap F \neq \{x\}$).

$\Longrightarrow\; \exists\, G \in \mathfrak{J}$ such that $x \in G$ and $x \in G \subseteq X - F$.

Corollary 2.4: If F is a closed in $\langle X, \mathfrak{J} \rangle$, then $X - F$ is an open set.

Proof: By Theorem 2.3 for each $x \notin F$, $\exists$ an open set G such that $x \in G \subseteq X - F$.

Thus $X - F = \bigcup_{x \in X - F} \{x\} = \bigcup \{G \in \mathfrak{J} \mid x \in G \text{ and } G \subseteq X - F\}$

Thus $X - F = \bigcup \{G \subseteq X - F \mid G \in \mathfrak{J} \text{ such that } x \in G \subseteq X - F\}$

Thus $X - F$ is an arbitrary union of open sets and hence $X - F$ is an open set.

Corollary 2.5: If $X - F$ is an open set in $\langle X, \mathfrak{J} \rangle$, then F is a closed set.

Proof: -To prove that F is a closed set in X i.e. to prove that $d(F) \subseteq F$.

Let us assume that $d(F) \nsubseteq F$. Then $\exists\; x \in d(F)$ such that $x \notin F$.

By Theorem 2.3, $\exists$ an open set G such that $x \in G \subseteq X - F$.

For this open set G containing x, we get $G \cap F - \{x\} = \emptyset - \{x\} = \emptyset$.

This shows that x is not a limit point of F. i.e. $x \notin d(F)$; a contradiction.

Hence, $d(F) \subseteq F$.Therefore F is a closed set.

Corollary 2.6: A set is closed subset of a topological space if and only if its complement is an open subset of the space.

Proof: -From the Corollary 2.4 and Corollary 2.5 the proof follows.

Corollary 2.7: The family $\mathcal{K}$ of all closed subsets in a topological space has the following properties:

1. The intersection of any number of members of $\mathcal{K}$ is member of $\mathcal{K}$.
2. The union of any finite number of members of $\mathcal{K}$ is a member of $\mathcal{K}$.
3. $X \in \mathcal{K}$ and $\emptyset \in \mathcal{K}$.

Theorem 2.8: Let $X \neq \emptyset$ and let $\mathcal{K}$ denotes the family of subsets of X satisfying the conditions in corollary 2.7. Then $\exists$ a unique topology $\mathfrak{J}$ on X for which the family $\mathcal{K}$ will be family of closed subsets of X.

Proof:- Define $\mathfrak{J} = \{X - F \mid F \in \mathcal{K}\}$. Then obviously $\mathfrak{J}$ is a topology on X.

To prove the uniqueness only.

Let $\mathfrak{J}'$ be another topology on X for which $\mathcal{K}$ is family of closed sets in X.

Then $G \in \mathfrak{J} \Leftrightarrow X - G \in \mathcal{K} \Leftrightarrow X - G$ is $\mathfrak{J}$ closed $\Leftrightarrow X - G$ is $\mathfrak{J}'$ closed (since $\mathfrak{J}$ and $\mathfrak{J}'$ have the same family of closed sets) $\Leftrightarrow G$ is open in $\langle X, \mathfrak{J}' \rangle$ i.e. $G \in \mathfrak{J}'$.

Thus $G \in \mathfrak{J} \Leftrightarrow G \in \mathfrak{J}'$.

Hence $\mathfrak{J} = \mathfrak{J}'$ and the uniqueness follows.

Example:- Let $\mathcal{K} = \{N\} \cup \{A \subseteq N \mid A \text{ is a finite set}\}$. Then $\mathcal{K}$ satisfies three conditions of corollary 2.7 and hence can be used to define a topology $\mathfrak{J}$ on N.

(I) $N \in \mathcal{K}$ (by definition) and $\emptyset \in \mathcal{K}$, as $\emptyset$ is always finite.

(II) Let $A, B \in \mathcal{K}$. If $A = N$ or $B = N$, then $A \cup B = N$ and we get $A \cup B \in \mathcal{K}$.
If $A \neq N$ and $B \neq N$, then A is finite and B is finite. Hence $A \cup B$ is finite and therefore $A \cup B \in \mathcal{K}$.
Thus $A, B \in \mathcal{K} \Longrightarrow A \cup B \in \mathcal{K}$.

(III) Let $A_\lambda \in \mathcal{K} \ \forall \lambda \in \Lambda$, where Λ is any indexing set .
Then $\bigcap_{\lambda \in \Lambda} A_\lambda \subseteq A_{\lambda_0}$ for some λ_0.
As A_{λ_0} is a finite set, we get $\bigcap_{\lambda \in \Lambda} A_\lambda \in \mathcal{K}$.

Thus the family $\mathcal{K}$ satisfies all the three conditions of corollary 2.7.

Define $\mathfrak{I} = \{X - F \mid F \in \mathcal{K}\}$. Then obviously $\mathfrak{I}$ is a topology on X for which $\mathcal{K}$ will form family of closed sets.

Remark: It can be observed that in discrete and indiscrete spaces the closed sets are same as open sets. But there also exists some non-trivial topological spaces in which closed sets are same as open sets.

e.g. Consider the topological space $\langle X, \mathfrak{I}\rangle$ where $X = \{a, b, c\}$ and $\mathfrak{I} = \{\emptyset, \{a\}, \{b, c\}, X\}$. The family of closed sets in $\langle X, \mathfrak{I}\rangle$ is $\mathcal{K} = \{\emptyset, \{a\}, \{b, c\}, X\}$, which is same as $\mathfrak{I}$.

Definition 2.9:-A subset in a topological space is said to be clo-open if it is both closed and open in that space.

$\emptyset$ and X are clo-open sets in any topological space.

Remark: Union of finite number of closed sets in a topological space is a closed set, but union of an infinite collection of closed sets in a topological space is not necessarily closed.

For this consider the topological space $\langle \mathbb{R}, \mathfrak{I}_u\rangle$.

Define $F_n = \left[\frac{1}{n}, 1\right]$, $\forall\ n \in \mathbb{N}$.

Then F_n is a closed set in $\langle \mathbb{R}, \mathfrak{I}_u\rangle$, $\forall\ n \in \mathbb{N}$.

As $\bigcup\limits_{n\in\mathbb{N}} F_n = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \left[\frac{1}{4}, 1\right] \cup \ldots\ldots = (0, 1]$, we get $\bigcup\limits_{n\in\mathbb{N}} F_n$ is not a closed set in $\langle \mathbb{R}, \mathfrak{I}_u\rangle$.

§3 Closure of a set

Definition 3.1: The closure of a set A in a topological space $\langle X, \mathfrak{I}\rangle$ is the intersection of all closed subsets of X containing A. This is denoted by $c(A)$ or $\bar{A}$.

Remarks:

(1) $\bar{A}$ is the smallest closed set containing A.

(2) $\bar{A}$ is a closed set(see corollary 1.10) containing A.

(3) A is a closed set if and only if $A = \bar{A}$. Hence $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$.

Theorem 3.2: For any set A in a topological space $\langle X, \mathfrak{J} \rangle$, $\bar{A} = A \cup d(A)$.

Proof:-

I) To prove that $\bar{A} \subseteq A \cup d(A)$.

Let $x \in \bar{A}$. To prove that $x \in A \cup d(A)$.

Assume that $x \notin A \cup d(A)$. Then $x \notin A$ and $x \notin d(A)$.

$x \notin d(A) \implies x$ is not a limit point of A.

$\implies \exists$ an open set G containing x such that $G \cap A = \emptyset$ or $G \cap A = \{x\}$.

As $x \notin A$ we get $G \cap A = \emptyset$. Hence $G \subseteq X - A$.

Further as $G \cap A = \emptyset$ and $G \in \mathfrak{J}$, no limit point of G will be a limit point of A.

But this will imply $G \subseteq X - d(A)$.

Thus $G \subseteq X - A$ and $G \subseteq X - d(A) \implies G \subseteq (X - A) \cap \big(X - d(A)\big)$

$\implies G \subseteq X - [A \cup d(A)]$.

Thus for each $x \notin A \cup d(A)$ i.e. for each $x \in X - [A \cup d(A)]$ there exists an open set G_x such that $x \in G_x$ and $G_x \subseteq X - [A \cup d(A)]$.

Hence $X - [A \cup d(A)] = \displaystyle\bigcup_{x \in X - [A \cup d(A)]} G_x$ is an open set.

Hence $X - [A \cup d(A)]$ is an open set in $\langle X, \mathfrak{J} \rangle$.

Therefore $[A \cup d(A)]$ is a closed set.

Obviously, $A \subseteq A \cup d(A)$. Hence $\bar{A} \subseteq A \cup d(A)$.

II) To prove that $A \cup d(A) \subseteq \bar{A}$.

Let $x \in A \cup d(A)$ *and let B be any closed set containing A.*

If $x \in A$, then $x \in B$ obviously.

If $x \in d(A)$, then $x \in d(B)$ $\big($since $A \subseteq B$ implies $d(A) \subseteq d(B)\big)$ and hence $x \in B$ as B is a closed set. Thus $x \in A \cup d(A)$ implies $x \in B$ for any closed set B.

Hence $x \in \bar{A}$.

This shows that $A \cup d(A) \subseteq \bar{A}$

From (I) and (II) the result follows.

Theorem 3.3: If E is a subset of a subspace $\langle X^*, \mathfrak{J}^* \rangle$ of a topological space $\langle X, \mathfrak{J} \rangle$, then $c^*(E) = X^* \cap c(E)$, where $c(E)$ =closure of E in $\langle X, \mathfrak{J} \rangle$ and $c^*(E)$ = closure of E in $\langle X^*, \mathfrak{J}^* \rangle$

Proof: Let $\mathcal{K}$ and $\mathcal{K}^*$ denote the family of closed sets in X and X^* respectively.

$$X^* \cap c(E) = X^* \cap \left(\bigcap\{F \mid F \in \mathcal{K} \text{ and } E \subseteq F\}\right) = \bigcap\{X^* \cap F \mid F \in \mathcal{K} \text{ and } E \subseteq F\}$$

$$= \bigcap\{F^* \mid F^* \in \mathcal{K}^* \text{ and } E \subseteq F^*\} = c^*(E)\,.$$

Remark: In any topological space $\langle X, \mathfrak{J}\rangle$ we have,

(1) $c(X) = X$

(2) $c(\emptyset) = \emptyset$

(3) $c\big(c(E)\big) = c(E)\quad for\ any\ E \subseteq X.$

(4) $A \subseteq B \implies c(A) \subseteq c(B)$ for any $A, B \subseteq X$.

(5) $c(A \cap B) = c(A) \cap c(B)$ for all $A, B \subseteq X$.

§4 Interior of a set

Definition 4.1: Let $\langle X, \mathfrak{J}\rangle$ be a topological space and $E \subseteq X$. The interior of E is the union of all open sets contained in E.

It is denoted by $i(E)\ or\ E^\circ$

Remarks:

(1) $i(E)$ is an open set in X and is the largest open set contained in E.

(2) E is open in X if and only if $i(E) = E$.

Examples:

(1) Let $\langle X, \mathfrak{J}\rangle$ be an indiscrete T – space with $|X| > 1$. Then for any $E \subset X$ we get $i(E) = \emptyset$.
For $E = X$, $i(E) = X$.

(2) Let $\langle X, \mathfrak{J}\rangle$ be a discrete topological space. $i(E) = E$ for each $E \subseteq X$.

Theorem 4.2: For any set E in a topological space $\langle X, \mathfrak{J}\rangle$, $i(E) = E'^{-\prime}$ (complement of closure of complement of E).

[A' = Complement of A in $\langle X, \mathfrak{J}\rangle$ and $\bar{A}$ = closure of A in $\langle X, \mathfrak{J}\rangle$.]

Proof: To prove $i(E) = E'^{-\prime}$.

Let $x \in i(E)$. Then $i(E)$ is an open set containing x and contained in E.

Hence $i(E) \cap (X - E) = \emptyset$.

Thus, $i(E) \cap (X - E) - \{x\} = \emptyset$.This shows that x is not a limit point of E.

Thus $x \in i(E) \implies x \notin (X - E)$ and $x \notin d[(X - E)]$. Hence $x \notin E' \cup d(E')$ i.e. $x \notin E'^{-}$.But then $x \in E'^{-\prime}$.This shows that $i(E) \subseteq E'^{-\prime}$ ________________ (1)

To prove $E'^{-\prime} \subseteq i(E)$. Let $x \in E'^{-\prime}$.

Then $x \notin E'^{-} \implies x \notin E' \cup d(E')$

$\implies x \notin E'$ and $x \notin d(E')$

$\implies x \in E$ and x is not a limit point of E'.

Hence $\exists$ an open set G in X such that $x \in G$ and $G \cap E' - \{x\} = \emptyset$.

This is possible only when $G \cap E' = \emptyset$ … (since $x \notin E' \implies G \cap E' \neq \{x\}$).

Thus $G \subseteq E$. By the definition of $i(E)$ we get $G \subseteq i(E)$.Hence $x \in i(E)$.

This shows that $E'^{-\prime} \subseteq i(E)$ ____________________________ (2)

From (1) and (2), we get,

$$i(E) = E'^{-\prime} .$$

Remarks.

In any topological space $\langle X, \mathfrak{J} \rangle$, $i(E) =$ Complement of the closure of the complement of E.

$i(E)$ equals the set of all those points of E which are not limit points of

$E' = X - E$.

$c(E)$ *equals the set of* complement of the interior of the complement of E.

Theorem 4.3: In any topological space $\langle X, \mathfrak{J} \rangle$ we have,

(6) $i(X) = X$

(7) $i(\emptyset) = \emptyset$

(8) $i(i(E)) = i(E)$ *for any* $E \subseteq X$.

(9) $A \subseteq B \implies i(A) \subseteq i(B)$ *for any* $A, B \subseteq X$.

(10) $i(A \cap B) = i(A) \cap i(B)$ for all $A, B \subseteq X$.

Proof: We prove the property (5) only.

$A \cap B \subseteq A$ and $A \cap B \subseteq B$.

$\Rightarrow\ i(A \cap B) \subseteq i(A)$ and $i(A \cap B) \subseteq i(B)$

$\Rightarrow\ i(A \cap B) \subseteq i(A) \cap i(B)$ ____ (i)

Let $x \in i(A) \cap i(B)$. Then $x \in i(A)$ and $x \in i(B)$.

$x \in i(A) \implies \exists\, G \in \mathfrak{J}$ such that $x \in G \subseteq A$.

$x \in i(B) \implies \exists\, H \in \mathfrak{J}$ such that $x \in H \subseteq B$.

But then $x \in G \cap H \subseteq A \cap B$.

As $G \cap H \in \mathfrak{J}$ we get $G \cap H \subseteq i(A \cap B)$. Hence $x \in i(A \cap B)$.

This shows that, $i(A) \cap i(B) \subseteq i(A \cap B)$ ________________________________ (ii)

From (i) and (ii), we get,

$$i(A \cap B) = i(A) \cap i(B).$$

Remark: $i(A) \cup i(B) \subseteq i(A \cup B)$ for $A, B \subseteq X$. But $i(A) \cup i(B) \neq i(A \cup B)$ in general. For this consider the topological space $\langle \mathbb{R}, \mathfrak{J}_u \rangle$. Take $A = [0,1)$ and $B = [1,2)$. Then $i(A) = (0,1)$ and $i(B) = (1,2)$. Hence $i(A) \cup i(B) = (0,2) - \{1\}$. $A \cup B = [0,2)$ and $i[A \cup B] = (0,2)$. This shows that $i[A \cup B] \neq i(A) \cup i(B)$.

Definition 4.4: Let $\langle X, \mathfrak{J} \rangle$ be a topological space, $E \subseteq X$ and $x \in X$. x is called an interior point of E if $\exists$ an open set G such that $x \in G \subseteq E$.

Remark: The set of all interior points of E is $i(E)$ for any subset E of a topological space $\langle X, \mathfrak{J} \rangle$.

§5 Exterior of a set

Definition 5.1: Let $\langle X, \mathfrak{J} \rangle$ be a topological space and $E \subseteq X$. The exterior of E is the set of interior points of the complement of E. This is denoted by $e(E)$.

Thus $e(E) = i(E') = i(X - E)$.

Remark: $i(E) = e(E')$.

Theorem 5.2: In any topological space $\langle X, \mathfrak{J} \rangle$ we have,

(1) $e(\emptyset) = X$

(2) $e(X) = \emptyset$

(3) $e(E) \subseteq E' = X - E \; for \; any \;\; E \subseteq X.$

(4) $e(E) = e[X - e(E)] \; for \; any \;\; E \subseteq X.$

(5) $e(A \cup B) = e(A) \cap e(B). \; for \; any \;\; A, B \subseteq X.$

Proof: Proofs of properties (1), (2) and (3) follows directly from the definition.

Proof of (4):

$$\begin{aligned} e[X - e(E)] &= i\big(e(E)\big) \text{ (by the definition)} \\ &= i\big(i(X - E)\big) \\ &= i(X - E) \\ &= e(E) \end{aligned}$$

Thus, $e(E) = e[X - e(E)]$.

Proof of (5):

$$\begin{aligned} e[A \cup B] &= i[X - (A \cup B)] \\ &= i[(X - A) \cap (X - B)] \\ &= i(X - A) \cap i(X - B) \\ &= e(A) \cap e(B) \end{aligned}$$

Thus $e(A \cup B) = e(A) \cap e(B)$.

§6 Boundary of set

Definition 6.1: Let $\langle X, \mathfrak{I} \rangle$ be a topological space and $E \subseteq X$. The boundary of set E is the set of all points interior to neither E nor $X - E$. This is denoted by $b(E)$ or frontier of E.

Theorem 6.1: In any topological space $\langle X, \mathfrak{I} \rangle$ for any $E \subseteq X$ we have

(1) $$\begin{aligned} b(E) &= X - [i(E) \cup i(X - E)] \\ &= [X - i(E)] \cap [X - i(X - E)] \\ &= [X - i(E)] \cap [X - e(E)] \\ &= X - [i(E) \cup e(E)] \end{aligned}$$

(2) $b(E) = b(X - E)$

$$\begin{aligned} b(E) &= X - [i(E) \cup i(X - E)] \\ &= X - [i[X - (X - E)] \cup i(X - E)] \\ &= b(X - E) \end{aligned}$$

(3) $b(E) = X - [i(E) \cup i(X-E)]$

$= [X - i(E)\,] \cap [X - i(X-E)]$

$= \bar{E} \cap \overline{(X-E)}$

§7 Solved Problems

Problem 1: If x is a limit point of a subset E of a topological space $\langle X, \mathfrak{J} \rangle$, what can be said about whether x is a limit point of E in the topological space $\langle X, \mathfrak{J}^* \rangle$ if $\mathfrak{J}^* \leq \mathfrak{J}$. What if $\mathfrak{J}^* \geq \mathfrak{J}$?

Solution: If $\mathfrak{J}^* \leq \mathfrak{J}$ then surely x is a limit point of E in $\langle X, \mathfrak{J}^* \rangle$ also [since any open set in $\langle X, \mathfrak{J}^* \rangle$ containing x will also be open in $\langle X, \mathfrak{J} \rangle$ containing x].

If $\mathfrak{J}^* \geq \mathfrak{J}$, then x need not be a limit point of E in $\langle X, \mathfrak{J}^* \rangle$. For this, consider $X = \{a, b\}$.
$\mathfrak{J} =$ indiscrete topology and $\mathfrak{J}^* =$ discrete topology on X. Then b is a limit point of $\{a\}$ in $\langle X, \mathfrak{J} \rangle$, but b is not a limit point of $\{a\}$ in $\langle X, \mathfrak{J}^* \rangle$.

Problem 2: Consider the topological space $\langle \mathbb{N}, \mathfrak{J} \rangle$ where,
$\mathfrak{J} = \{\emptyset\} \cup \{A_n \mid n = 1,2, \dots\}$ where $A_n = \{n, n+1, n+2, \dots\}$, $\forall\, n \in \mathbb{N}$.

Find (1) $d(E)$ where E is an infinite subset of $\mathbb{N}$.

(2) $d(E)$ where E is a finite subset of $\mathbb{N}$ and $E \neq \{1\}$.

(3) $d(E)$ where $E = \{1\}$.

Solution:

(1) Let E be any infinite set and $n \in \mathbb{N}$.

Then $n \in A_m$ for all $m \leq n$. $E \cap A_m - \{n\} \neq \emptyset \quad \forall\, m \leq n$. Hence n is a limit point of E.

As this is true for any $n \in \mathbb{N}$, we get $d(E) = \mathbb{N}$.

(2) Let E is a finite subset of $\mathbb{N}$. Let $E = \{x_1, x_2, \dots, x_n\}$. Let $m = max\,\{x_1, x_2, \dots, x_n\}$.

Then for any A_p , $p \leq m$, $A_p \cap E - \{p\} \neq \emptyset$. Hence each $p \leq m$ is a limit point of E.

Hence $d(E) = \{1,2, \dots, m\}$.

(3) Let $E = \{1\}$. Then for 1, A_1 is the open set containing 1.

$A_1 \cap E - \{1\} = \{1\} - \{1\} = \emptyset$. For any $n \neq 1$, $A_n \cap E - \{1\} = \emptyset - \{1\} = \emptyset$.

Hence $n \neq 1$ is not a limit point of $E = \{1\}$.

Thus no point of $\mathbb{N}$ will be a limit point of $\{1\}$. Hence $d(\{1\}) = \emptyset$.

Problem 3:- Let $\langle X, \mathfrak{J} \rangle$ be a p −inclusion topology $(p \in X)$ (see Example 6, Unit 1). Find $\bar{A}$ for $A = \{b\}$.

Solution:- We know that $\mathfrak{J} = \{\emptyset\} \cup \{G \subseteq X \mid p \in G\}$.

Consider $x \in X$ and an open set G containing x. Then $p \in G$ (by definition of $\mathfrak{J}$) and $G \cap \{p\} - \{x\} \neq \emptyset$ for each $x \neq p$ as $p \in G \cap \{p\} - \{x\}$ where $p \neq x$. But this shows that each $x \neq p$ is a limit point of $A = \{p\}$. Hence by the Theorem 3.2,

$\bar{A} = A \cup d(A) = \{p\} \cup (X - \{p\}) = X$.

Problem 4:- Let $\langle X, \mathfrak{J} \rangle$ be a co-finite topological space, where X is an uncountable set. Show that for infinite countable subset A of X, $\bar{A} = X$.

Solution:- By the definition, $\mathfrak{J} = \{\emptyset\} \cup \{G \subseteq X \mid X - G$ is finite$\}$.

Thus closed set in X must be finite. As A is not finite, A is not closed set in $\langle X, \mathfrak{J} \rangle$. Hence only closed set containing A is X. As $\bar{A}$ = the smallest closed set containing A, we get $\bar{A} = X$.

Problem 5:- Find the derived set of (a, b) $[a < b]$ in $\mathbb{R}$ relative to,

(i) Discrete topology.

(ii) Usual topology $\mathfrak{J}_u$.

(iii) Lower limit topology.

(iv) Indiscrete topology.

Solution: -

(i) $d\big((a, b)\big) = \emptyset$ relative to discrete topology in $\mathbb{R}$ (see Example 1 in 1.2).

(ii) $d\big((a, b)\big) = [a, b]$ relative to usual topology $\mathfrak{J}_u$ on $\mathbb{R}$.

$a \in (a - r, a + r)$ and $(a - r, a + r) \in \mathfrak{J}_u$ for any $r > 0$.

$(a - r, a + r) \cap (a, b) - \{a\} \neq \emptyset \quad \forall\, r > 0$. This shows that a is a limit point of (a, b). Similarly, we can prove that b is a limit point of (a, b) in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$. Further any $x \in (a, b)$ will obviously a limit point of (a, b). Further $x \notin (a, b)$ will not be a limit point of (a, b). Hence $d\big((a, b)\big) = [a, b]$ relative to usual topology on $\mathbb{R}$.

(iii) Let $\mathfrak{J}$ denote the lower limit topology on $\mathbb{R}$.

Then $\mathfrak{J} = \{\emptyset\} \cup \{[a,b) \mid a,b \in \mathbb{R} \text{ and } a < b\}$. Now a is a limit point of (a,b) as $a \in [a\,,\ a+\epsilon)$ and $(a\,,\ a+\epsilon) \in \mathfrak{J}\ \forall\, \epsilon > 0$ and $[a\,,\ a+\epsilon) \cap (a\,,\ b) - \{a\} \neq \emptyset$ for any $\epsilon > 0$. This shows that a is a limit point of $(a\,,\ b)$ relative to $\mathfrak{J}$.

Obviously, any $p \in (a\,,\ b)$ will be limit point of $(a\,,\ b)$ relative to $\mathfrak{J}$. For $b \in \mathbb{R}$, the open set containing b is of the form $[b\,,\ b+\epsilon)$ for $\epsilon > 0$ and

$[b\,,\ b+\epsilon) \cap (a\,,\ b) - \{b\} = \emptyset - \{b\} = \emptyset$. Hence b is not a limit point of (a,b).

Similarly any $x \notin (a,b)$ will not be a limit point of (a,b) relative to $\mathfrak{J}$. Hence, $d\big((a,b)\big) = [a,b)$ relative to $\mathfrak{J}$.

(iv) Let $\mathfrak{J}$ be an indiscrete topology on $\mathbb{R}$. Then $d\big((a,b)\big) = \mathbb{R}$ relative to indiscrete topology (see Example 2 in 1.2)

Problem 6:-Let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of $\langle X, \mathfrak{J} \rangle$. Then a subset A of Y is closed in Y if and only if there exists a set F closed in X such that $A = F \cap Y$.

Solution: - Let a subset A of Y be closed in Y. Hence $Y - A$ is open in Y.

Hence $Y - A = G \cap Y$ for some $G \in \mathfrak{J}$.

But then $A = (X - G) \cap Y$ will imply $A = F \cap Y$ where $F = X - G$ is closed set in X.

Similarly we can prove the converse.

Exercises

1) In a topological space $\langle \mathbb{R}, \mathfrak{J} \rangle$ where,
   $\mathfrak{J} = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{(a,\infty) \mid a \in \mathbb{R}\}$ find all $\mathfrak{J}$ - closed subsets of $\mathbb{R}$.

2) Let $X = \{a,b,c,d,e\}$. Define $\mathfrak{J} = \{\emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}, X\}$. Show that $\mathfrak{J}$ is a topology on X and find all $\mathfrak{J}$ - closed subsets of X.

3) Consider the topological space $\langle \mathbb{R}, \mathfrak{J}_u \rangle$.
   Define $A = (a,b)$, $B = [a,b)$ and $C = [a,b]$. Which of the following sets are neither open nor closed:
   (a) A (b) B (c) $A \cap B$ (d) $A \cap C$

4) Show that $i[\,\overline{A \cap B}\,] = i(\bar{A}) \cap i(\bar{B})$, where A and B are open sets in $\langle X, \mathfrak{I} \rangle$.

5) Verify the following properties of $\boldsymbol{i}$, $\boldsymbol{e}$ and $\boldsymbol{b}$ for any sets A, B and E:
   (i) $c(E) = E \cup b(E)$, $i(E) = E - b(E)$.
   (ii) $X = i(E) \cup b(E) \cup e(E)$ where $i(E) \cap b(E) \cap e(E) = \emptyset$.
   (iii) $b\big(i(E)\big) \subseteq b(E)$, $b\big(c(E)\big) \subseteq b(E)$ (give an example where these sets are not equal).
   (iv) $b(A \cup B) \subseteq b(A) \cup b(B)$, $i(A \cup B) \supseteq i(A) \cup i(B)$ (give an example where these sets are not equal).
   (v) $b(E) = \emptyset$ if and only if E is both open and closed.
   (vi) If A and B are open, $i\big(c(A \cap B)\big) = i\big(c(A)\big) \cap i\big(c(B)\big)$.

Unit 4

Different ways of defining topologies

§1 Closure operator.

§2 Interior operator.

§3 Exterior operator.

§4 Neighbourhood system.

Unit 4: *Different ways of defining topologies*

§1 Closure operator

Definition 1.1: Let X be any non-empty set. By a closure operator c* on X we mean a function

$c^*: \wp(X) \longrightarrow \wp(X)$ satisfying the following conditions:

(1) $c^*(\emptyset) = \emptyset$
(2) $A \subseteq c^*(A)$
(3) $c^*(c^*(A)) = c^*(A)$
(4) $c^*(A \cup B) = c^*(A) \cup c^*(B)$

for all $A, B \in \wp(X)$.

Example 1.2: Let $\langle X, \mathfrak{J} \rangle$ be any T – space. Define $c^*: \wp(X) \longrightarrow \wp(X)$ by $c^*(A) = \bar{A}$ = closure of A in $\langle X, \mathfrak{J} \rangle$. Then c* is a closure operator on X.

Theorem 1.3: Let c* be a closure operator defined on X. Let $\mathcal{F} = \{F \subseteq X \mid c^*(F) = F\}$ and

$\mathfrak{J} = \{X - F \mid F \in \mathcal{F}\}$. Then $\mathfrak{J}$ is a topology on X and $c^*(A) = \bar{A}$ = closure of A in $\langle X, \mathfrak{J} \rangle$, for any $A \subseteq X$.

Proof: I] To prove that $\mathfrak{J}$ is a topology on X.

(i) $c^*(\emptyset) = \emptyset$ (by definition of c*) $\Longrightarrow X - \emptyset \in \mathfrak{J} \Longrightarrow X \in \mathfrak{J}$.
$X \subseteq c^*(X)$ (by definition of c*). We get $c^*(X) = X \Longrightarrow X - X \in \mathfrak{J} \Longrightarrow \emptyset \in \mathfrak{J}$.

(ii) Let $A, B \in \mathfrak{J}$. Then $c^*(X - A) = X - A$ and $c^*(X - B) = X - B$ (by definition of $\mathfrak{J}$).

$$\begin{aligned} c^*[X - (A \cap B)] &= c^*[(X - A) \cup (X - B)] \\ &= c^*(X - A) \cup c^*(X - B) \text{ ...... (definition of c*)} \\ &= (X - A) \cup (X - B) \text{ ...... } (A, B \in \mathfrak{J}) \\ &= X - (A \cap B) \end{aligned}$$

This shows that $A \cap B \in \mathfrak{J}$.
Thus $A, B \in \mathfrak{J} \Longrightarrow A \cap B \in \mathfrak{J}$.

(iii) Let $G_\lambda \in \mathfrak{J}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set .

To prove that $\bigcup_{\lambda\in\Lambda} G_\lambda \in \mathfrak{J}$.

First note that $A \subseteq B \implies c^*(A) \subseteq c^*(B)$, for $A, B \in \wp(X)$.

$$A \subseteq B \implies A \cup B = B \implies c^*(A \cup B) = c^*(B)$$
$$\implies c^*(A) \cup c^*(B) = c^*(B)$$
$$\implies c^*(A) \subseteq c^*(B)$$

Thus $A \subseteq B \implies c^*(A) \subseteq c^*(B)$.

Now $G_\lambda \in \mathfrak{J} \implies c*(X - G_\lambda) = X - G_\lambda\ \forall\, \lambda \in \Lambda$.

$$G_\lambda \subseteq \bigcup_{\lambda\in\Lambda} G_\lambda \implies (X - G_\lambda) \supseteq \left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right), \quad \forall\, \lambda \in \Lambda$$

$$\implies c^*(X - G_\lambda) \supseteq c^*\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right), \quad \forall\, \lambda \in \Lambda$$

$$\implies (X - G_\lambda) \supseteq c^*\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right), \quad \forall\, \lambda \in \Lambda$$

$$\implies \bigcap_{\lambda\in\Lambda}(X - G_\lambda) \supseteq c^*\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right)$$

$$\implies \left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right) \supseteq c^*\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right)$$

But by definition of c*,

$$\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right) \subseteq c^*\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right)$$

Hence

$$c^*\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right) = \left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right)$$

But this shows that

$$\left(X - \bigcup_{\lambda\in\Lambda} G_\lambda\right) \in \mathcal{F} \quad \implies \quad \bigcup_{\lambda\in\Lambda} G_\lambda \in \mathfrak{J}.$$

From (i), (ii) and (iii) we get $\mathfrak{J}$ is a topology on X. Hence $\langle X, \mathfrak{J}\rangle$ is a T – space.

II] To prove that $c^*(A) = \bar{A}$ = closure of A in $\langle X, \mathfrak{J} \rangle$.

By definition of c*, $c^*[c^*(A)] = c^*(A)$. Hence $c^*(A)$ is closed set in $\langle X, \mathfrak{J} \rangle$ (by definition of $\mathfrak{J}$ and $\mathcal{F}$). By definition of c*, $A \subseteq c^*(A)$. Thus $c^*(A)$ is a closed set containing A. Let $\exists$ a closed set B in $\langle X, \mathfrak{J} \rangle$ containing A.

Then $A \subseteq B \implies c^*(A) \subseteq c^*(B) \implies c^*(A) \subseteq B$ ….(since B is closed $c^*(B) = B$; by definition of $\mathfrak{J}$). Thus $c^*(A) \subseteq B$. But this shows that $c^*(A)$ is the smallest closed set containing A. Hence, $c^*(A) = \bar{A}$, the closure of A in $\langle X, \mathfrak{J} \rangle$.

§2 Interior operator

Definition 2.1: Let X be any non-empty set. By an interior operator i^* on X we mean a function

$i^*: \wp(X) \longrightarrow \wp(X)$ satisfying the following conditions:

(1) $i^*(X) = X$

(2) $i^*(A) \subseteq A$

(3) $i^*\big(i^*(A)\big) = i^*(A)$

(4) $i^*(A \cap B) = i^*(A) \cap i^*(B)$

for all $A, B \in \wp(X)$.

Example 2.2: Let $\langle X, \mathfrak{J} \rangle$ be any T – space. Define $i^*: \wp(X) \longrightarrow \wp(X)$ by $i^*(A) = A^0$ = interior of A in $\langle X, \mathfrak{J} \rangle$. Then i^* is an interior operator on X.

Theorem 2.3: Let i^* be an interior operator defined on X. Let $\mathfrak{J} = \{A \subseteq X \mid i^*(A) = A\}$. Then $\mathfrak{J}$ is a topology on X and $i^*(A) = A^\circ$ = interior of A in $\langle X, \mathfrak{J} \rangle$ for any $A \subseteq X$.

Proof: I] To prove that $\mathfrak{J}$ is a topology on X.

(i) $i^*(X) = X$ (by definition of i^*) $\implies X \in \mathfrak{J}$.

$i^*(\emptyset) \subseteq \emptyset$ (by definition of i^*) $\implies \emptyset \in \mathfrak{J}$.

(ii) Let $A, B \in \mathfrak{J}$. Then $i^*(A) = A$ and $i^*(B) = B$ (by definition of $\mathfrak{J}$).

$i^*[(A \cap B)] = i^*(A) \cap i^*(B)$ …… (definition of c*)

$= A \cap B$ …… $(A, B \in \mathfrak{I})$

This shows that $A \cap B \in \mathfrak{I}$.

Thus $A, B \in \mathfrak{I} \implies A \cap B \in \mathfrak{I}$.

(iii) Let $G_\lambda \in \mathfrak{I}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set . To prove that $\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{I}$.

First note that $A \subseteq B \implies i^*(A) \subseteq i^*(B)$.

$$A \subseteq B \implies A \cap B = A \implies i^*(A \cap B) = i^*(A)$$
$$\implies i^*(A) \cap i^*(B) = i^*(A)$$
$$\implies i^*(A) \subseteq i^*(B)$$

Thus $A \subseteq B \implies i^*(A) \subseteq i^*(B)$.

Now $G_\lambda \in \mathfrak{I} \implies i^*(G_\lambda) = G_\lambda\ \ \forall\, \lambda \in \Lambda$.

$$G_\lambda \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda \implies i^*(G_\lambda) \subseteq i^*\left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) \quad \forall\, \lambda \in \Lambda$$

$$\implies G_\lambda \subseteq i^*\left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) \quad \forall\, \lambda \in \Lambda \ \ (\text{since } i^*(G_\lambda) = G_\lambda)$$

$$\implies \bigcup_{\lambda \in \Lambda} G_\lambda \subseteq i^*\left(\bigcup_{\lambda \in \Lambda} G_\lambda\right)$$

But by definition of i^*,

$$i^*\left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$$

Hence

$$i^*\left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) = \bigcup_{\lambda \in \Lambda} G_\lambda$$

But this shows that

$$\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{I}.$$

From (i), (ii) and (iii) we get $\mathfrak{I}$ is a topology on X. Hence $\langle X, \mathfrak{I} \rangle$ is a T – space.

II] To prove that $i^*(A) = A^\circ =$ interior of A in $\langle X, \mathfrak{I} \rangle$.

Fix up $A \subseteq X$. By definition of i^*, $i^*(A) \subseteq A$ and $i^*[i^*(A)] = i^*(A)$. Hence $i^*(A) \in \mathfrak{J}$.

Thus $i^*(A) \subseteq A$ implies that $i^*(A)$ is an open set contained in A. Let $\exists$ an open set B in $\langle X, \mathfrak{J} \rangle$ contained in A.

Then $B \subseteq A \implies i^*(B) \subseteq i^*(A) \implies B \subseteq i^*(A)$ ….(since B is open $i^*(B) = B$). Thus $i^*(A)$ is the largest open set contained in A. Hence, $i^*(A) = A^\circ =$ the interior of A in $\langle X, \mathfrak{J} \rangle$.

§3 Exterior operator

Definition 3.1: Let X be any non-empty set. By an exterior operator e^* on X we mean a function

$e^*: \wp(X) \longrightarrow \wp(X)$ satisfying the following conditions:

(1) $e^*(\emptyset) = X$ and $e^*(X) = \emptyset$

(2) $e^*(A) \subseteq X - A$

(3) $e^*\big(X - e^*(A)\big) = e^*(A)$

(4) $e^*(A \cup B) = e^*(A) \cap e^*(B)$

for all $A, B \in \wp(X)$.

Example 3.2: Let $\langle X, \mathfrak{J} \rangle$ be any T – space. Define $e^*: \wp(X) \longrightarrow \wp(X)$ by $e^*(A) =$ exterior of A in $\langle X, \mathfrak{J} \rangle$ for each $A \in \wp(X)$. Then e^* is an exterior operator on X.

Theorem 3.3: Let e^* be an exterior operator defined on X. Then there exists a unique topology $\mathfrak{J}$ on X such that $e^*(A) = e(A) =$ the exterior of A in $\langle X, \mathfrak{J} \rangle$ for any $A \subseteq X$.

Proof:- Define $\mathfrak{J} = \{G \subseteq X \mid e(X - G) = G\}$.

[I] To prove that $\mathfrak{J}$ is a topology on X.

(i) By definition of e^*, $e^*(\emptyset) = X$ and $e^*(X) = \emptyset$

$$\implies e^*(X - \emptyset) = \emptyset \text{ and } e^*(X - X) = X \ \ldots \text{(definition of } e^*\text{, condition (3))}$$

$$\implies \emptyset \in \mathfrak{J} \text{ and } X \in \mathfrak{J}$$

(ii) Let $A, B \in \mathfrak{J}$. To prove that $A \cap B \in \mathfrak{J}$.

$A, B \in \mathfrak{J} \implies e^*(X - A) = A$ and $e^*(X - B) = B$

$e^*[X - (A \cap B)] = e^*[(X - A) \cup (X - B)]$

$$= [e^*(X - A)] \cap [e^*(X - B)] \ \ldots \text{(definition of } e^*\text{, condition (4))}$$

$= A \cap B$ ………………… (since $A, B \in \mathfrak{J}$)

Thus $e^*[X - (A \cap B)] = A \cap B$. Hence $A \cap B \in \mathfrak{J}$.

(iii) Let $G_\lambda \in \mathfrak{J}\ \forall\ \lambda \in \Lambda$, where Λ is any indexing set . To prove that $\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{J}$.

$G_\lambda \in \mathfrak{J} \implies e^*(X - G_\lambda) = G_\lambda \quad \forall\ \lambda \in \Lambda$.

First we prove that $A \subseteq B \implies e^*(A) \supseteq e^*(B)$

$$A \subseteq B \implies A \cup B = B \implies e^*(A \cup B) = e^*(B)$$

$$\implies e^*(A) \cap e^*(B) = e^*(B)$$

$$\implies e^*(B) \subseteq e^*(A)$$

Thus $A \subseteq B \implies e^*(B) \subseteq e^*(A)$.

Now

$$G_\lambda \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda \implies (\mathrm{X} - G_\lambda) \supseteq \left(\mathrm{X} - \bigcup_{\lambda \in \Lambda} G_\lambda\right) \quad \forall\ \lambda \in \Lambda$$

$$\implies e^*(\mathrm{X} - G_\lambda) \subseteq e^*\left(\mathrm{X} - \bigcup_{\lambda \in \Lambda} G_\lambda\right) \quad \forall\ \lambda \in \Lambda$$

$$\implies G_\lambda \subseteq e^*\left(X - \bigcup_{\lambda \in \Lambda} G_\lambda\right) \quad \forall\ \lambda \in \Lambda \ \ (\text{since } e^*(X - G_\lambda) = G_\lambda)$$

$$\implies \bigcup_{\lambda \in \Lambda} G_\lambda \subseteq e^*\left(X - \bigcup_{\lambda \in \Lambda} G_\lambda\right)$$

But by definition of e^*,

$$e^*\left(X - \bigcup_{\lambda \in \Lambda} G_\lambda\right) \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$$

Hence

$$e^*\left(X - \bigcup_{\lambda \in \Lambda} G_\lambda\right) = \bigcup_{\lambda \in \Lambda} G_\lambda$$

But this shows that

$$\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{J}.$$

From (i), (ii) and (iii) we get $\mathfrak{J}$ is a topology on X. Hence $\langle X, \mathfrak{J} \rangle$ is a T – space.

II] To prove that $e^*(A) = e(A) =$ the exterior of A in $\langle X, \mathfrak{I} \rangle$ for any $A \subseteq X$.

By definition of e^*, $e^*\big(X - e^*(A)\big) = e^*(A)$. Hence, $e^*(A) \in \mathfrak{I}$. Again by definition of e^*, $e^*(A) \subseteq X - A$. Thus $e^*(A)$ is an open set contained in $X - A$. Let B be any open set contained in $X - A$. Then $B \subseteq X - A \implies A \subseteq X - B$

$$\implies e^*(A) \supseteq e^*(X - B)$$

$$\implies e^*(A) \supseteq B \text{ .... (Since } B \in \mathfrak{I})$$

Thus $e^*(A)$ is the largest open set contained in $X - A$. Hence by definition of exterior, $e^*(A) = e(A) =$ the exterior of A in $\langle X, \mathfrak{I} \rangle$ for any $A \subseteq X$.

§4 Neighbourhood system

Definition 4.1: Let $\langle X, \mathfrak{I} \rangle$ be any T – space and $x \in X$. A neighbourhood of a point x is any subset of X which contains an open set containing the point x.

Example 4.2: Let $X = \{a, b, c, d\}$ and $\mathfrak{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then in the topological space $\langle X, \mathfrak{I} \rangle$, $\{b, c\}$ is a neighbourhood of b. However, $\{b, c\}$ is not a neighbourhood of c.

Remarks:

(1) In topological space $\langle X, \mathfrak{I} \rangle$, any $G \in \mathfrak{I}$, is a neighbourhood of each of its points.

(2) If N is a neighbourhood of a point $x \in X$ then any superset of N is also a neighbourhood of x.

(3) Each point $x \in X$ is contained in some neighbourhood.

Theorem 4.3: Let X be any non-empty set. Let there be associated with each point x of set X, a collection of subsets, called neighbourhoods, subject to the conditions:

(1) Every point of X is contained in at least one neighbourhood, and each point is contained in each of its neighbourhood.

(2) The intersection of any two neighbourhoods of a point is a neighbourhood of that point.

(3) Any set, which contains a neighbourhood of a point, is itself a neighbourhood of that point.

(4) If N is a neighbourhood of a point x, then there exists a neighbourhood N^* of x such that N is a neighbourhood of each point of N^*.

Let $\mathfrak{I} = \{G \subseteq X \mid G$ is neighbourhood of each of its points$\}$.

Then $\mathfrak{I}$ is a topology on X and $N^*(x) = N(x) =$ the collection of all neighbourhoods of x in $\langle X, \mathfrak{I}\rangle$.

Proof:- I] To prove that $\mathfrak{I}$ is a topology on X.

(i) $\emptyset \in \mathfrak{I}$, since obviously it is a neighbourhood of each of its points.

We know that, by (1), any x is contained in at least one neighbourhood and this neighbourhood is contained in X. Therefore, by (3), X is a neighbourhood of x. Thus as X is neighbourhood of each $x \in X$, $X \in \mathfrak{I}$.

(ii) Let $A, B \in \mathfrak{I}$. Let $x \in A \cap B$.

$x \in A$ and $A \in \mathfrak{I} \Rightarrow A$ is a neighbourhood of x …… (by definition of $\mathfrak{I}$) .

$\Rightarrow A \in N^*(x)$.

Similarly, $x \in B$ and $B \in \mathfrak{I} \Rightarrow B$ is a neighbourhood of x

$\Rightarrow B \in N^*(x)$.

$A \in N^*(x)$ and $B \in N^*(x) \Rightarrow A \cap B \in N^*(x)$ …… (by (2))

Hence, $A \cap B$ is a neighbourhood of x. Thus as $A \cap B$ is a neighbourhood of each $x \in A \cap B$, $A \cap B \in \mathfrak{I}$.

(iii) Let $G_\lambda \in \mathfrak{I}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set .

To prove that $\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{I}$.

Let $x \in \bigcup_{\lambda \in \Lambda} G_\lambda$. *Then* $x \in G_\lambda$ *for some* $\lambda \in \Lambda$.

By data, $G_\lambda \in \mathfrak{I} \Rightarrow G_\lambda$ is a neighbourhood of x.

Hence, $G_\lambda \in N^*(x)$.

As $G_\lambda \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$, *we get* $\bigcup_{\lambda \in \Lambda} G_\lambda \in N^*(x)$ … … by (3)

Hence $\bigcup_{\lambda \in \Lambda} G_\lambda$ is a neighbourhood of x. As this true for any $x \in \bigcup_{\lambda \in \Lambda} G_\lambda$,

we get , $\bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{I}$.

From (i), (ii) and (iii), $\mathfrak{I}$ is a topology on X and hence $\langle X, \mathfrak{I} \rangle$ is a T – space.

II] To prove that $N^*(x) = N(x)$, where $N(x)$ is the collection of all neighbourhoods of x in $\langle X, \mathfrak{I} \rangle$, for $x \in X$.

Fix up $x \in X$.

$N \in N(x) \implies N$ is a neighbourhood of x in $\langle X, \mathfrak{I} \rangle$.

$\implies \exists\, G \in \mathfrak{I}$ such that $x \in G \subseteq N$

As G is an open set, G is a neighbourhood of x. Hence, $G \in N^*(x)$.

As $G \subseteq N$, we get $N \in N^*(x)$ … (by (3)). Thus $N \in N(x) \implies N \in N^*(x)$.

Therefore $N(x) \subseteq N^*(x)$.

Now suppose that $N \in N^*(x)$.

Define $G = \{x \in X \mid N$ is neighbourhood of $x\}$.

Now $x \in G \implies N$ is a neighbourhood of x

$\implies x \in N$.

Therefore $G \subseteq N$.

Let $y \in G$. Then N is a neighbourhood of $y \implies N \in N^*(y)$. By (4), there exists $N^* \subseteq X$ such that $N^* \in N^*(y)$ and if $z \in N^*$, then $N \in N^*(z)$. But then, by definition of $G,\ z \in G$.

Hence $N^* \subseteq G$ and by (3) , $G \in N^*(y)$.

Thus $y \in G \implies G \in N^*(y)$. Hence, G is a neighbourhood of each of its points. Hence, $G \in \mathfrak{I}$.

Thus given $N \in N^*(X)$, $G \in \mathfrak{I}$ such that $G \subseteq N$. Hence $G \in N(x)$. This shows that $N^*(x) \subseteq N(x)$. Combining both inclusions, we get $N^*(x) = N(X)$.

Exercises

1) Define a closure operator c * on X. Show that c* induces a unique topology $\mathfrak{I}$ on X such that the $\mathfrak{I}$ - closure of A = c* (A) for any A ⊆ X.
2) Define an interior operator i * on X. Show that i* induces a unique topology $\mathfrak{I}$ on X such that the $\mathfrak{I}$ - interior of A = i* (A) for any A ⊆ X.
3) Define a closure operator e * on X. Show that e* induces a unique topology $\mathfrak{I}$ on X *such that the* $\mathfrak{I}$ - exterior of A = e* (A) for any A ⊆ X.

Unit 5

Continuous functions and Homeomorphisms

Unit 5: *Continuous functions and Homeomorphisms*

§1 Definitions and examples.

Definition 1.1: A function f, mapping a topological space $\langle X, \mathfrak{J}\rangle$ into a topological space $\langle X^*, \mathfrak{J}^*\rangle$ is said to be continuous at $x \in X$ if for every open set G^* containing $f(x)$ there is an open set G containing x such that $f(G) \subseteq G^*$.

Definition 1.2: A function f, mapping a topological space $\langle X, \mathfrak{J}\rangle$ into a topological space $\langle X^*, \mathfrak{J}^*\rangle$ is said to be continuous on a set $E \subseteq X$ if it is continuous at each point of E.

Examples 1.3:

(1) Let $f: X \longrightarrow X^*$ be a function. Let $a \in X^*$ be any fixed point.
Define $f(x) = a$, for each x in X. f is continuous at each $x \in X$ since for every open set G^* containing $f(x) = a$, there is an open set G = X containing x such that $f(G) \subseteq G^*$. Hence f is continuous on X .

(2) Let $f: X \longrightarrow X^*$ be a function. Let $a \in X$ such that $\{a\} \in \mathfrak{J}$. Then f is continuous at a.
Let $G^* \in \mathfrak{J}^*$ such that $f(a) \in G^*$. Then $a \in f^{-1}(G^*) \implies \{a\} \subseteq f^{-1}(G^*)$. Define $G = \{a\}$. Then $G \in \mathfrak{J}$ such that $a \in G$ and $f(G) \subseteq G^*$. Hence f is continuous at a.

Remarks:

(1) If $\langle X, \mathfrak{J}\rangle$ is a discrete topological space, then $\{x\} \in \mathfrak{J}$, $\forall\, x \in X$. Hence, by Example 1.4, f is continuous at each $x \in X$ i.e. f is continuous on X.
Thus, any function defined on discrete topological space is always continuous.
(2) Converse of the Example 1.4 need not be true i.e. is continuous at $x = a$ in X need not imply $\{a\} \in \mathfrak{J}$. For this consider the following example.
$X = \{1,2,3,4\}$. $\mathfrak{J} = \{\emptyset, \{1\}, \{1,2\}, \{2,3,4\}, X\}$. Define $f: X \longrightarrow X$ by, $f(1) = 2$, $f(2) = 4$, $f(3) = 2, f(4) = 3$. Then f is continuous at $x = 4$, but $\{4\} \notin \mathfrak{J}$.

§2 Characterizations

Theorem 2.1: Let $\langle X, \mathfrak{J}\rangle$ and $\langle X^*, \mathfrak{J}^*\rangle$ be topological spaces and $f: X \longrightarrow X^*$. f is continuous on X if and only if the inverse image of an open set in X^* is an open set in X.

Proof: Only if part .

Let $f: X \longrightarrow X^*$ be continuous on X. Let $G^* \in \mathfrak{J}^*$. To prove that $f^{-1}(G^*) \in \mathfrak{J}$.

Let $x \in f^{-1}(G)$. Then by assumption, f is continuous at x. Hence there exists an open set G in X such that $f(G) \subseteq G^*$. But then $x \in G \subseteq f^{-1}(G^*)$ will imply x is an interior point of $f^{-1}(G^*)$. As any $x \in f^{-1}(G^*)$ is its interior point $f^{-1}(G^*)$ is an open set in X.

If part .

To prove that f is continuous on X.

Fix up any $x \in X$. Select any open set G^* in X^* containing $f(x)$. By assumption, $f^{-1}(G^*)$ is an open set in X.

$f(x) \in G^* \Longrightarrow x \in f^{-1}(G^*)$. Define $G = f^{-1}(G^*)$.

Then we get $x \in G$ and $f(G) = f[f^{-1}(G^*)] \subseteq G^*$. Hence f is continuous at x. As f is continuous at each $x \in X$, we get f is continuous on X.

Examples 2.2:

I. Let $\mathfrak{J}$ = co-finite topology on $\mathbb{R}$.

$\mathfrak{J}_u$ = usual topology on $\mathbb{R}$.

(1) $i: \langle \mathbb{R}, \mathfrak{J}\rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J}_u\rangle$ be an identity map. Then $i^{-1}(0,1) = (0,1) \notin \mathfrak{J}$ as $\mathbb{R} - (0,1) = (-\infty, 0] \cup [1, \infty)$ is not finite. Thus, though $(0,1) \in \mathfrak{J}_u$, $i^{-1}(0,1) \notin \mathfrak{J}$. Hence the identity map $i: \langle \mathbb{R}, \mathfrak{J}\rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J}_u\rangle$ is not continuous.

(2) Let $i: \langle \mathbb{R}, \mathfrak{J}_u\rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J}\rangle$ be an identity map. Let $G \in \mathfrak{J}$. Then $\mathbb{R} - G$ is finite. Hence $i^{-1}(\mathbb{R} - G)$ is finite subset of $\langle \mathbb{R}, \mathfrak{J}_u\rangle$. Hence $\mathbb{R} - G$ is closed in $\langle \mathbb{R}, \mathfrak{J}_u\rangle$. Hence $G \in \mathfrak{J}_u$. Thus given $G \in \mathfrak{J}$, $G = i^{-1}(G) \in \mathfrak{J}_u$. Hence i is continuous.

II. Let $\mathfrak{J}$ = co-countable topology on $\mathbb{R}$.

$\mathfrak{J}_u$ = usual topology on $\mathbb{R}$.

(1) Let $i: \langle \mathbb{R}, \mathfrak{J}\rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J}_u\rangle$ be an identity map. $(0,1) \in \mathfrak{J}_u$. Then $i^{-1}(0,1) = (0,1)$. As

$\mathbb{R} - (0,1) = (-\infty, 0] \cup [1, \infty)$, $\mathbb{R} - (0,1)$ is not a countable subset of $\mathbb{R}$.

Hence $(0,1) \notin \mathfrak{J}$. Therefore $i: \langle \mathbb{R}, \mathfrak{J} \rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J}_u \rangle$ is not a continuous map.

(2) Let $i: \langle \mathbb{R}, \mathfrak{J}_u \rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J} \rangle$ be an identity. Let $G = \mathbb{R} - \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ \ldots\right\}$.

Then $G \in \mathfrak{J}$ as $\mathbb{R} - G = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ \ldots\right\}$ is a countable set. Now $i^{-1}(G) = G \notin \mathfrak{J}_u$.

[$\mathbb{R} - G = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ \ldots\right\}$ is not a closed in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ as 0 is the limit point of $\mathbb{R} - G$ and $0 \notin \mathbb{R} - G$] . Hence $i: \langle \mathbb{R}, \mathfrak{J}_u \rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J} \rangle$ *is not continuous*.

Remark: Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two topological spaces. Let $f: \langle X, \mathfrak{J}_1 \rangle \longrightarrow \langle Y, \mathfrak{J}_2 \rangle$ be a continuous map. Then

(1) $f: \langle X, \mathfrak{J}_1^* \rangle \longrightarrow \langle Y, \mathfrak{J}_2 \rangle$ is a continuous map if $\mathfrak{J}_1^* \geq \mathfrak{J}_1$.

(2) $f: \langle X, \mathfrak{J}_1 \rangle \longrightarrow \langle Y, \mathfrak{J}_2^* \rangle$ is a continuous map if $\mathfrak{J}_2^* \leq \mathfrak{J}_2$.

Theorem 2.3: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be topological spaces and $f: X \longrightarrow X^*$. f is continuous on X if and only if the inverse image of a closed set in X^* is a closed set in X.

Proof: Only if part .

Let $f: X \longrightarrow X^*$ be continuous and let F^* be closed set in X^*. Then $X^* - F^*$ is a open set in X^*. Hence by Theorem 2.1, $f^{-1}(X^* - F^*)$ is open in X i.e. $X - f^{-1}(F^*)$ is open in X. Hence $f^{-1}(F^*)$ is closed in X.

If part

To prove that f is continuous on X. Let G^* be any open set in X^*. Then $X^* - G^*$ is closed set in X^*. Hence $f^{-1}(X^* - G^*)$ is closed set in X, by assumption . Therefore $X - f^{-1}(G^*)$ is closed set in X.

Hence $f^{-1}(G^*)$ is an open set in X. Thus inverse image of an open set in X^* is an open set in X. Hence by Theorem 2.1, f is continuous on X.

Theorem 2.4: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be topological spaces. $f: X \longrightarrow X^*$ is continuous if and only if $f[c(E)] \subseteq c^*[f(E)]$ for any $E \subseteq X$.

[$c(E)$ = closure of E in $\langle X, \mathfrak{J} \rangle$ and $c^*[f(E)]$ = closure of $f(E)$ in $\langle X^*, \mathfrak{J}^* \rangle$]

Proof: Only if part.

Let $f: X \longrightarrow X^*$ be continuous and let $E \subseteq F$. We know that $E \subseteq f^{-1}[f(E)]$.

Hence $E \subseteq f^{-1}[f(E)] \subseteq f^{-1}\big[c^*\big(f(E)\big)\big]$ $\big($since $f(E) \subseteq c^*\big(f(E)\big)$ always$\big)$.

As $c^*\big(f(E)\big)$ is closed set in X^* and f is continuous function on X, $f^{-1}\big[c^*\big(f(E)\big)\big]$ is a closed set in X (see Theorem 2.3). Hence $c(E) \subseteq f^{-1}\big[c^*\big(f(E)\big)\big]$ i.e. $f[c(E)] \subseteq c^*\big(f(E)\big)$.

If part.

To prove that f is continuous on X.

Let F^* be any closed set in X^*. Define $E = f^{-1}(F^*)$. Then by assumption

$f[c(f^{-1}(F^*))] \subseteq c^*[f(f^{-1}(F^*))]$.

But $f[f^{-1}(F^*)] \subseteq F^*$ always. Hence $c^*\big[\,f[f^{-1}(F^*)]\big] \subseteq c^*(F^*) = F^*$.

Thus we get , $f[c(f^{-1}(F^*))] \subseteq F^*$. Hence $c(f^{-1}(F^*)) \subseteq f^{-1}(F^*)$.

As $f^{-1}(F^*) \subseteq c(f^{-1}(F^*))$ always, we get $c(f^{-1}(F^*)) = f^{-1}(F^*)$.

Hence $f^{-1}(F^*)$ is a closed set in X. Hence by Theorem 2.3, f is a continuous function.

Theorem 2.5: Let $\langle X, \mathfrak{J}\rangle$ and $\langle X^*, \mathfrak{J}^*\rangle$ be two topological spaces and $f: X \longrightarrow X^*$. f is continuous on X if and only if $f^{-1}[i^*(E^*)] \subseteq i[f^{-1}(E^*)]$ for every $E^* \subseteq X^*$.

$[i^*(E^*) =$ interior of E^* in X^* and $i[f^{-1}(E^*)] =$ interior of $f^{-1}(E^*)$ in $X]$

Proof: Only if part.

Let $f: X \longrightarrow X^*$ be continuous and let $E^* \subseteq X^*$. Then $i^*(E^*)$ is an open set in X^*. Hence by Theorem 2.1, $f^{-1}[i^*(E^*)]$ is open in X. As $i^*(E^*) \subseteq E^*$, $f^{-1}[i^*(E^*)] \subseteq f^{-1}(E^*)$.

Hence, $f^{-1}[i^*(E^*)] \subseteq i[f^{-1}(E^*)]$.

If part .

To prove that $f: X \longrightarrow X^*$ is continuous on X. Let G^* be any open set in X^*. Then $i^*(\,G^*) = \,G^*$.

By assumption, $f^{-1}[i^*(G^*)] \subseteq i[f^{-1}(G^*)]$ i.e. $f^{-1}(G^*) \subseteq i[f^{-1}(G^*)]$. But always $i[f^{-1}(G^*)] \subseteq f^{-1}(G^*)$. Hence $i[f^{-1}(G^*)] = f^{-1}(G^*)$. This shows that $f^{-1}(G^*)$ is an open set in X. Hence, by Theorem 2.1, f is continuous function on X.

Theorem 2.6: Let $\langle X, \mathfrak{J}\rangle$ and $\langle X^*, \mathfrak{J}^*\rangle$ be two topological spaces. $i: X \longrightarrow X$ be identity map. i is continuous on X if and only if $\mathfrak{J}^* \leq \mathfrak{J}$.

Proof: Only if part .

Let $i: X \longrightarrow X$ be continuous on X. To prove $\mathfrak{J}^* \leq \mathfrak{J}$. Let $G^* \in \mathfrak{J}^*$. As $i: X \longrightarrow X$ is continuous, $i^{-1}(G^*) \in \mathfrak{J}$ i.e. $G^* \in \mathfrak{J}$. This shows that $\mathfrak{J}^* \leq \mathfrak{J}$.

If part .

Let $\mathfrak{I}^* \leq \mathfrak{I}$. To prove the identity map $i: X \longrightarrow X$ is continuous. Let $G^* \in \mathfrak{I}^*$. $i^{-1}(G^*) = G^*$. As $\mathfrak{I}^* \leq \mathfrak{I}$ we get $i^{-1}(G^*) = G^* \in \mathfrak{I}$. Thus inverse image of any open set in $\langle X^*, \mathfrak{I}^* \rangle$ is an open set in $\langle X, \mathfrak{I} \rangle$. Hence i is continuous, by Theorem 2.1.

Example: Let $\mathfrak{I}_u$ = usual topology on $\mathbb{R}$ and $\mathfrak{I}$ = co-finite topology on $\mathbb{R}$.

Let $i: \langle \mathbb{R}, \mathfrak{I}_u \rangle \longrightarrow \langle \mathbb{R}, \mathfrak{I} \rangle$ be an identity map Then i is continuous as $\mathfrak{I} \leq \mathfrak{I}_u$.

§3 Properties

Theorem 3.1: Let $f: X \longrightarrow X^*$ and $g: X^* \longrightarrow X^{**}$ be continuous maps. Then $g \circ f: X \longrightarrow X^{**}$ is continuous [i.e. composition of two continuous functions is a continuous function]

Proof:- Let G^{**} be any open set in X^{**}. As $g: X^* \longrightarrow X^{**}$ is continuous, $g^{-1}(G^{**})$ is open in X^*. Again, as $f: X \longrightarrow X^*$ is continuous, $f^{-1}[g^{-1}(G^{**})]$ is open in X. i.e. $(g \circ f)^{-1}[G^{**}] \in X$ for every $G^{**} \in \mathfrak{I}^{**}$. Hence $g \circ f$ is a continuous map.

Theorem 3.2: Let $f: X \longrightarrow X^*$ is continuous map and $E \subseteq X$. The restriction of f to E is also a continuous map.

Proof:- Let $g: E \longrightarrow X^*$ be a restriction of f to E i.e. $g(x) = f(x)$, $\forall\, x \in E$. To prove g is continuous on E. Let G^* be any open set in X^*. Then $g^{-1}[G^*] = E \cap f^{-1}[G^*]$. As f is continuous, $f^{-1}[G^*]$ is an open set in X. Hence $E \cap f^{-1}[G^*]$ is an open set in E. But this shows that g is continuous on E.

Theorem 3.3: Let $\langle X, \mathfrak{I} \rangle$ and $\langle \mathbb{R}, \mathfrak{I}_u \rangle$ be a topological spaces and $E \subseteq X$. $\chi_E : X \longrightarrow \mathbb{R}$ denotes the characteristic function on E i.e.

$$\chi_E(x) = \begin{cases} 1\,, if \;\; x \in E \\ 0\,, if \;\; x \notin E \end{cases}$$

Then χ_E is continuous on X if and only if E is both open and closed in X.

Proof:- Let $G^* \in \mathfrak{I}_u$. Then,

$$\chi_E^{-1}(G^*) = \begin{cases} E & if\ 1 \in G^*\ and\ 0 \notin G^* \\ X - E & if\ 0 \in G^*\ and\ 1 \notin G^* \\ X & if\ 0 \in G^*\ and\ 1 \in G^* \\ \emptyset & if\ 0 \notin G^*\ and\ 1 \notin G^* \end{cases}$$

Thus as χ_E is continuous on X, $\chi_E^{-1}(G^*) \in \mathfrak{J}$ i.e. $E \in \mathfrak{J}$ and $X - E \in \mathfrak{J}$. Thus E and $X - E$ are open in X. Hence, E must be both open and closed in X.

Conversely, if E is both open and closed then $\chi_E^{-1}(G^*) \in \mathfrak{J}$ for any $G^* \in \mathfrak{J}_u$. Hence χ_E will be continuous, by Theorem 2.1.

Theorem 3.4: Let $\langle X, \mathfrak{J}\rangle$ and $\langle X^*, \mathfrak{J}^*\rangle$ be two topological spaces. If $\mathfrak{J}^*$ is the indiscrete topology on X^*, then any function $f: X \longrightarrow X^*$ is continuous.

Proof:- As $\mathfrak{J}^*$ is the indiscrete topology on X^*, the only open sets in X^* are $\emptyset$ and X^*. And $f^{-1}(X^*) = X$ and $f^{-1}(\emptyset) = \emptyset$, shows that f is continuous on X, by Theorem 2.1.

Definition 3.5: Let $\langle X, \mathfrak{J}\rangle$ be a topological space. A subset E of X is said to be dense in itself if every point of E is a limit point of E i.e. $E \subseteq d(E)$.

Theorem 3.6: Let $\langle X, \mathfrak{J}\rangle$ and $\langle X^*, \mathfrak{J}^*\rangle$ be topological spaces. $f: X \to X^*$ be one-one, continuous map. f maps every dense in itself subset of X onto dense in itself subset of X^*.

Proof:- Let $E \subseteq X$ and let E be dense in itself in X. To prove that $f(E)$ is dense in itself in X^* i.e. to prove that each point of $f(E)$ is its limit point. Let $x^* \in f(E)$. Then $\exists\, x \in E$ such that $f(x) = x^*$. $x \in E \implies x$ is a limit point of E.

Let G^* be any open set containing x^*. Then $f^{-1}(G^*)$ is an open set in X containing x, since f is continuous. As x is a limit point of E, $f^{-1}(G^*) \cap E - \{x\} \neq \emptyset$.

Let $z \in f^{-1}(G^*) \cap E - \{x\}$. Then $z \neq x \implies f(z) \neq f(x)$, since f is one-one.

Further $f(z) \in G^*$ and $f(z) \in f(E)$.

Thus $f(z) \in G^* \cap f(E) - \{f(x)\}$ i.e. $f(z) \in G^* \cap f(E) - \{x^*\}$.

But this shows that x^* is the limit point of $f(E)$.

Thus each point of $f(E)$ is its limit point. Hence $f(E)$ is dense-in-itself in X^*.

§4 Homeomorphism

Definition 4.1: Let $\langle X, \mathfrak{J}\rangle$ and $\langle X^*, \mathfrak{J}^*\rangle$ be two topological spaces. $f: X \longrightarrow X^*$ is an open mapping if image of every open set in X is open in X^*.

Definition 4.2: Let $\langle X, \mathfrak{I} \rangle$ and $\langle X^*, \mathfrak{I}^* \rangle$ be two topological spaces. $f: X \longrightarrow X^*$ is a closed mapping if image of every closed set in X is a closed set in X^*.

Examples:

(I) Let $X = \{a, b, c\}$, $\mathfrak{I} = \{\emptyset, \{a\}, X\}$, $X^* = \{p, q, r\}$ and $\mathfrak{I}^* = \{\emptyset, \{p\}, \{p, r\}, X^*\}$.

(1) Define $f: X \longrightarrow X^*$ by, $f(a) = p$, $f(b) = q$, $f(c) = r$.Then f is an open map (Note that f is not a continuous map).

(2) Define $g: X \longrightarrow X^*$ by $g(a) = q$, $g(b) = q$, $g(c) = q$. Then g is a closed map (Note that g is continuous map).

(II) Let $\langle X, \mathfrak{I} \rangle$ be any topological space. Let $X^* = \{a, b, c\}$ and $\mathfrak{I}^* = \{\emptyset, \{a\}, \{a, c\}, X^*\}$.

(1) Define $\theta: X \longrightarrow X^*$ by, $\theta(x) = a$, $\forall \ x \in X$. Then θ is an open map but not a closed map.

(2) Define $\psi: X \longrightarrow X^*$ by $\psi(x) = b$, $\forall \ x \in X$. Then ψ is a closed map but not an open map.

Remarks:

(1) As θ is continuous but not closed, we get continuous map need not be closed map. Similarly as ψ is continuous but not open, we get continuous map need not be an open map.

(2) Open map need not be continuous, as f is open but not continuous. Similarly, closed mapping need not be continuous.

Theorem 4.3: The identity mapping of $\langle X, \mathfrak{I} \rangle$ onto $\langle X, \mathfrak{I}^* \rangle$ is open if and only if $\mathfrak{I}^* \geq \mathfrak{I}$.

Proof:- Only if part .

Let $i: \langle X, \mathfrak{I} \rangle \longrightarrow \langle X, \mathfrak{I}^* \rangle$ be open. Then $\forall \ G \in \mathfrak{I}$, $i(G) \in \mathfrak{I}^*$

i.e. $G \in \mathfrak{I} \implies G \in \mathfrak{I}^* \ (since \ i(G) = G)$. Hence $\mathfrak{I}^* \geq \mathfrak{I}$.

If part .

Let $\mathfrak{I}^* \geq \mathfrak{I}$. Then for any $G \in \mathfrak{I}$, $i(G) = G \in \mathfrak{I}^*$. Hence i is an open map.

Theorem 4.4: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be T – spaces. A mapping $f: X \longrightarrow X^*$is open if and only if $f[i(E)] \subseteq i^*[f(E)]$ for any $E \subseteq X$.

Proof: Only if part.

Let $f: X \longrightarrow X^*$is open and $E \subseteq X$. As $i(E)$ is an open set in X, $f[i(E)]$ is an open set in X^*. Further $i(E) \subseteq E$ implies $f[i(E)] \subseteq f(E)$. Hence, as $f[i(E)]$ is an open set contained in $f(E)$, we get $f[i(E)] \subseteq i^*[f(E)]$.

If part.

To prove $f: X \longrightarrow X^*$ is open. Let $G \in \mathfrak{J}$. Then $i(G) = G$. By data, $f[i(G)] \subseteq i^*[f(G)]$
i.e. $f(G) \subseteq i^*[f(G)]$.
As $i^*[f(G)] \subseteq f(G)$ always, we get, $i^*[f(G)] = f(G)$. Hence $f(G)$ is an open set in X^*. This shows that f is open.

Theorem 4.5: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be any topological spaces. $f: X \longrightarrow X^*$ is closed if and only if $f[c(E)] \supseteq c^*[f(E)]$.

Proof:- Only if part.

Let $f: X \longrightarrow X^*$ be a closed mapping and $E \subseteq X$. $c(E)$ is a closed set in X. Hence, $f[c(E)]$ is a closed set in X^*. Now $E \subseteq c(E) \implies f(E) \subseteq f[c(E)] \implies c^*[f(E)] \subseteq f[c(E)]$.

If part.

To prove $f: X \longrightarrow X^*$ is closed mapping. Let F be any closed set in X. Then $c(F) = F$. By assumption, $f[c(F)] \supseteq c^*[f(F)]$. Therefore $f(F) \supseteq c^*[f(F)]$.
But always $f(F) \subseteq c^*[f(F)]$. Hence $f(F) = c^*[f(F)]$.
This shows that f is a closed map.

Definition 4.6: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be two topological spaces. $f: X \longrightarrow X^*$ is a homeomorphism if f one-one, onto, continuous and open mapping.

Definition 4.7: Two topological spaces $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ are said to be homeomorphic if there exists a homeomorphism $f: X \longrightarrow X^*$ [or $f: X^* \longrightarrow X$].

Definition 4.8: A property of sets, which is preserved under a homeomorphism, is called a topological property.

Theorem 4.9: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be two topological spaces. Let $f: X \longrightarrow X^*$ be bijective mapping. The following statements are equivalent :

(1) f is a homeomorphism (or f is continuous and open mapping).

(2) f and f^{-1} both are continuous.

(3) f is continuous and closed mapping.

Proof:- (1) $\Rightarrow$ (2)

$f: X \longrightarrow X^*$ be continuous and open mapping.

To prove that f^{-1} is continuous.

$f^{-1}: X^* \longrightarrow X$. f^{-1} is one-one and onto. Let $G \in \mathfrak{J}$. As $f: X \longrightarrow X^*$ is open, we get $f(G) \in \mathfrak{J}^*$ i.e. $[f^{-1}]^{-1}(G) \in \mathfrak{J}^*$. But this shows that for any $G \in \mathfrak{J}$, $[f^{-1}]^{-1}(G) \in \mathfrak{J}^*$. Hence f^{-1} is continuous.

(2) $\Rightarrow$ (1)

Let f and f^{-1} both are continuous. To prove that f is an open map. Let $G \in \mathfrak{J}$.

Then $f^{-1}: X^* \longrightarrow X$ being continuous, $[f^{-1}]^{-1}(G) \in \mathfrak{J}^*$ i.e. $f(G) \in \mathfrak{J}^*$.

This shows that f is an open map.

(1) $\Rightarrow$ (3)

Let $f: X \longrightarrow X^*$ be continuous and open mapping.

To prove that f is closed.

Let F be a closed set in X. Then $X - F$ is an open set in X. Hence $f(X - F)$ is open in X^*.

But $f(X - F) = X^* - f(E)$, as f is onto.

But this shows that $f(E)$ is closed set in X^*.

Hence, f is closed map.

(3) $\Rightarrow$ (1)

Let $f: X \longrightarrow X^*$ be a closed map. To prove that f is open map.

Let $G \in \mathfrak{J}$. Then $X - G$ is closed set in X. Hence $f(X - G)$ is closed in X^*.

But $f(X - G) = X^* - f(G)$, f being onto. Hence $f(G)$ is open in $\langle X^*, \mathfrak{J}^* \rangle$.

This shows that f is an open map.

Thus (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3) .Hence the result .

Theorem 4.10: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be two topological spaces. Let $f: X \longrightarrow X^*$ be one-one, onto mapping. Then f is a homeomorphism if and only if $f[i(E)] = i^*[f(E)]$ for any $E \subseteq X$.

Proof:- $f: X \longrightarrow X^*$ is continuous $\Leftrightarrow$ $i^*[f(E)] \subseteq f[i(E)] \quad \forall E \subseteq X$ (see Theorem 2.5).

$f: X \longrightarrow X^*$ is an open mapping $\Leftrightarrow$ $f[i(E)] \subseteq i^*[f(E)] \quad \forall E \subseteq X$ (see Theorem 4.4).

Hence, the bijective map $f: X \longrightarrow X^*$ is a homeomorphism if and only if

$f[i(E)] = i^*[f(E)] \quad \forall E \subseteq X$.

Theorem 4.11: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be two topological spaces. Let $f: X \longrightarrow X^*$ be one-one, onto mapping. Then f is a homeomorphism if and only if $f[c(E)] = c^*[f(E)]$ for any $E \subseteq X$.

Proof: $f: X \longrightarrow X^*$ is continuous $\Leftrightarrow$ $f[c(E)] \subseteq c^*[f(E)] \quad \forall E \subseteq X$ (see Theorem 2.4).

$f: X \longrightarrow X^*$ is a closed mapping $\Leftrightarrow$ $f[c(E)] \supseteq c^*[f(E)] \quad \forall E \subseteq X$ (see Theorem 4.5).

Hence, the bijective map $f: X \longrightarrow X^*$ is a homeomorphism if and only if

$f[c(E)] = c^*[f(E)] \quad \forall E \subseteq X$.

§ 5 Solved Problems

Problem 1: Let $X = \{a, b, c\}$, $\mathfrak{J} = \{\emptyset, \{a\}, X\}$, $X^* = \{p, q, r\}$ *and* $\mathfrak{J}^* = \{\emptyset, \{p\}, \{p, r\}, X^*\}$. Let $f: X \longrightarrow X^*$ be defined by $f(a) = p$, $f(b) = q$, $f(c) = r$. Check the continuity of the function f.

Solution: [I] Continuity of f at a:-

The open set containing $f(a)$ in X^* are $\{p\}$ and $\{p, r\}$ and X^*.

Case (1): $G^* = \{p\}$.

Take $G = \{a\}$. Then $f(G) = \{p\} \subseteq G^*$.

Case (2): $G^* = \{p, r\}$.

Take $G = \{a\}$. Then $f(G) = \{p\} \subseteq G^*$.

Case (3): $G^* = X^*$.

Take $G = \{a\}$. Then $f(G) = \{p\} \subseteq G^*$.

Hence f is continuous at $a \in X$.

[II] Continuity of f at b:-

The open set containing $f(b) = q$ is X^* only.

Hence, in this case for $G^* = X^*$, select $G = X$ and we get $f(G) \subseteq G^*$.

Hence f is continuous at $b \in X$.

[III] Continuity of f at c:-

The open set containing $f(c) = r$ in X^* are $\{p, r\}$ and X^*.

Case (1): $G^* = \{p, r\}$.

As the only open set containing c is X and $f(X) \nsubseteq G^*$ $(\because f(X) = X^*)$ we get f is not continuous at $x = c$.

From [I] , [II] and [III] we get f is not continuous on X.

Problem 2: Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be two topological spaces. $\mathfrak{B}$ is a base for $\mathfrak{J}$. If $f: X \longrightarrow Y$ is a mapping such that $\{f(B) \mid B \in \mathfrak{B}\}$ is a base for $\mathfrak{J}^*$. Then show that f is an open map.

Solution:- Let $G \in \mathfrak{J}$. By definition of base, $G = \bigcup\{B_\lambda \mid \lambda \in \Lambda\}$ where Λ is any indexing set.

Hence, $f(G) = f\left[\bigcup\{B_\lambda \mid \lambda \in \Lambda\}\right]$

$\Rightarrow f(G) = \bigcup\{f(B_\lambda) \mid \lambda \in \Lambda\}$

$\Rightarrow f(G) \in \mathfrak{J}^*$.

Thus for any $G \in \mathfrak{J}$, we get $f(G) \in \mathfrak{J}^*$. Hence f is an open mapping.

Problem 3. Let $\mathfrak{J}$ = indiscrete topology on $\mathbb{R}$.

$\mathfrak{J}_1$ = discrete topology on $\mathbb{R}$.

$\mathfrak{J}_u$ = usual topology on $\mathbb{R}$.

Show that no two topological spaces $\langle \mathbb{R}, \mathfrak{J} \rangle$, $\langle \mathbb{R}, \mathfrak{J}_1 \rangle$ and $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ are homeomorphic.

Solution: -

(1) Let if possible there exists $f: \langle \mathbb{R}, \mathfrak{J} \rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J}_u \rangle$ such that f is a homeomorphism. Then f must be a constant map. Hence, f is not a bijective map. Hence, f is not a homeomorphism. Hence $\langle \mathbb{R}, \mathfrak{J} \rangle$ and $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ are not homeomorphic.

(2) Let if possible there exists $g: \langle \mathbb{R}, \mathfrak{J}_1 \rangle \longrightarrow \langle \mathbb{R}, \mathfrak{J}_u \rangle$ such that g is a homeomorphism. Then $\{x\} \in \mathfrak{J} \Rightarrow g(\{x\}) \in \mathfrak{J}_u \quad \forall\, x \in \mathbb{R}$. But $g(\{x\}) = \{g(x)\} \notin \mathfrak{J}_u$ being a singleton set; a contradiction (since g is open). Hence $\langle \mathbb{R}, \mathfrak{J}_1 \rangle$ and $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ are not homeomorphic.

Hence three spaces $\langle \mathbb{R}, \mathfrak{J} \rangle$, $\langle \mathbb{R}, \mathfrak{J}_1 \rangle$ and $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ are not homeomorphic.

Remark:- Let $\langle X, \mathfrak{J}\rangle$ and $\langle X, \mathfrak{J}^*\rangle$ be two topological spaces and $i: \langle X, \mathfrak{J}\rangle \to \langle X, \mathfrak{J}^*\rangle$ be an identity map. Then i is $(\mathfrak{J} - \mathfrak{J}^*)$ continuous if and only if $\mathfrak{J} \geq \mathfrak{J}^*$. Similarly the identity map $i: \langle X, \mathfrak{J}^*\rangle \to \langle X, \mathfrak{J}\rangle$ is continuous if and only if $\mathfrak{J}^* \geq \mathfrak{J}$.
Therefore identity map need not be continuous.

Problem 4: Let $\langle X, \mathfrak{J}\rangle$ and $\langle Y, \mathfrak{J}^*\rangle$ be two topological spaces. Let $f: X \longrightarrow Y$ be continuous at $x \in X$. If $\{x_n\}$ is a sequence of points of X, converging to x, then show that the image sequence $\{f(x_n)\}$ in Y, converges to $f(x)$ in Y.
Solution: Let $G^* \in \mathfrak{J}^*$ such that $f(x) \in G^*$. $f: X \longrightarrow Y$ is continuous. Hence $f^{-1}(G^*)$ is an open set in $\langle X, \mathfrak{J}\rangle$. As $x \in f^{-1}(G^*)$ and $x_n \to x$, $\exists\, N$ such that $x_n \in f^{-1}(G^*)$ for $n \geq N$. But then $f(x_n) \in G^*$, for $n \geq N$. This shows that $f(x_n) \to f(x)$ in Y.

Problem 5: Let f be a mapping of topological space $\langle X, \mathfrak{J}\rangle$ onto a set Y.
Define $\mathfrak{J}^* = \{G \subseteq Y \mid f^{-1}(G) \in \mathfrak{J}\}$. Then show that

(1) $\mathfrak{J}^*$ is a topology on Y.
(2) $f: \langle X, \mathfrak{J}\rangle \longrightarrow \langle Y, \mathfrak{J}^*\rangle$ is a continuous function.
(3) $\mathfrak{J}^*$ is the largest topology on Y for which $f: X \longrightarrow Y$ is continuous.
(4) $F \subseteq Y$ is closed in $\langle Y, \mathfrak{J}^*\rangle$ if and only if $f^{-1}(F)$ is closed in $\langle X, \mathfrak{J}\rangle$.

Solution:

(1) To prove that $\mathfrak{J}^*$ is a topology on Y.

(i) $f^{-1}(\emptyset) = \emptyset\,,\ \ \emptyset \in \mathfrak{J} \implies \emptyset \in \mathfrak{J}^*$.
$f^{-1}(Y) = X\,,\ \ X \in \mathfrak{J} \implies Y \in \mathfrak{J}^*$ (since f is onto)

(ii) Let $A, B \in \mathfrak{J}^*$. Then $f^{-1}(A) \in \mathfrak{J}$ and $f^{-1}(B) \in \mathfrak{J}$.
Therefore, $f^{-1}(A) \cap f^{-1}(B) \in \mathfrak{J}$ i.e. $f^{-1}(A \cap B) \in \mathfrak{J}$.
But this shows that $A \cap B \in \mathfrak{J}^*$.

(iii) $A_\lambda \in \mathfrak{J}^*\ \ \forall\, \lambda \in \Lambda$, where Λ is any indexing set . Then $f^{-1}(A_\lambda) \in \mathfrak{J}\ \ \forall\, \lambda \in \Lambda$.
$\mathfrak{J}$ being a topology, $\displaystyle\bigcup_{\lambda\in\Lambda} f^{-1}(A_\lambda)\ \in \mathfrak{J}\ \ i.e.\ \ f^{-1}\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] \in \mathfrak{J}$.
But this shows that $\displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda\ \in \mathfrak{J}^*$.

From (i), (ii) and (iii) we get, $\mathfrak{J}^*$ is a topology on Y.

(2) To prove $f: \langle X, \mathfrak{J} \rangle \longrightarrow \langle Y, \mathfrak{J}^* \rangle$ is a continuous function.

Let $G \in \mathfrak{J}^*$. Then by definition of $\mathfrak{J}^*$, $f^{-1}(G) \in \mathfrak{J}$. Hence f is continuous.

(3) Let $\mathfrak{J}_1$ denote a topology on Y such that $f: \langle X, \mathfrak{J} \rangle \longrightarrow \langle Y, \mathfrak{J}_1 \rangle$ is continuous function.

To prove that $\mathfrak{J}_1 \subseteq \mathfrak{J}^*$.

Let $G \in \mathfrak{J}_1$. Then by continuity of f, $f^{-1}(G) \in \mathfrak{J}$. But then by definition of $\mathfrak{J}^*$, $G \in \mathfrak{J}^*$.

Thus $G \in \mathfrak{J}_1 \Longrightarrow G \in \mathfrak{J}^*$. Hence $\mathfrak{J}_1 \subseteq \mathfrak{J}^*$.

This shows that, $\mathfrak{J}^*$ is the largest topology on Y for which $f: X \longrightarrow Y$ is continuous.

(4) $F \subseteq Y$ is closed in $\langle Y, \mathfrak{J}^* \rangle$

$\Leftrightarrow Y - F \in \mathfrak{J}^*$.

$\Leftrightarrow f^{-1}(Y - F) \in \mathfrak{J}$.

$\Leftrightarrow X - f^{-1}(F) \in \mathfrak{J}$.

$\Leftrightarrow f^{-1}(F)$ is closed in $\langle X, \mathfrak{J} \rangle$.

Exercises

(I) Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be topological spaces and $f: X \longrightarrow X^*$. Prove that the following statements are equivalent.

1) f is continuous on X.
2) The inverse image of an open set in X^* is an open set in X.
3) The inverse image of a closed set in X^* is a closed set in X.
4) $f[c(E)] \subseteq c^*[f(E)]$ for any $E \subseteq X$.
5) $f^{-1}[i^*(E^*)] \subseteq i[f^{-1}(E^*)]$ for every $E^* \subseteq X^*$.

(II) Let $\langle X, \mathfrak{J} \rangle$ and $\langle Y, \mathfrak{J}^* \rangle$ be two topological spaces. Let $f: X \longrightarrow Y$ be continuous at $x \in X$. If $\{x_n\}$ is a sequence of points of X, converging to x, then show that the image sequence $\{f(x_n)\}$ in Y, converges to $f(x)$ in Y. Is the converse true ? Justify your answer.

(III) Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be topological spaces and $f: X \longrightarrow X^*$. Prove that the following statements are equivalent.

1) f is a homeomorphism .
2) $f[i(E)] = i^*[f(E)]$ for any $E \subseteq X$.
3) $f[c(E)] = c^*[f(E)]$ for any $E \subseteq X$.

(IV) Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be any topological spaces. Show that a mapping $f: X \longrightarrow X^*$ is closed if and only if $f[c(E)] \supseteq c^*[f(E)]$.

(V) Let $\langle X, \mathfrak{J} \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ be T – spaces. Show that a mapping $f: X \longrightarrow X^*$is open if and only if $f[i(E)] \subseteq i^*[f(E)]$ for any $E \subseteq X$.

(VI) Show by an example that the image an open set E of a space X under a continuous function $f: X \longrightarrow X^*$ is not necessarily an open set in the space $\langle X^*, \mathfrak{J}^* \rangle$.

(VII) Show by an example that the image a closed set E of a space X under a continuous function $f: X \longrightarrow X^*$ is not necessarily a closed set in the space $\langle X^*, \mathfrak{J}^* \rangle$.

(VIII) Let A_1 and A_2 are closed sets in $\langle X, \mathfrak{J} \rangle$ such that $A_1 \cup A_2 = X$. Let $\langle X^*, \mathfrak{J}^* \rangle$ be another topological space and let $f: X \longrightarrow X^*$ be a mapping such that the restriction of f to each of the subspaces A_1 and A_2 is continuous. Show that f must be continuous.

Unit 6

Compact spaces

Unit 6: *Compact spaces*

§1 Definition and Examples

Definition 1.1: Let $\langle X, \mathfrak{I} \rangle$ be a topological space and $E \subseteq X$. A family $\{G_\lambda \mid \lambda \in \Lambda\}$ of subsets of X is said to form an open cover of E if $E \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$ and $G_\lambda \in \mathfrak{I}$, for each $\lambda \in \Lambda$.

Definition 1.2: If some finite sub-collection of the given covering of a set E is also a covering of E, then we say that the covering is reducible to a finite sub-covering.

Definition 1.3: A subset E of a topological space is said to be compact if every open covering of E is reducible to a finite sub-covering of E.

When X itself is a compact subset of $\langle X, \mathfrak{I} \rangle$, we say that $\langle X, \mathfrak{I} \rangle$ is compact.

Examples 1.4:

<u>Compact spaces</u>

1) Any subset of an indiscrete topological space $\langle X, \mathfrak{I} \rangle$ is compact, as $\{X\}$ is the only open cover for any $E \subseteq X$.

2) Any finite subset of any topological space $\langle X, \mathfrak{I} \rangle$ is compact.

3) Co-finite topological space is compact.

Let $\langle X, \mathfrak{I} \rangle$ be a co-finite topological space (with X an infinite set). Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be any open cover of X.

Fix up any G_{λ_0} for some $\lambda_0 \in \Lambda$. Then $G_{\lambda_0} \in \mathfrak{I} \implies X - G_{\lambda_0}$ is finite set.

Let $X - G_{\lambda_0} = \{x_1, x_2, \dots, x_n\}$.As $X = \bigcup_{\lambda \in \Lambda} G_\lambda$, find $G_{\lambda_i} \in \{G_\lambda \mid \lambda \in \Lambda\}$ such that $x_i \in G_{\lambda_i}$ $\forall\, i$, $1 \leq i \leq n$.

As $X = G_{\lambda_0} \cup \left(X - G_{\lambda_0}\right)$, we get $X = G_{\lambda_0} \cup G_{\lambda_1} \cup \dots \cup G_{\lambda_n}$.

Thus any open cover $\{G_\lambda \mid \lambda \in \Lambda\}$ of X contains a finite sub-cover.

Hence $\langle X, \mathfrak{I} \rangle$ is compact.

4) Fort's space is compact.

Let X be an uncountable set and let ∞ be the fixed point of X.

$\mathfrak{J} = \{G \subseteq X \mid \infty \notin G\} \cup \{G \subseteq X \mid \infty \in G \text{ and } X - G \text{ is finite}\}$.

$\langle X, \mathfrak{J} \rangle$ is a T – space – Fort's space (see Unit (1) §1.2 Example (10)).

To prove that $\langle X, \mathfrak{J} \rangle$ is a compact.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be any open cover for X .

As $X = \bigcup_{\lambda \in \Lambda} G_\lambda$ and $\infty \in X$, we get $\infty \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

By definition of $\mathfrak{J}$, $G_{\lambda_0} \in \mathfrak{J} \implies X - G_{\lambda_0}$ is finite subset of X.

Let $X - G_{\lambda_0} = \{x_1, x_2, \dots, x_n\}$. Select $G_{\lambda_i} \in \{G_\lambda \mid \lambda \in \Lambda\}$ such that $x_i \in G_{\lambda_i}\ \forall\, i,\ \ 1 \le i \le n$

Thus $X = G_{\lambda_0} \cup \left(X - G_{\lambda_0}\right) \subseteq G_{\lambda_0} \cup G_{\lambda_1} \cup \dots \cup G_{\lambda_n}$.

Thus $X = G_{\lambda_0} \cup G_{\lambda_1} \cup \dots \cup G_{\lambda_n}$.

This shows that any open cover $\{G_\lambda \mid \lambda \in \Lambda\}$ of X has a finite sub-cover.

Hence $\langle X, \mathfrak{J} \rangle$ is a compact.

5) $\langle X, \mathfrak{J} \rangle$ is a compact space where $\mathfrak{J}$ is the $p -$ exclusion topology $(p \in X)$ on X .

i.e. $\mathfrak{J} = \{X\} \cup \{G \subseteq X \mid p \notin G\}$.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be any open cover for X.

As $X = \bigcup_{\lambda \in \Lambda} G_\lambda$ and $p \in X$, we get $p \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

By definition of $\mathfrak{J}$, $G_{\lambda_0} = X$. Thus the open cover $\{G_\lambda \mid \lambda \in \Lambda\}$ of X has a finite sub-cover $\{G_{\lambda_0}\}$. Hence $\langle X, \mathfrak{J} \rangle$ is a compact.

<u>Non compact spaces</u>.

1) Any infinite discrete topological space $\langle X, \mathfrak{J} \rangle$ is not compact, as the open cover $\{\{x\} \mid x \in X\}$ has no finite sub-cover for X.

2) $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is not compact.

$\{(-n, n) \mid n \in \mathbb{N}\}$ forms an open cover for $\mathbb{R}$. Let this cover contains a finite sub-cover.

Let $\mathbb{R} = (-n_1\,,\ n_1) \cup (-n_2\,,\ n_2) \cup \dots \cup (-n_k\,,\ n_k)$.

Let $m = \max\{n_1, n_2, \dots, n_k\}$.

Then $m \in \mathbb{R}$ but $m \notin (-n_1\,,\ n_1) \cup (-n_2\,,\ n_2) \cup \dots \cup (-n_k\,,\ n_k)$ a contradiction. Hence the open cover $\{(-n, n) \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ has no finite sub-cover.Hence $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is not compact.

§2 Characterizations and Properties

Theorem 2.1: Let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of $\langle X, \mathfrak{J} \rangle$ and $E \subseteq Y$. Then E is compact subset of $\langle Y, \mathfrak{J}^* \rangle$ if and only if E is a compact subset of $\langle X, \mathfrak{J} \rangle$.

Proof: -Only if part.

Let $E \subseteq Y$ be compact in $\langle Y, \mathfrak{J}^* \rangle$. To prove that $E \subseteq X$ is compact in $\langle X, \mathfrak{J} \rangle$.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be any $\mathfrak{J}$ – open cover of E in X .

Define $G_\lambda^* = G_\lambda \cap Y$ for each $\lambda \in \Lambda$. Then $G_\lambda^* \in \mathfrak{J}^* \quad \forall \lambda$ and $E \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda^*$ shows that ,

$\{G_\lambda^* \mid \lambda \in \Lambda\}$ forms an $\mathfrak{J}^*$ - open cover for E.

As E is compact in $\langle Y, \mathfrak{J}^* \rangle$, the $\mathfrak{J}^*$ - open cover $\{G_\lambda^* \mid \lambda \in \Lambda\}$ of E has a finite sub-cover.

Let $E \subseteq \bigcup_{i=1}^{n} G_{\lambda_i}^*$ .Then $E \subseteq \bigcup_{i=1}^{n} G_{\lambda_i}$.

This shows that E is compact in $\langle X, \mathfrak{J} \rangle$.

If part .

Let $E \subseteq Y$ is compact in $\langle X, \mathfrak{J} \rangle$.

To prove that E is compact in $\langle Y, \mathfrak{J}^* \rangle$.

Let $\{G_\lambda^* \mid \lambda \in \Lambda\}$ be an open cover of E in $\langle Y, \mathfrak{J}^* \rangle$.

Then for each $\lambda \in \Lambda$, $G_\lambda^* \in \mathfrak{J}^* \implies G_\lambda^* = G_\lambda \cap Y$ for some $G_\lambda \in \mathfrak{J}$.

As $E \subseteq Y$ we get $E \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$ .But this in turns shows that $\{G_\lambda \mid \lambda \in \Lambda\}$ forms an open cover for E in $\langle X, \mathfrak{J} \rangle$. As E is compact in $\langle X, \mathfrak{J} \rangle$, there exists a finite sub-cover for open cover $\{G_\lambda \mid \lambda \in \Lambda\}$.

Let $E \subseteq \bigcup_{i=1}^{n} G_{\lambda_i}$. But then $E = E \cap Y \subseteq E \subseteq \left[\bigcup_{i=1}^{n} G_{\lambda_i}\right] \cap Y = E \subseteq \bigcup_{i=1}^{n} [G_{\lambda_i} \cap Y] = E \subseteq \bigcup_{i=1}^{n} G_{\lambda_i}^*$.

Thus any open cover $\{G_\lambda^* \mid \lambda \in \Lambda\}$ of E has a finite sub-cover for E.

Hence E is compact in $\langle Y, \mathfrak{J}^* \rangle$.

Remark: -Being compact is an absolute property i.e. the property of being compact for a set does not depend on the subspace in which it is contained.

Definition 2.2: A family of sets is said to have finite intersection property (f.i.p. in short) if every finite sub-family of the family has a non-empty intersection.

Theorem 2.3: A topological space $\langle X, \Im \rangle$ is compact if and only if every family of closed sets having the finite intersection property has a non-empty intersection.

Proof: Only if part.

Let $\langle X, \Im \rangle$ be compact and let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a family of closed sets in X satisfying f.i.p.

To prove that $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.

Let $\bigcap_{\lambda \in \Lambda} F_\lambda = \emptyset$ .Then $\left[X - \bigcap_{\lambda \in \Lambda} F_\lambda\right] = X \implies \bigcup_{\lambda \in \Lambda} [X - F_\lambda] = X$.

Thus the family $\{X - F_\lambda \mid \lambda \in \Lambda\}$ forms an open cover for X. As X is compact, this open cover has finite sub-cover.

Let $X = \bigcup_{i=1}^{n} \left(X - F_{\lambda_i}\right)$. But then $\bigcap_{i=1}^{n} F_{\lambda_i} = \emptyset$ $-$ a contradiction to assumption.

Hence $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.

If part .

Let any family of closed sets in $\langle X, \Im \rangle$ have f.i.p.

To prove that X is compact.Let X is not compact. Then there exist an open cover $\{G_\lambda \mid \lambda \in \Lambda\}$ of X such that $X \neq \bigcup_{i=1}^{n} G_{\lambda_i}$ for any finite n .

Hence $\bigcap_{i=1}^{n} \left(X - G_{\lambda_i}\right) \neq \emptyset$ for any finite n.

Thus the family $\{X - G_\lambda \mid \lambda \in \Lambda\}$ of closed sets in X satisfy f.i.p. Hence by assumption

$\bigcap_{\lambda \in \Lambda} (X - G_\lambda) \neq \emptyset$ $i.e.$ $X \neq \bigcup_{\lambda \in \Lambda} G_\lambda$; a contradiction.

Hence our assumption is wrong. Therefore X must be compact space.

Theorem 2.4: A T – space X is compact if and only if every basic open cover of X has a finite subcover.

Proof: Let X be a compact. Then every open cover of X has a finite subcover. In particular, every basic open cover of X must have a finite subcover.

Conversely, suppose that every basic open cover of X has a finite subcover and let

$$C = \{G_\lambda \in \mathfrak{I} \mid \lambda \in \Lambda\}$$

be any open cover of X. If

$$\mathfrak{B} = \{B_\alpha \mid \alpha \in \Delta\}$$

be any open base for X , then each G_λ is union of some members of $\mathfrak{B}$ and the totality of all such members of $\mathfrak{B}$ is evidently a basic open cover of X. By hypothesis this collection of members of $\mathfrak{B}$ has a finite subcover, say

$$\{B_{\alpha_i} \mid i = 1 , 2 , \dots , n\} .$$

For each B_{α_i} in this finite subcover, we can select a G_λ , from C such that $B_{\alpha_i} \subset G_{\lambda_i}$.

It follows that the finite subcollection

$$\{G_{\lambda_i} \mid i = 1 , 2 , \dots , n\} ,$$

Which arises in this way is a subcover of C.

Hence X is compact.

Theorem 2.5: Every closed subset of a compact space is compact (i.e. compactness is closed hereditary)

Proof: Let $\langle X, \mathfrak{I} \rangle$ be a compact space and E be a closed subset of X.

To prove that E is compact.Let $\{G_\lambda \in \mathfrak{I} \mid \lambda \in \Lambda\}$ be any open cover of E.

As $X = E \cup (X - E) \subseteq \left[\bigcup \{G_\lambda \mid \lambda \in \Lambda\} \right] \cup (\mathrm{X} - \mathrm{E})$ shows that $\bigcup \{G_\lambda \mid \lambda \in \Lambda\} \cup (\mathrm{X} - \mathrm{E})$ forms an open cover for X. As X is compact, this open cover has a finite sub-cover.

Let $X = \left[\bigcup\limits_{\mathrm{i}=1}^{\mathrm{n}} \{G_{\lambda_i} \mid \lambda_i \in \Lambda\} \right] \cup (\mathrm{X} - \mathrm{E})$. Then surely, $E \subseteq \bigcup\limits_{\mathrm{i}=1}^{\mathrm{n}} G_{\lambda_i}$.

Thus the open cover $\{G_\lambda \in \mathfrak{I} \mid \lambda \in \Lambda\}$ of E contains a finite sub-cover. Hence E is compact (by Theorem 2.1).

Remarks:

(1) Being a compact space is a closed hereditary property, but it is not a hereditary property. For this, consider the following example.

Let $\langle X, \mathfrak{I}\rangle$ be Fort's space. Then $\langle X, \mathfrak{I}\rangle$ is compact space (see §1.4 Example (4)).

Let $Y = X - \{\infty\}$. Then the relative topology $\mathfrak{I}^*$ on Y is the discrete topology on Y (by the definition of $\mathfrak{I}$).

Hence $\langle Y, \mathfrak{I}^*\rangle$ is not a compact space (since Y is an uncountable set). Thus we get a subspace of a compact space need not be compact. Hence being a compact space is not a hereditary property.

(2) Converse of Theorem 2.4 need not be true i.e. compact subset of a compact space need not be closed. For this, consider the following example.

Let $\langle X, \mathfrak{I}\rangle$ be an indiscrete topological space with $|X| \geq 2$. We know $\langle X, \mathfrak{I}\rangle$ is compact.

Let $\emptyset \subset E \subset X$. Then E is a compact subset of X as $\{x\}$ is the only open cover for E. But E is not closed in $\langle X, \mathfrak{I}\rangle$.

(3) By Theorem 2.4 closed subset of a compact space is compact but there may exists an open set in a compact space which is compact (i.e. closed sets are not the only compact subsets in a compact space). For this, consider the following example:

Let X be an infinite set and let $\mathfrak{I}$ be a co-finite topology on X. Then $\langle X, \mathfrak{I}\rangle$ is a compact space (see §1.4 Example (3)). Fix up any $x \in X$ and define $Y = X - \{x\}$. Then Y is proper subset of X and $Y \in \mathfrak{I}$.

Claim that Y is compact.

Let $\{G_\lambda \in \mathfrak{I} \mid \lambda \in \Lambda\}$ be any open cover of Y. Fix up any G_{λ_0}, $\lambda_0 \in \Lambda$. Then $X - G_{\lambda_0}$ is a finite subset of X and hence it contains finite number of elements of Y.

Let $y_1, y_2, \ldots, y_n \in \left(X - G_{\lambda_0}\right) \cap Y. As\ Y \subseteq \bigcup_{\lambda\in\Lambda} G_\lambda$, we can select G_{λ_i}

$\in \{G_\lambda \in \mathfrak{I} \mid \lambda \in \Lambda\}$ such

that $y_i \in G_{\lambda_i}$ $\forall\, i$, $1 \leq i \leq n$ .Thus $Y \subseteq G_{\lambda_0} \cup G_{\lambda_1} \cup \ldots \cup G_{\lambda_n}$.

This shows that the open cover $\{G_\lambda \in \mathfrak{I} \mid \lambda \in \Lambda\}$ of Y contains a finite sub-cover.

Hence Y is compact.

Thus in a compact space $\langle X, \mathfrak{I}\rangle$ there exists a compact open subset Y in X.

(4) Union of any two compact sets of a T-space is a compact set but intersection of any two compact ts of a T-space need not be a compact set.

For this consider the following example.

Let X be an infinite set. Let a , b $\in X$. $Define\ \mathfrak{J} = \{ X\} \cup \{G \mid G \subseteq X - \{ a, b \}\}$.

$i.e. \mathfrak{J} = \{X\} \cup \{G \subseteq X \mid a \notin G \text{ and } b \notin G\}$.Then $\langle X, \mathfrak{J} \rangle$ be a topological space.

Define A = X – {a} and B = X –{ b }.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be any open cover for A.

As $A \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$ and $a \in A$, we get $a \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

By definition of $\mathfrak{J}$, $G_{\lambda_0} = X$. Thus the open cover $\{G_\lambda \mid \lambda \in \Lambda\}$ of A has a finite sub-cover $\{ G_{\lambda_0}\}$. Hence A is a compact subset of X.

Similarly, we can prove that B is a compact subset of X.

Now A ∩ B = X – { a , b } is not compact as the open cover {{ x } | $x \in$ A ∩ B} of A ∩ B has no finite sub-cover. This shows that intersection of any two compact sets of a T-space need not be a compact set.

Theorem 2.6: An intersection of closed compact sets of a T-space is a closed compact set.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be compact and let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a family of closed compact sets in X.

To prove that $\bigcap_{\lambda \in \Lambda} F_\lambda$ is a closed compact set in X.

Obviously, $\bigcap_{\lambda \in \Lambda} F_\lambda$ is a closed set in X as F_λ is a closed set for each $\lambda \in \Lambda$.

Now $\bigcap_{\lambda \in \Lambda} F_\lambda \subseteq F_\lambda$ shows that

that $\bigcap_{\lambda \in \Lambda} F_\lambda$ is a closed subset of a compact set F_λ .

Hence $\bigcap_{\lambda \in \Lambda} F_\lambda$ is a compact set (see Theorem 2.4).

Theorem 2.7: Let $\langle X, \mathfrak{J} \rangle$ be a topological space. Let $\mathfrak{J}^* \leq \mathfrak{J}$. Then $\langle X, \mathfrak{J}^* \rangle$ is a compact space.

Proof: Let $\{G_\lambda \in \mathfrak{J}^* \mid \lambda \in \Lambda\}$ be an $\mathfrak{J}^*$- open cover for X.

As $\mathfrak{J}^* \leq \mathfrak{J}$, $\{G_\lambda \in \mathfrak{J}^* \mid \lambda \in \Lambda\}$ is also $\mathfrak{J}$ - open cover for X.

As $\langle X, \mathfrak{J} \rangle$ is compact, $\exists$ a finite sub-cover say $\left\{G_{\lambda_i} \in \mathfrak{J}^* \mid \lambda_i \in \Lambda, 1 \leq i \leq n\right\}$ for X.

But this in turns shows that $\langle X, \mathfrak{J}^* \rangle$ is a compact space.

Remark: Let $\langle X, \mathfrak{J}^* \rangle$ be a compact space and $\mathfrak{J}^* \leq \mathfrak{J}$. Then $\langle X, \mathfrak{J} \rangle$ need not be a compact space. For this, consider the following example.

Let X be any infinite set and

$$\mathfrak{J}^* = \text{indiscrete topology on } X.$$

$$\mathfrak{J} = \text{discrete topology on } X .$$

Then $\mathfrak{J}^* \leq \mathfrak{J}$. $\langle X, \mathfrak{J}^* \rangle$ is a compact space but $\langle X, \mathfrak{J} \rangle$ is not a compact space.

Theorem 2.8: Let $\langle X, \mathfrak{J} \rangle$ and $\langle Y, \mathfrak{J}^* \rangle$ be any two topological spaces. Let $\langle X, \mathfrak{J} \rangle$ be a compact space and let $f: X \to Y$ be onto, continuous map. Then $\langle Y, \mathfrak{J}^* \rangle$ is compact.

Proof:- Let $\{G_\lambda^* \in \mathfrak{J}^* \mid \lambda \in \Lambda\}$ be any open cover for Y.

As $Y \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda^*$ and f is onto we get $X = f^{-1}\left[\bigcup_{\lambda \in \Lambda} G_\lambda^*\right] = \bigcup_{\lambda \in \Lambda} f^{-1}(G_\lambda^*)$

As f is continuous and $G_\lambda^* \in \mathfrak{J}^*$, $\forall\, \lambda \in \Lambda$; we get $f^{-1}(G_\lambda^*) \in \mathfrak{J}$, $\forall\, \lambda \in \Lambda$.

Hence $\{ f^{-1}(G_\lambda^*) \in \mathfrak{J} \mid \lambda \in \Lambda\}$ forms an open cover for X. As X is compact this open cover has a finite sub-cover. Let $X = \bigcup_{i=1}^{n} f^{-1}\left(G_{\lambda_i}^*\right)$. But then

$Y = \bigcup_{i=1}^{n} G_{\lambda_i}^*$ shows that the open cover $\{G_\lambda^* \in \mathfrak{J}^* \mid \lambda \in \Lambda\}$ of Y

has a finite sub-cover. Hence $\langle Y, \mathfrak{J}^* \rangle$ is compact.

Corollary 2.9: Being a compact space is a topological property.

Corollary 2.10: Let f be a continuous map of $\langle X, \mathfrak{J} \rangle$ into $\langle Y, \mathfrak{J}^* \rangle$. f maps every compact subset of X onto a compact subset of Y.

Proof: $-E$ is a compact subset of X. Restriction of f on the subspace E of X is continuous onto map on the subspace $f(E)$ of Y. Hence by Theorem2.7, $f(E)$ is a compact space of Y.

§3 Special examples: $\langle \mathbb{R}, \mathfrak{J}_u \rangle$

Though $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ itself is not a compact space, compact sets in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ are of special importance.

Theorem 3.1: Any closed and bounded interval in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is compact.

Proof:- Let $I = [a, b]$ be any closed bounded interval in $\mathbb{R}$. To prove that I is compact.

Let I is not compact. Then $\exists$ an open cover $\{G_\lambda \in \mathfrak{J}_u \mid \lambda \in \Lambda\}$ of I which has no finite sub-cover.

Consider the two closed intervals $\left[a\ ,\ \frac{b-a}{2}\right]$ and $\left[\frac{b-a}{2}\ ,\ b\right]$. Obviously both of these closed bounded intervals have no finite sub-cover for the given open cover $\{G_\lambda \in \mathfrak{J}_u \mid \lambda \in \Lambda\}$.

Denote by I, the closed interval among these two which has no finite sub-cover. Again bisect I into two closed intervals. Label I_2 , the interval which has no finite sub-cover. Continuing this process we get a sequence $\{I_n\}$ of intervals such that $I \supset I_1 \supset I_2 \supset \cdots$ and length of $I_n \longrightarrow 0$ as $n \to \infty$. Hence by Canter's intersection theorem,

$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} I_n$. Then $x \in \mathbb{R} \implies x \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

$G_{\lambda_0} \in \mathfrak{J}_u$ and $x \in G_{\lambda_0} \implies \exists\ r > 0$ such that $(x - r\ ,\ x + r) \subseteq G_{\lambda_0}$. Select n so large that $I_n \subseteq (x - r\ ,\ x + r)$. But then $I_n \subseteq G_{\lambda_0}$ shows that I_n has a finite sub-cover for the given cover$\{G_\lambda \in \mathfrak{J}_u \mid \lambda \in \Lambda\}$. This contradicts the choice of I_n . Hence, our assumption is wrong. Hence, I is compact subset of $\langle \mathbb{R}, \mathfrak{J}_u \rangle$.

Theorem 3.2: Any compact subset of $\mathbb{R}$ is closed in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$.

Proof:- Let A be any compact subset of $\mathbb{R}$.

To prove that A is closed in $\mathbb{R}$.

Select $x \in \mathbb{R} - A$ and $a \in A$. Then $x \neq a$. Let $d(x, a) = r > 0$.

Then $\left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right) \cap \left(a - \frac{r}{2}\ ,\ a + \frac{r}{2}\right) = \emptyset$.

Define $G_a = \left(a - \frac{r}{2}\ ,\ a + \frac{r}{2}\right)$, $\forall\ a \in A$ and $G_x = \left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right)$, $\forall\ x \in \mathbb{R} - A$.

Then $\{G_a \in \mathfrak{J}_u \mid a \in A\}$ will form an open cover for A. As A is compact, this open cover has a finite sub-cover.

Let $A \subseteq \bigcup_{i=1}^{n} G_{a_i}$ .Find the corresponding G_{x_i} , $1 \leq i \leq n$.

Then $x \in \bigcap_{i=1}^{n} G_{x_i} \subseteq \mathbb{R} - A$.

Define $x_m = min\ \{x_1\ ,\ x_2\ , \dots , x_n\}$. Then $x \in (x_m - r\ ,\ x_m + r) \subseteq \mathbb{R} - A$.

But this shows that each $x \in \mathbb{R} - A$ is its interior point. Hence, $\mathbb{R} - A$ is an open set.

Therefore A is a closed set in $\mathbb{R}$.

Theorem 3.3: Heine - Borel Theorem.

A subset A of $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is compact if and only if A is bounded and closed.

Proof: Only if part.

Let A be a compact subset of $\mathbb{R}$.

To prove that A is closed and bounded.

For each $a \in A$, define $G_a = (a - 1, a + 1)$. Then $\{G_a \in \mathfrak{J}_u \mid a \in A\}$ will form an open cover for A. As A is compact, this open cover has a finite sub-cover for A .

Let $A \subseteq \bigcup_{i=1}^{n} G_{a_i}$ .Thus $\{a_1, a_2, \dots , a_n\} \subseteq A$ (by construction).

Let $m = min\ \{a_1, a_2, \dots , a_n\}$ and $M = max\ \{a_1, a_2, \dots , a_n\}$.

Then $G_{a_1} \cup G_{a_2} \cup \dots \cup G_{a_n} \subseteq [m - 1, m + 1]$ will imply $A \subseteq [m - 1, m + 1]$.

Hence A is bounded subset of $\mathbb{R}$.

If part .

Let A be a closed bounded subset of $\mathbb{R}$. Then $A \subseteq [m\ , M]$. By Theorem 3.1, $[m\ , M]$ is compact.

Thus A is a closed subset of a compact space $[m\ , M]$ (w.r.t. relative topology).

Hence A is compact (see Theorem 2.4).

Theorem 3.4: Cantor set C in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is compact.

Proof: We know that the Cantor set C is given by,

$$C = \bigcap_{n=0}^{\infty} F_n \,, \text{ where}$$

$F_0 = [0,1]$,

$F_1 = \left[0\,,\,\frac{1}{3}\right] \cup \left[\frac{2}{3}\,,1\right]$,

$F_2 = \left[0\,,\,\frac{1}{9}\right] \cup \left[\frac{2}{9}\,,\,\frac{1}{3}\right] \cup \left[\frac{2}{3}\,,\frac{1}{9}\right] \cup \left[\frac{8}{9}\,,\,1\right]$

- -

Thus, each F_n is union of 2^n disjoint closed intervals each of length $\frac{1}{3^n}$.

Each F_n is a closed set in $\mathbb{R}$ and hence $C = \bigcap_{n=0}^{\infty} F_n$ is a closed set.

As $C \subseteq [0\,,1]$, C is a bounded closed subset of $\langle \mathbb{R}, \mathfrak{I}_u \rangle$. Hence by Theorem 3.3, C is a compact set.

§4 One point compactification

We know that every topological space need not be compact (see Example 1.4 (6)). But for given non-compact space $\langle X, \mathfrak{I} \rangle$ we can construct a compact space $\langle X^*, \mathfrak{I}^* \rangle$ such that X is homeomorphic with some dense subspace of X. This compact space $\langle X^*, \mathfrak{I}^* \rangle$ is called compactification of the space X. If $X^* = X \cup \{\infty\}$ for some object $\infty \notin X$, then compactification of X is called one-point compactification.

The topology $\mathfrak{I}^*$ on $X^* = X \cup \{\infty\}$, $\infty \notin X$ for which we get a one-point compactification $\langle X^*, \mathfrak{I}^* \rangle$ of $\langle X, \mathfrak{I} \rangle$ is explained in the following theorem.

Theorem 4.1: Let $\langle X, \mathfrak{I} \rangle$ be a non-compact space. $X^* = X \cup \{\infty\}$, where $\infty \notin X$. Define $\mathfrak{I}^*$ as $\mathfrak{I}^* = \{G \subseteq X^* \mid G \in \mathfrak{I}\} \cup \{G \subseteq X^* \mid X^* - G$ is a closed compact subset of $X\}$. Then

1) $\mathfrak{I}^*$ is topology on X.
2) $\langle X, \mathfrak{I} \rangle$ is a subspace of $\langle X^*, \mathfrak{I}^* \rangle$
3) $\langle X^*, \mathfrak{I}^* \rangle$ is a compact space.
4) $\langle X, \mathfrak{I} \rangle$ is dense subspace of $\langle X^*, \mathfrak{I}^* \rangle$.

This $\langle X^*, \mathfrak{I}^* \rangle$ is a one-point compactification of $\langle X, \mathfrak{I} \rangle$.

Proof: -

1) To prove that $\mathfrak{I}^*$ is topology on X.

$\mathfrak{I}^* = \{G \subseteq X^* \mid G \in \mathfrak{I}\} \cup \{G \subseteq X^* \mid X^* - G$ is a closed compact subset of $X\}$.

Then $\mathfrak{I}^* = \mathfrak{I} \cup \mathfrak{I}_1$, where $\mathfrak{I}_1 = \{G \subseteq X^* \mid X^* - G$ is a closed compact subset of $X\}$.

Further, note that $G \in \mathfrak{I}_1 \Leftrightarrow \infty \in G$ and $G \in \mathfrak{I} \Leftrightarrow \infty \notin G \quad \forall\, G \subseteq X^*$.

Again, if $\infty \in G, G \subseteq X^*$, then $X^* - G = X^* \cap G' = [X \cup \{\infty\}] \cap G' = [X \cap G'] \cup [\{\infty\} \cap G'] = X \cap G'$

($\because\ \infty \in G \Longrightarrow \infty \notin G'$).

Thus $X^* - G = X - G \quad \forall\, G \subseteq X^*$ such that $\infty \in G$.

(i) $\emptyset \in \mathfrak{I}^*$ as $\emptyset \in \mathfrak{I}$.

$X^* \in \mathfrak{I}^*$ as $X^* - X^* = \emptyset$ is a closed compact subset of X.

(ii) Let $A, B \in \mathfrak{I}^*$. To prove that $A \cap B \in \mathfrak{I}^*$.

Case 1: $\infty \in A \cap B$.

Then $\infty \in A\,, A \in \mathfrak{I}^* \Longrightarrow X - A$ is a closed compact subset of X.

Similarly, $X - B$ is a closed compact subset of X. Hence $(X - A) \cup (X - B)$ is a closed, compact subset of X (since union of two compact sets is a compact set and union of two closed sets is a closed set). But as $X - (A \cap B) = (X - A) \cup (X - B)$ we get $A \cap B \in \mathfrak{I}_1$ and hence $A \cap B \in \mathfrak{I}^*$.

Case 2: $\infty \notin A \cap B$.

Then either $\infty \notin A$ or $\infty \notin B$.

Sub case (1): Suppose $\infty \notin A$ and $\infty \in B$.

Then $A \in \mathfrak{I}$ and $B \in \mathfrak{I}_1$.

Hence $A \in \mathfrak{I}$ and $X^* - B$ is closed compact subset of X. Hence $X - [X^* - B]$ is an open set in X. But
$$
\begin{aligned}
X - [X^* - B] &= X - [X^* \cap B'] \\
&= X \cap [X^* \cap B']' \\
&= X \cap [(X^*)' \cup (B')'] \\
&= X \cap [\emptyset \cup B] \\
&= X \cap B.
\end{aligned}
$$

As $A \in \mathfrak{I}$ and $X \cap B \in \mathfrak{I}$, we get $A \cap (X \cap B) \in \mathfrak{I}$ i.e. $(A \cap X) \cap B \in \mathfrak{I}$ i.e. $A \cap B \in \mathfrak{I}$. Hence $A \cap B \in \mathfrak{I}^*$.

Sub case (2): $\infty \in A$ or $\infty \notin B$.

As in sub case (1) we can show that $A \cap B \in \mathfrak{I}^*$.

Sub case (3): $\infty \notin A$ and $\infty \notin B$.

$\infty \notin A$ and $A \in \mathfrak{J}^* \implies A \in \mathfrak{J}$.

$\infty \notin B$ and $B \in \mathfrak{J}^* \implies B \in \mathfrak{J}$.

$\mathfrak{J}$ being topology on X, $A \cap B \in \mathfrak{J}$ and hence $A \cap B \in \mathfrak{J}^*$.

(iii) Let $A_\lambda \in \mathfrak{J}^*\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set.

To prove that, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}^*$.

Case (1): $\infty \notin \bigcup_{\lambda \in \Lambda} A_\lambda$.

Then $\infty \notin A_\lambda$ for each $\lambda \in \Lambda$. As $\infty \notin A_\lambda$ *and* $A_\lambda \in \mathfrak{J}^*$ we get $A_\lambda \in \mathfrak{J}$, $\forall\, \lambda \in \Lambda$.

Hence $\mathfrak{J}$ being topology on X, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

Hence in this case, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}^*$.

Case (2): $\infty \in \bigcup_{\lambda \in \Lambda} A_\lambda$.

Then $X^* - \bigcup_{\lambda \in \Lambda} A_\lambda = X - \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda \in \Lambda} (X - A_\lambda)$.

As $X - A_\lambda$ is a closed set in X, we get $\bigcap_{\lambda \in \Lambda} (X - A_\lambda)$ is a closed set.

Select $\lambda_0 \in \Lambda$ such that $\infty \in A_{\lambda_0}$.

As $A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$ *we get* $\left[X - \bigcup_{\lambda \in \Lambda} A_\lambda\right] \subseteq X - A_{\lambda_0}$.

As $\infty \in A_{\lambda_0}$, $A_{\lambda_0} \in \mathfrak{J}_1$, $X - A_{\lambda_0}$ is a closed, compact subset of X.

Again $X - \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda \in \Lambda} (X - A_\lambda)$.

As each $X - A_\lambda$ is a closed set in $\mathfrak{J}$, we get $\bigcap_{\lambda \in \Lambda} (X - A_\lambda)$ is a closed set in $\mathfrak{J}$.

Hence $X - \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda \in \Lambda} (X - A_\lambda)$ is a closed set in X which contained in a compact space $X - A_{\lambda_0}$.

Hence $X - \bigcup_{\lambda \in \Lambda} A_\lambda$ is compact (see Theorem 2.5).Thus $X - \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}_1$ and hence

$$\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}^* \text{ whenever } A_\lambda \in \mathfrak{J}^* \quad \forall \lambda \in \Lambda.$$

Thus from both the cases we get,

$$A_\lambda \in \mathfrak{J}^* \ \forall \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}^*.$$

From (i), (ii) and (iii) is a topology on X^*. Hence $\langle X^*, \mathfrak{J}^* \rangle$ is a topological space.

2) To prove $\langle X, \mathfrak{J} \rangle$ is a subspace of $\langle X^*, \mathfrak{J}^* \rangle$.

$X \subset X^*$. The relative topology $\mathfrak{J}_X^*$ on X is given by, $\mathfrak{J}_X^* = \{G \cap X \mid G \in \mathfrak{J}^*\}$. .

To prove that $\mathfrak{J}_X^* = \mathfrak{J}$.

Let $G \in \mathfrak{J}$. Then $G \subseteq X$ and $G \in \mathfrak{J}^*$ (since $\mathfrak{J} \subseteq \mathfrak{J}^*$).

Hence $G = G \cap X, G \in \mathfrak{J}^* \Rightarrow G \in \mathfrak{J}_X^*$. Hence $\mathfrak{J} \subseteq \mathfrak{J}_X^*$ ________________ (i)

Now let $G^* \in \mathfrak{J}_X^*$. Then $G^* = G \cap X$ for some $G \in \mathfrak{J}_X^* = \mathfrak{J} \cup \mathfrak{J}_1$.

Case (1): $G \in \mathfrak{J}$. Then $G \subseteq X$ and hence $G^* = G \cap X = G$.

This shows that $G^* \in \mathfrak{J}$.

Case (2): $G \in \mathfrak{J}_1$. Then $X^* - G$ is a closed, compact subset of X.

Hence $X - [X^* - G]$ is open in X.

$X - [X^* - G] = X \cap G$ is open in X.

As $G^* = G \cap X$, we get $G^* \in \mathfrak{J}$.

Thus from case (1) and case (2), $G^* \in \mathfrak{J}_X^* \Rightarrow G^* \in \mathfrak{J}$.

Hence $\mathfrak{J}_X^* \subseteq \mathfrak{J}$ ______________________ (ii)

From (i) and (ii), $\mathfrak{J} = \mathfrak{J}_X^*$.

Thus the relative topology $\mathfrak{J}_X^*$ on X coincides with the topology $\mathfrak{J}$ on X.

Hence $\langle X, \mathfrak{J} \rangle$ is a subspace of $\langle X^*, \mathfrak{J}^* \rangle$.

3) To prove that $\langle X^*, \mathfrak{J}^* \rangle$ is a compact space.

Let $\{G_\lambda\}$ be any open cover of X^*.

Hence $X^* = \bigcup_\lambda G_\lambda \Rightarrow \infty \in G_{\lambda_0}$ for some λ_0.

Hence $X^* - G_{\lambda_0}$ is closed compact subset of X. As $X^* - G_{\lambda_0} \subseteq X^*$ we get,

$X^* - G_{\lambda_0} \subseteq \bigcup_\lambda G_\lambda$ .Thus $\{G_\lambda\}$ forms an open cover for $X^* - G_{\lambda_0}$

and $X^* - G_{\lambda_0}$is compact subset of X^*.

$X^* - G_{\lambda_0} \subseteq \bigcup_{i=1}^{n} G_{\lambda_i}$ .But then $X^* = G_{\lambda_0} \cup G_{\lambda_1} \cup \ldots \cup G_{\lambda_n}$.

This shows that the open cover $\{G_\lambda\}$ of X^* has a finite sub-cover.

Hence $\langle X^*, \mathfrak{J}^* \rangle$ is compact.

4) To prove that X is dense in X^*.

X is non-compact subset of X^*. Hence X is not a closed subset of $\langle X^*, \mathfrak{J}^* \rangle$ (since $\langle X^*, \mathfrak{J}^* \rangle$ is compact and closed subset of a compact subset is compact). Hence $X^* - X$ is not open in X^*. As $X^* - X = \{\infty\}$ we get $\{\infty\}$ is not open in X^*. Hence for any open set G containing ∞ we get $G \cap X - \{\infty\} \neq \emptyset$ (since $G \neq \{\infty\} \implies G$ contains some $x \in X$). But this shows that ∞ is a limit point of X. Hence $\bar{X} = X \cup d(X) = X \cup \{\infty\} = X^*$. This shows that X is a dense in X^*.

Remarks:

(1) The one-point compactification $\langle X^*, \mathfrak{J}^* \rangle$ of $\langle X, \mathfrak{J} \rangle$ is also known Alenander's compactification of $\langle X, \mathfrak{J} \rangle$. The point ∞ is called the point at infinity.

(2) We know that $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is not compact. Compactification of this space is obtained by adding two points denoted by ∞ and$-\infty$ and properly introducing topology $\mathfrak{J}^*$ on $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$. Thus $\langle \mathbb{R}^*, \mathfrak{J}^* \rangle$ is the compactification of $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ but it is not the one-point compactification.

§5 Locally compact spaces

Definition 5.1: A topological space is $\langle X, \mathfrak{J} \rangle$ is a locally compact space if each point $x \in X$ has a compact neighbourhood.

Examples 5.2:

1) $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a locally compact space.

For each $x \in \mathbb{R}$, $[x - r\ ,\ x + r]$ is a compact neighbourhood of x ($r > 0$) [see Theorem …]

2) Let $\langle X, \mathfrak{J} \rangle$ be the discrete topological space. Then $\{x\}$ is a compact neighbourhood of each $x \in X$. Hence $\langle X, \mathfrak{J} \rangle$ is locally compact space.

Theorem 5.3: Every compact space is locally compact.

Proof: Let $\langle X, \mathfrak{J} \rangle$ be a compact space. Then for any $x \in X$, X itself is a compact neighbourhood of x. Hence X is locally compact.

Remark: Converse of Theorem 5.3 need not be true. i.e. every locally compact space need not be compact.

For this, consider the T – space $\langle \mathbb{R}, \mathfrak{J}_u \rangle$. $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is not compact but $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is locally compact. Also any infinite discrete space is locally compact but not compact.

Theorem 5.4: Closed subset of a locally compact space is locally compact.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a locally compact space and let F be any closed subset of X . To prove that the subspace $\langle F, \mathfrak{J}^* \rangle$ is locally compact.

Let $x \in F$. As $x \in X$ and X is locally compact, $\exists$ a compact neighbourhood say T of x in $\langle X, \mathfrak{J} \rangle$. As $F \cap T$ is a closed subset of a compact space T, we get $T \cap F$ is compact neighbourhood of x in F. Hence $\langle F, \mathfrak{J}^* \rangle$ is locally compact.

Theorem 5.5: Being a locally compact space is a topological property.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a locally compact space.

Let $\langle X, \mathfrak{J} \rangle$ be any topological space and let $f: X \to Y$ be a homeomorphism.

To prove that Y is locally compact space.

Let $y \in Y$. As f is onto, $\exists\ x \in X$ such that $f(x) = y$. As $x \in X$ and X is locally compact, $\exists$ a compact neighbourhood of say N of x. As f is continuous, onto, $f(N)$ is compact subset of Y containing y. Hence $f(N)$ is compact neighbourhood of $f(x) = y$ ($f(N)$ is a neighbourhood of y as f is an open map).

This shows that Y is locally compact.

Remark: Continuous image of a locally compact space need not be locally compact.

For this, consider the following example:

Let $\langle X, \mathfrak{J} \rangle$ be non-locally compact space. Let $\mathfrak{J}^*$ denote the discrete topology on X. Then $\langle X, \mathfrak{J}^* \rangle$ is a locally compact space. Let $i : X \to X$ be identity map. Then i is $\mathfrak{J}^* - \mathfrak{J}$ continuous, onto and one-one. But $i(X) = X$ is not locally compact.

This shows that continuous image of a locally compact space need not be locally compact.

§6 Countably compact spaces

Definition 6.1: A T – space $\langle X, \mathfrak{J} \rangle$ is said to be countably compact if any infinite subset of X has a limit point.

Example 6.2: Let $X = \mathbb{N}$, the set of all natural numbers. Let $\mathfrak{B} = \left\{ \{2n - 1 \,, 2n\} \mid n \in \mathbb{N} \right\}$.

Then $\mathfrak{B}$ is a base for some topology say $\mathfrak{J}$ on X. This T – space $\langle X, \mathfrak{J} \rangle$ is countably compact.

Let A be any infinite subset of X. Let p be the smallest number in A.

Case (1): p is an even number. Let $p = 2m$. Then $2m - 1$ is a limit point of A, for, the only basic open set containing $2m - 1$ is $G = \{2m - 1 \,,\ 2m\}$ and $2m \in G \cap A - \{2m - 1\}$ which implies $G \cap A - \{2m - 1\} \neq \emptyset$. Hence in this case $2m - 1$ is the limit point of A.

Case (2):): p is an odd number. Let $p = 2m - 1$. Then the only basic open set containing p is $H = \{2m - 1 \,,\ 2m\}$. $2m$ is the limit point of A, for, $2m - 1 \in H \cap A - \{2m\}$ which implies $H \cap A - \{2m\} \neq \emptyset$.

Thus from either the case we conclude that A has a limit point in X. Therefore $\langle X, \mathfrak{J} \rangle$ is countably compact space.

Theorem 6.3: Every compact space is countably compact.

Proof: Assume that there exists a compact space $\langle X, \mathfrak{J} \rangle$, which not countably compact. Hence, there exists an infinite set A of X that has no limit point in X. Thus, each $x \in X$ is not a limit point of A. But then for each $x \in X$, there exists an open set G_x containing x such that $G_x \cap A - \{x\} = \emptyset$. Hence either $G_x \cap A = \emptyset$ or $G_x \cap A = \{x\},\ \forall\, x \in X$. Again as $\{G_x\}_{x \in X}$ forms an open cover for X, it must have a finite sub-cover (X being a compact space).

Let $X = \displaystyle\bigcup_{i=1}^{n} G_{x_i}$.

Thus $A = X \cap A = \left[\bigcup_{i=1}^{n} G_{x_i}\right] \cap A = \bigcup_{i=1}^{n} [G_{x_i} \cap A]$.

As $G_{x_i} \cap A = \{x_i\}$ or $G_{x_i} \cap A = \emptyset,\ \forall\, i,\ 1 \leq i \leq n$;

$\bigcup_{i=1}^{n} [G_{x_i} \cap A]$ must be a finite set.

Hence A is a finite subset of X; a contradiction.

Thus, our assumption is wrong. This proves that every compact space is countably compact.

Example 6.4: By Theorem 6.3 we immediately get

(i) co-finite topological space is a countably compact space.

(ii) Fort's space is a countably compact space.

Remark: Converse of Theorem 6.3 need not be true. i.e. **every countably compact space need not be a compact space.**

For this consider the topological space given in Example 6.2.

The T – space defined in Example 7.2 is countably compact but not a compact space. Since the open cover $\{\{2n-1\,,2n\}\}$ of ℕ has no finite sub-cover.

Theorem 6.5: Any closed subset of a countably compact space is countably compact.

Proof:- Let A be any closed subset of X. To prove that A is countably compact.

i.e. To prove that any infinite subset E of A has a limit point in A.

$E \subseteq A \implies d(E) \subseteq d(A)$ and A is closed $\implies d(A) \subseteq A$. Hence $d(E) \subseteq A$. $E \subseteq X$ and X is countably compact $\implies E$ has a limit point say p in X. But then $p \in d(E) \implies p \in A$. This in turn shows that A is countably compact.

Theorem 6.6: Being countably compact space is a topological property.

Proof:- Let $\langle X, \mathfrak{I}\rangle$ and $\langle Y, \mathfrak{I}^*\rangle$ be two T – spaces. Let $\langle X, \mathfrak{I}\rangle$ be a countably compact space and let $f{:}\,XY$ be a homeomorphism. To prove that Y is countably compact space. Let A be any infinite subset of Y. Then f being one-one and onto, $f^{-1}(A) = B$ is an infinite subset of X. As X is countably compact, B has a limit point in X say p. Claim that $f(p) \in Y$ is a limit point of A. Let

$G^* \in \mathfrak{J}^*$ such that $f(p) \in G^*$. Then $G = f^{-1}(G^*) \in \mathfrak{J}$ and $p \in G$. As p is a limit point of B, we get $G \cap B - \{p\} \neq \emptyset$. But this will imply $G^* \cap A - \{f(p)\} \neq \emptyset$. Hence $f(p)$ is a limit point of A in Y. Therefore Y is countably compact. Homeomorphic image of a countably compact space being countably compact, the result follows.

Exercises

1) Show that Fort's space is compact.
2) Show that Cantor set C in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is compact.
3) Show that $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is not compact.
4) Explain in detail what do you mean by one-point compactification of $\langle X, \mathfrak{J} \rangle$
5) Prove that:
    i. Being compact space is a topological property..
    ii. Being countably compact space is a topological property.
    iii. Being locally compact space is a topological property.
6) Prove or disprove the following statements.
    i. Being countably compact space is a hereditary property.
    ii. Every compact space is countably compact.
    iii. Every countably compact space is compact.
    iv. Continuous image of a locally compact space is locally compact.
    v. Every compact space is locally compact.
    vi. Every locally compact space is compact.
    vii. A subset A of $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is compact if A is bounded and closed.
    viii. Any compact subset of $\mathbb{R}$ is closed in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$.
    ix. Closed sets are not the only compact subsets in a compact space
    x. Let $\langle X, \mathfrak{J}^* \rangle$ be a compact space and $\mathfrak{J}^* \leq \mathfrak{J}$. Then $\langle X, \mathfrak{J} \rangle$ is a compact space.
7) Prove that a topological space $\langle X, \mathfrak{J} \rangle$ is compact if and only if every family of closed sets having the finite intersection property has a non-empty intersection.

Unit 7

Connected Spaces

Unit 7: Connected Spaces

§1 Separated sets

Definition 1.1: Let $\langle X, \mathfrak{J} \rangle$ be a T – space. The subsets A and B of X are said to be separated in X if

(i) $A \neq \emptyset$ and $B \neq \emptyset$.

(ii) $A \cap B = \emptyset$.

(iii) $A \cap d(B) = \emptyset$ and $B \cap d(A) = \emptyset$.

Example 1.2:

$A = (1,4)$ and $B = (5,8)$ are separated sets in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$.

Remarks:

1) Conditions (ii) and (iii) can be combined into the following single condition

$$(*)\ldots\ (A \cap \bar{B}) \cup (B \cap \bar{A}) = \emptyset$$

This condition is known as Hausdorff Lenne's condition.

2) Any two disjoint, non-empty closed sets in any T – space are separated sets.

3) Any two disjoint, non-empty open sets in a T – space are separated sets. Let A and B be both open, non-empty disjoint sets in X .

$A \cap B = \emptyset \implies A \subseteq X - B \implies \bar{A} \subseteq \overline{X - B} = X - B \implies \bar{A} \cap B \subseteq (X - B) \cap B = \emptyset$.

Hence $\bar{A} \cap B = \emptyset$. Similarly $\bar{B} \cap A = \emptyset$.

Therefore A and B are separated sets.

4) If A and B are separated sets in $\langle X, \mathfrak{J} \rangle$ and if C and D are non-empty subsets of X such that $C \subseteq A$ and $D \subseteq B$, then C and D are also separated sets.

Theorem 1.3: If A and B are separated sets in $\langle X, \mathfrak{J} \rangle$, then A and B are both open and closed in $A \cup B$ and conversely.

Proof: Let A and B be separated sets in $\langle X, \mathfrak{J} \rangle$. Hence $A \neq \emptyset$ and $B \neq \emptyset$ and

$(A \cap \bar{B}) \cup (B \cap \bar{A}) = \emptyset$ ($\bar{A} = cl_X\, A$) .

Let $Y = A \cup B$. Then

$cl_Y\ A = cl_X\ A \cap Y$ … (By Theorem 3.3 in Unit 3)

$= cl_X\ A \cap (A \cup B)$

$= [cl_X\ A \cap A] \cup [cl_X\ A \cap B]$

$= A \cup \emptyset$ … (since A and B are separated sets)

$= A$.

This shows that A is closed in Y.

Similarly, we can show that B is closed in Y. As $A \cap B = \emptyset$ and $A \cup B = Y$, A and B are complements of each other in Y.

Hence A and B both are open in Y.

Conversely, let A and B are both open and closed in $A \cup B$.

To prove that A and B are separated in X.

By data $A \neq \emptyset$ and $B \neq \emptyset$ and $A \cap B = \emptyset$. Let $Y = A \cup B$.

$A = cl_Y\ A = cl_X\ A \cap Y$ (see Theorem 3.3 in Unit 3)

$= cl_X\ A \cap (A \cup B)$

$= [cl_X\ A \cap A] \cup [cl_X\ A \cap B]$

$= A \cap [cl_X\ A \cap B]$

Thus $A = A \cap [cl_X\ A \cap B]$Hence, $cl_X\ A \cap B \subseteq A$.

But then $cl_X\ A \cap B \subseteq A \cap B = \emptyset$, will imply $cl_X\ A \cap B = \emptyset$ i.e. $\bar{A} \cap B = \emptyset$ ($\bar{A} = cl_X\ A$).

Similarly, we can prove that $cl_X\ B \cap A = \bar{B} \cap A = \emptyset$.

As Hausdorff Lenne's condition is satisfied by A and B we get A and B are separated sets in X.

Theorem 1.4: Let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of a T – space $\langle X, \mathfrak{J} \rangle$ and $A, B \subseteq Y$. A , B are $\mathfrak{J}^*$ separated if and only if A , B are $\mathfrak{J}$ – separated (i.e. A , B are separated in $\langle Y, \mathfrak{J}^* \rangle$ if and only if A , B are separated in $\langle X, \mathfrak{J} \rangle$).

Proof: First note that,

$$
\begin{aligned}
[cl_Y\ A\ \cap B] \cup [cl_Y\ B\ \cap A] &= \big[[cl_X\ A\ \cap Y] \cap B\big] \cup \big[[cl_X\ B \cap Y] \cap A\big] \\
&= \big[cl_X\ A\ \cap [Y \cap B]\big] \cup \big[cl_X\ B \cap [Y \cap A]\big] \\
&= [cl_X\ A\ \cap B] \cup [cl_X\ B\ \cap A] \quad (since A, B \subseteq Y).
\end{aligned}
$$

Thus $[cl_Y\ A\ \cap B] \cup [cl_Y\ B\ \cap A] = \emptyset \iff [cl_X\ A\ \cap B] \cup [cl_X\ B\ \cap A] = \emptyset$.

Thus if A and B are non-empty disjoint sets in Y (and hence in X) are separated in $\langle Y, \mathfrak{J}^* \rangle$ if and only if A and B are separated in $\langle X, \mathfrak{J} \rangle$.

§2 Connected sets

Definition 2.1: Two separated sets A and B said to form a separation of E in a topological space $\langle X, \mathfrak{J} \rangle$ if $E = A \cup B$.

We denote this by $E = A \mid B$.

Definition 2.2: Let $\langle X, \mathfrak{J} \rangle$ be a T – space. A subset E of X is said to be connected if it has no separation in $\langle X, \mathfrak{J} \rangle$.
i.e. E is connected if E cannot be expressed as union of two disjoint, non-empty sets satisfying the Hausdorff Lenne's condition.

Examples 2.3:

(1) $\emptyset$ and $singleton\ sets$ are connected sets in any topological space.
(2) In a discrete topological space $\langle X, \mathfrak{J} \rangle$, $\{x\}$ $(x \in X)$ are the only connected sets ($|X| \geq 2$).
(3) Any interval in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is connected.
(4) Any indiscrete topological space is connected.
(5) Let $X = \{a, b\}$ and $\mathfrak{J} = \{\emptyset, \{a\}, X\}$. Then $\langle X, \mathfrak{J} \rangle$ is a connected space.

Theorem 2.4: Let $\langle X, \mathfrak{J} \rangle$ be a topological space and let A and B be non-empty subsets of X. The following statements are equivalent.

(1) $X = A \mid B$.
(2) $X = A \cup B$ and $\bar{A} \cap \bar{B} = \emptyset$.
(3) $X = A \cup B$, $A \cap B = \emptyset$ and A , B are both closed in X.
(4) $B = X - A$ and A is both open and closed in X.
(5) $B = X - A$ and $b(A) = \emptyset$ $(b(A) =$ boundary of A).
(6) $X = A \cup B$, $A \cap B = \emptyset$ and A, B both are open in X.

Proof: (1) $\Longrightarrow$ (2)

Let $X = A \mid B$. Then $X = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset, B \neq \emptyset$, $A \cap d(B) = \emptyset$ and $B \cap d(A) = \emptyset$.

To prove that $\bar{A} \cap \bar{B} = \emptyset$.

But $\bar{A} \cap \bar{B} = [A \cup d(A)] \cap [B \cup d(B)]$

$= (A \cap B) \cup (A \cap d(B)) \cup (B \cap d(A)) \cup [d(A) \cap d(B)]$

$= d(A) \cap d(B)$

Hence $x \in \bar{A} \cap \bar{B} \implies x \in d(A) \cap d(B)$

$\implies x \in d(A)$ and $x \in d(B)$

$\implies x \notin B \; and \; x \notin A$ (since $A \cap d(B) = \emptyset$ and $B \cap d(A) = \emptyset$)

$\implies x \in X - B \; and \; x \notin A$.

$\implies x \in A \; and \; x \notin A$ (since $A \cup B = X$, $A \cap B = \emptyset$).

which is a contradiction.

Hence $\bar{A} \cap \bar{B} = \emptyset$.

(2) ⟹ (3)

$A \subseteq \overline{A}$ and $B \subseteq \overline{B} \implies A \cap B \subseteq \bar{A} \cap \bar{B} = \emptyset \implies A \cap B = \emptyset$.

Now, $\bar{A} \cap \bar{B} = \emptyset \implies \overline{A} \subseteq X - \overline{B}$

$\implies \overline{A} \subseteq X - B$ (since $B \subseteq \overline{B} \implies X - \overline{B} \subseteq X - B$)

$\implies \overline{A} \subseteq A$ ($A \cup B = X$, $A \cap B = \emptyset \implies A = X - B$)

$\implies A = \overline{A}$.

Hence, A is a closed set in X. Similarly, we can prove that B is closed in X.

(3) ⟹ (4)

$A \cup B = X$, $A \cap B = \emptyset \implies A = X - B$.

As B is closed, $A = X - B$ is open. Thus A is both open and closed.

(4) ⟹ (5)

$b(A) = \overline{A} \cap \overline{(X - A)}$

$= A \cap (X - A)$ (since A is both open and closed $\implies X - A$ is also both open and closed)

$= \emptyset$

(5) ⟹ (6)

Let $b(A) = \emptyset$. We know that $\bar{A} = A^\circ \cup b(A)$.

As $b(A) = \emptyset$, $\bar{A} = A^\circ$. As $A^\circ \subseteq A \subseteq \bar{A}$, we get $\bar{A} = A = A^\circ$. Hence A is open.

As $B = X - A$, B is also open in X.

(6) ⟹ (1)

Let $X = A \cup B$, $A \cap B = \emptyset$ and A, B both are open in X. As A and B are complements of each other, A and B are closed in X. Hence $(A \cap \bar{B}) \cup (B \cap \bar{A}) = (A \cap B) \cup (B \cap A) = \emptyset$.
Thus $X = A \mid B$.

Remarks:

1) The space of rationals $\mathbb{Q}$ with relative topology is disconnected.
   Fix any real number α . Define $A = \{x \in \mathbb{Q} \mid x > \alpha\}$ and $B = \{x \in \mathbb{Q} \mid x < \alpha\}$.
   Then A and B both are non-empty disjoint subsets of $\mathbb{Q}$. Further both are open in $\mathbb{Q}$ w.r.t. the relative topology on $\mathbb{Q}$ [$\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is T – space and $\mathbb{Q} \subseteq \mathbb{R}$].
2) The space of irrational numbers $\mathbb{Q}'$ with relative topology is disconnected space.

Theorem 2.5: Let $\langle X, \mathfrak{J} \rangle$ be a T – space. If a connected set C has a non-empty intersection with both a set E and the complement of E in $\langle X, \mathfrak{J} \rangle$, then C has a non-empty intersection with the boundary of E.

Proof: To prove that $C \cap b(E) \neq \emptyset$ ($b(E)$ = boundary of E in $\langle X, \mathfrak{J} \rangle$).

Let $C \cap b(E) = \emptyset$.

$C = C \cap X = C \cap [E \cup (X - E)]$.

Hence, $C = [C \cap E] \cup [C \cap (X - E)]$ ________________________ ____(1)

$[C \cap E] \cap [C \cap (X - E)] = C \cap [E \cap (X - E)] = C \cap \emptyset = \emptyset$.

Hence, $[C \cap E] \cap [C \cap (X - E)] = \emptyset$ ________________________ _____(2)

Now, $[C \cap E] \cap \overline{[C \cap (X - E)]} \subseteq [C \cap \bar{E}\,] \cap \overline{(X - E)}$

$= C \cap \left[\bar{E} \cap \overline{(X - E)}\,\right]$

$= C \cap b(E)$ (since $b(E) = \bar{E} \cap \overline{(X - E)}$)

$= \emptyset$ …… (by assumption)

Thus $[C \cap E] \cap \overline{[C \cap (X - E)]} = \emptyset$ ________________________________ (3)

Similarly, we can prove that $[C \cap (X - E)] \cap \overline{[C \cap E]} = \emptyset$ _______(4)

Hence $\left\{[C \cap E] \cap \overline{[C \cap (X - E)]}\right\} \cup \left\{[C \cap (X - E)] \cap \overline{[C \cap E]}\right\}$

$= \emptyset \cup \emptyset = \emptyset$__________ ________ (5)

From (1), (2) and (5) we get,

$C = (C \cap E) \mid [C \cap (X - E)]$ (as by hypothesis$(C \cap E) \neq \emptyset$ and $[C \cap (X - E)] \neq \emptyset$).

This is a contradiction, as C is connected set.

Thus our assumption is wrong. Hence $C \cap b(E) \neq \emptyset$.

Theorem 2.6: Continuous image of a connected space is a connected space.

Proof: Let t $\langle X, \mathfrak{J} \rangle$ be a connected space. Let $\langle Y, \mathfrak{J}^* \rangle$ be any T – space and $f: X \longrightarrow Y$ be a continuous function. To prove that Y is a connected space.

Let if possible Y is not connected. Let $Y = A \mid B$. Then $Y = A \cup B$, and A, B are non-empty, disjoint open sets of Y (by Theorem 1.3). As f is onto, $f^{-1}(Y) = X$ and hence $X = f^{-1}(A) \cup f^{-1}(B)$. f being onto and continuous, $f^{-1}(A)$ and $f^{-1}(B)$ both are non-empty disjoint open sets in X. But this shows that $X = f^{-1}(A) \mid f^{-1}(B)$; a contradiction. Hence our assumption is wrong i.e. Y must be a connected space.

Corollary 2.7: Homeomorphic image of a connected space is a connected space.

Theorem 2.8: Let E be a subset of the subspace $\langle X^*, \mathfrak{J}^* \rangle$ of a T – space $\langle X, \mathfrak{J} \rangle$. E is $\mathfrak{J}^*$ connected if and only if E is $\mathfrak{J}$ connected.

Proof:- Let $A , B \subseteq X^* \subseteq X$.

Denote $c^*(A)$ = closure of A in $\langle X^*, \mathfrak{J}^* \rangle$ and $c(A)$ = closure of A in $\langle X, \mathfrak{J} \rangle$.

Then $c^*(A) = c(A) \cap X^*$ (see Theorem 3.3 in Unit 3).

$$\begin{aligned} \text{Then } [A \cap c(B)] \cup [B \cap c(A)] &= [(A \cap X^*) \cap c(B)] \cup [(B \cap X^*) \cap c(A)] \\ &= \big[A \cap [X^* \cap c(B)]\big] \cup \big[B \cap [X^* \cap c(A)]\big] \\ &= [A \cap c^*(B)] \cup [B \cap c^*(A)] \end{aligned}$$

Thus for $A , B \subseteq X^* \subseteq X$ we get ,

$[A \cap c(B)] \cup [B \cap c(A)] = \emptyset \quad \Leftrightarrow \quad [A \cap c^*(B)] \cup [B \cap c^*(A)] = \emptyset$.

Thus the set E has separation in $\langle X^*, \mathfrak{J}^* \rangle$ if and only if $\exists\ A , B \subseteq E$ such that $E = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset$ and $B \neq \emptyset$ and $[A \cap c^*(B)] \cup [B \cap c^*(A)] = \emptyset$.

$\Leftrightarrow \exists\ A , B \subseteq E$ such that $E = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset$ and $B \neq \emptyset$ and $[A \cap c(B)] \cup [B \cap c(A)] = \emptyset$.

$\Leftrightarrow$ E has separation in $\langle X, \mathfrak{J} \rangle$.

Theorem 2.9: Let C be a connected subset of the T – space $\langle X, \mathfrak{I} \rangle$.

Let $X = A \mid B$. Then either $C \subseteq A$ or $C \subseteq B$.

Proof:-X $= A \mid B \Rightarrow X = A \cup B,\ A \cap B = \emptyset$, $A \neq \emptyset$ and $B \neq \emptyset$.

Hence $C = C \cap X = C \cap (A \cap B) = (C \cap A) \cup (C \cap B)$.

Again $(C \cap A) \cup (C \cap B) = C \cap (A \cap B) = \emptyset$.

Now $\left[(C \cap A) \cap \overline{(C \cap B)}\right] \cup \left[(C \cap B) \cap \overline{(C \cap A)}\right] \subseteq (A \cap \bar{B}) \cup (B \cap \bar{A}) = \emptyset$.

$\Rightarrow \left[(C \cap A) \cap \overline{(C \cap B)}\right] \cup \left[(C \cap B) \cap \overline{(C \cap A)}\right] = \emptyset$.

Thus $C = (C \cap A) \mid (C \cap B)$, if$(C \cap A) \neq \emptyset$ and$(C \cap B) \neq \emptyset$. But as C is connected C has no separation. Hence either $(C \cap A) = \emptyset$ or $(C \cap B) = \emptyset$. Thus either $C \subseteq X - A = B$ or$C \subseteq X - B = A$ and hence the result follows.

Corollary 2.10: If C is a connected set in a T – space $\langle X, \mathfrak{I} \rangle$ and if $C \subseteq E \subseteq \bar{C}$, then E is connected in $\langle X, \mathfrak{I} \rangle$.

Proof:- To prove that E is connected set.

Let if possible E is not connected set. Then E must have a separation, say $E = A \mid B$.

As C is a connected subset of X and $E \subseteq X$, C is connected subset of E (by Theorem 2.8).

But then $C \subseteq A$ or $C \subseteq B$(by Theorem 2.9). Let us assume that $C \subseteq A$.

Then $\bar{C} \cap B \subseteq \bar{A} \cap B = \emptyset$ will imply $\bar{C} \cap B = \emptyset$. But $B \subseteq E \subseteq \bar{C}$ implies $\bar{C} \cap B = B$.

Thus $B = \emptyset$; a contradiction. Hence our assumption is wrong. This proves that E must be connected set.

Remark: By taking E = $\bar{C}$ in particular in Corollary 2.10, we get

if C is connected, then $\bar{C}$ is connected.

Corollary 2.11: Let $\langle X, \mathfrak{I} \rangle$ be a topological space such that any two points of a set $E \subseteq X$ are contained in same connected subset of E. Then E is connected.

Proof:- To prove that E is a connected set.

Let if possible E be not connected. Let $E = A \mid B$. Then $E = A \cup B,\ A \cap B = \emptyset$,

$A \neq \emptyset$ and $B \neq \emptyset$. As $A \neq \emptyset$ and $B \neq \emptyset$, select $a \in A$ and $b \in B$. As $A \cap B = \emptyset$, $a \neq b$ in X.

By assumption, $\exists$ a connected set C containing both a and b. By Theorem 2.8, $C \subseteq A$ or $C \subseteq B$.

Let $C \subseteq A$. Then $b \in A \cap B = \emptyset$; a contradiction. Hence our assumption is wrong. This shows that E must be a connected set.

Corollary 2.12: -The union of any family $\{C_\lambda\}$ of connected sets having a non-empty intersection, is a connected set.

Proof: Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a family of connected sets such that, $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$.

To prove that $E = \bigcup_{\lambda \in \Lambda} C_\lambda$ is connected set.

Let E is not connected. Then $E = A \mid B$.

Let $x \in \bigcap_{\lambda \in \Lambda} C_\lambda$ ...$\left(\text{since } \bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset\right)$. Then $x \in C_\lambda, \forall \lambda \in \Lambda,\ C_\lambda \subseteq E \implies x \in E = A \mid B$.

As $A \cap B = \emptyset$, either $x \in A$ or $x \in B$.

Without loss of generality, assume that $x \in A$. Thus $x \in C_\lambda \cap A \implies C_\lambda \cap A \neq \emptyset, \forall \lambda \in \Lambda$.

C_λ is connected subset of X and $E \subseteq X \implies C_\lambda$ is connected subset of E (by Theorem 2.8).

As C_λ is connected subset of E and $E = A \mid B$, $C_\lambda \subseteq A$ or $C_\lambda \subseteq B$. But as $A \cap B = \emptyset$, we get $C_\lambda \subseteq A, \forall \lambda \in \Lambda$ or $C_\lambda \subseteq B, \forall \lambda \in \Lambda$.

Hence, $C_\lambda \subseteq A, \forall \lambda \in \Lambda$ as $A \cap B = \emptyset$.

Thus $E = \bigcup_{\lambda \in \Lambda} C_\lambda \subseteq A$. But then $E = A \cup B \implies A \subseteq E$.

Hence, $E = A$. Therefore $B = \emptyset$; a contradiction.

Thus $E = \bigcup_{\lambda \in \Lambda} C_\lambda$ is connected set.

§3 Solved Problems

Problem1: Let $\langle X, \mathfrak{J} \rangle$ be a topological space and $E \subseteq X$. If $E = A \mid B$ and E is closed subset of X, then A and B are closed in X.

Solution: $E = A \mid B \implies A$ and B are proper non-empty sets of E and both are open and closed in E (by Theorem 2.4 (4)). Hence $A = E \cap F$ for some closed set F in X. As E itself is a closed set in X, we get $E \cap F = A$ is closed in X. Similarly we can prove that B is closed in X.

Problem 2: Show that in a connected topological space, every non-empty proper subset has a non-empty boundary.

Solution: - Let $E \neq \emptyset,\ E \subset X$.

To prove $b(E) \neq \emptyset$. Let $b(E) = \emptyset$.Then $\bar{E} \cap \overline{(X-E)} = b(E) = \emptyset$.

Hence, $E \cap \overline{(X-E)} = \emptyset$ and $\bar{E} \cap (X-E) = \emptyset$.

$E \neq \emptyset,\ E \neq X \quad \Longrightarrow \quad X - E \neq \emptyset\,, X - E \neq X.$

$E \cup (X-E) = X \; and \; E \cap (X-E) = \emptyset$.

Hence $X = E \mid (X-E)$, a contradiction.Hence, $b(E) \neq \emptyset$.

Problem3: If $\langle X, \mathfrak{J} \rangle$ is a connected topological space and $\mathfrak{J}^* \leq \mathfrak{J}$, then show that $\langle X, \mathfrak{J}^* \rangle$ is connected.

Solution:- Let $\langle X, \mathfrak{J}^* \rangle$ be not connected. Hence there exist proper, non-empty subsets A and B of X such that $A, B \in \mathfrak{J}^*$ and $X = A \cup B$, $A \cap B = \emptyset$. But as $\mathfrak{J}^* \leq \mathfrak{J}$, we get A and B are proper non-empty subset of X such that $A, B \in \mathfrak{J},\ X = A \cup B,\ A \cap B = \emptyset$.

But this shows that $\langle X, \mathfrak{J} \rangle$ is not a connected set; a contradiction.

Hence $\langle X, \mathfrak{J}^* \rangle$ is connected space.

Problem4: For a topological space $\langle X, \mathfrak{J} \rangle$ show that following statements are equivalent.

1) X is connected.
2) X cannot be written as disjoint union of two non-empty closed sets.
3) X cannot be written as disjoint union of two non empty open sets.
4) The only clo-open sets are $\emptyset$ and X. (clo-open = both open and closed)
5) Every non-empty proper subset of X has a non-empty boundary.

Solution:- The result follows immediately by definition and by Theorem 2.4.

Problem 5: Let E be a subset of a topological space $\langle X, \mathfrak{J} \rangle$.If E is connected , then E is not the union of any two non-empty sets A and B such that $\bar{A} \cap \bar{B} = \emptyset$.

Solution: Let E be connected. Let if possible $E = A \cup B, A \neq \emptyset, B \neq \emptyset$ and $\bar{A} \cap \bar{B} = \emptyset$.

$\bar{E} = \overline{(A \cup B)} = \bar{A} \cup \bar{B}$. $A \neq \emptyset \implies \bar{A} \neq \emptyset$ and $B \neq \emptyset \implies \bar{B} \neq \emptyset$.

$\bar{A} \cap \bar{\bar{B}} = \bar{A} \cap \bar{B} = \emptyset$ and $\bar{B} \cap \bar{\bar{A}} = \bar{B} \cap \bar{A} = \emptyset$.

Hence $\bar{E} = \bar{A} \mid \bar{B}$; a contradiction (since E is connected $\Longrightarrow$ $\bar{E}$ is connected)

Hence E is not a union of any two non-empty sets A and B such that $\bar{A} \cap \bar{B} = \emptyset$.

Exercises

Prove or disprove the following statements.

1) Closure of a connected set in $\langle X, \mathfrak{J} \rangle$ is a connected set.
2) Any two disjoint sets are separated in $\langle X, \mathfrak{J} \rangle$.
3) Any two separated sets are disjoint in $\langle X, \mathfrak{J} \rangle$.
4) Union of two connected sets in $\langle X, \mathfrak{J} \rangle$is a connected sets in $\langle X, \mathfrak{J} \rangle$.
5) X is connected if and only if X cannot be written as disjoint union of two non-empty closed sets.
6) X is connected if and only if X cannot be written as disjoint union of two non-empty open sets.
7) X is connected if and only if the only clo-open sets are $\emptyset$ and X.
8) X is connected if and only if every non-empty proper subset of X has a non-empty boundary.
9) Let C be a connected subset in $\langle X, \mathfrak{J} \rangle$ and $X = A \mid B$. Then either $C \subseteq A$ or $C \subseteq B$
10) X is connected if and only if it has non-empty proper subsets which are both open and closed.

Unit 8

First axiom spaces

Unit 8: *First Axiom Spaces*

§1 Definition and Examples

Definition 1.1: Let $\langle X, \mathfrak{I}\rangle$ be a topological space. $\langle X, \mathfrak{I}\rangle$ is said to be first axiom space (or f.a.s. in short) if it satisfies the following first axiom of countability.

For each point $x \in X$, there exists a countable family $\{B_n(x)\}_{n \in N}$ of open sets such that $x \in B_n(x)$ for each $x \in N$ and for any open set G containing x, $\exists\ n_0 \in N$ such that , $x \in B_n(x) \subseteq G$.

The family $\{B_n(x)\}_{n \in N}$ is called a countable local base at x.

Examples 1.2

(1) Every discrete topological space is first axiom space**.**

Let $\langle X, \mathfrak{I}\rangle$ be a discrete topological space. Then $\{\{x\}\}$ forms a countable local base at x. Hence $\langle X, \mathfrak{I}\rangle$ is a first axiom space.

(2) $\langle \mathbb{R}, \mathfrak{I}_u\rangle$ is a first axiom space**.**

Fix up any $\in\ \mathbb{R}$. For each $n \in N$ define $B_n(x) = \left(x - \frac{1}{n}\ , x + \frac{1}{n}\right)$. Then $\{B_n(x)\}_{n \in N}$ forms a countable family of open set in $\langle \mathbb{R}, \mathfrak{I}_u\rangle$.

Further if G is open set in $\mathbb{R}_u$ containing x , then by the definition of $\mathfrak{I}_u$,

$\exists\ r > 0$ such that $\left(x - \frac{1}{r}\ , x + r\right) \subseteq G$. Select n_0 so large, such that $\frac{1}{n_0} < r$.

Then $\left(x - \frac{1}{n_0}\ , x + \frac{1}{n_0}\right) \subseteq (x - r\ , x + r) \subseteq G \quad \Rightarrow B_{n_0} \subseteq G$.

Hence the family $\{B_n(x)\}_{n \in N}$ forms a countable local base of $x \in \mathbb{R}$. Therefore $\langle \mathbb{R}, \mathfrak{I}_u\rangle$ is a first axiom space.

(3) Every metric space is first axiom space.

Let $\langle X, d\rangle$ be a topological space and $\mathfrak{I}$ be the topology on X induced by the metric d. Fix up any $x \in X$. Define $B_n(x) = S\left(x, \frac{1}{n}\right)$ for any $n \in N$. Then by the definition of $\mathfrak{I}$, $B_n(x) \in \mathfrak{I}, \forall\, n \in N. x \in B_n(x)$ for each $n \in N$.

Let $G \in \mathfrak{I}$ such that $x \in G$. Then by the definition of $\mathfrak{I}, \exists\, r > 0$ such that $s(x, r) \subseteq G$. Select n so large that $\frac{1}{n_0} < r$.

Then $x \in S\left(x, \dfrac{1}{n_0}\right) \subseteq S(x, r) \subseteq G$.

Thus given $G \in \mathfrak{I}$ containing x, $\exists\ n_0 \in N$ such that $x \in B_{n_0}(x) \subseteq G$. Hence the countable family $\{B_n(x)\}$ of open sets in $\langle X, \mathfrak{I}\rangle$ forms a countable local base at x. This shows that $\langle X, d\rangle$ is a first axiom space.

(4) Co-finite topological space $\langle X, \mathfrak{I}\rangle$ (with X an infinite set) is a non-first axiom space.

Let $\langle X, \mathfrak{I}\rangle$ be a first axiom space and $x \in X$. Then $\exists$ a countable local base $\{B_n(x)\}$ at x. $B_n(x) \in \mathfrak{I}, \forall\, n \in N \implies X - B_n(x)$ is a closed set in $\langle X, \mathfrak{I}\rangle$, $\forall\, n \in N$

$\implies X - B_n(x)$ is a finite set in $\langle X, \mathfrak{I}\rangle$, $\forall\, n \in N$.

$$\implies \bigcup_{n=1}^{\alpha} [X - B_n(x)] \text{ is a countable subset of } X.$$

… (since countable union of countable sets is countable.)

Hence $X \neq \bigcup_{n=1}^{\infty} [X - B_n(x)]$.

Select $y \in X$ such that $y \notin \bigcup_{n=1}^{\infty} [X - B_n(x)]$ and $y \neq x$.

But then $y \in X - \left[\bigcup_{n=1}^{\infty} [X - B_n(x)]\right] \implies y \in \bigcup_{n=1}^{\infty} B_n(x)$

Now then $G = X - \{y\}$. Then G is an open set $\langle X, \mathfrak{I}\rangle$ containing x (since x $\neq y$). Hence $\exists\ n_0 \in N$ such that $x \in B_{n_0}(x) \subseteq X - \{y\}$.

$y \in B_{n_0}(x) \implies y \in X - \{y\}$, a contradiction .

Hence $\langle X, \mathfrak{I}\rangle$ is a non-first axiom space.

(5) Co-countable topological space defined on an uncountable set is non-first axiom space**.**

Let $\langle X, \mathfrak{I} \rangle$ be a first axiom space and $x \in X$. Then $\exists$ a countable local base $\{B_n(x)\}$ at x.

$B_n(x) \in \mathfrak{I} \ \forall\, n \in N \implies X - B_n(x)$ is a closed set in $\langle X, \mathfrak{I} \rangle, \forall\, n \in N$

$\implies X - B_n(x)$ is a finite set in $\langle X, \mathfrak{I} \rangle$, $\forall\, n \in N$.

$$\implies \bigcup_{n=1}^{\alpha} [X - B_n(x)] \text{ is a countable subset of X.}$$

… (since countable union of countable sets is countable).

Hence $X \neq \bigcup_{n=1}^{\infty} [X - B_n(x)]$.

Select $y \in X$ such that $y \notin \bigcup_{n=1}^{\infty} [X - B_n(x)]$ and $y \neq x$.

But then $y \in X - \left[\bigcup_{n=1}^{\infty} [X - B_n(x)] \right] \implies y \in \bigcup_{n=1}^{\infty} B_n(x)$.

Now then $G = X - \{ y \}$. Then G is an open set $\langle X, \mathfrak{I} \rangle$ containing x (since $x \neq y$).

Hence $\exists\ n_0 \in N$ such that $x \in B_{n_0}(x) \subseteq X - \{ y \}$.

$y \in B_{n_0}(x) \implies y \in X - \{ y \}$, a contradiction .

Hence $\langle X, \mathfrak{I} \rangle$ is a non-first axiom space.

(6) Fort's space is a non-first axiom space.

Let $\langle X, \mathfrak{I} \rangle$ be a Fort's space. X is an uncountable set, ∞ is a fixed point of X and $\mathfrak{I} = \{ A \subseteq X \,/\, \infty \notin A\} \cup \{ A \subseteq X \,/\, \infty \in A$ and $X - A$ is finite$\}$.

Assume that $\langle X, \mathfrak{I} \rangle$ is a first axiom space. Hence, there exists a countable local base say $\{ B_n(\infty)\}_{n \in N}$, $\forall\ n \in N$.

We get $\bigcup_{n=1}^{\infty} B_n(\infty) \in \mathfrak{I}$.

As $\infty \in \bigcup_{n=1}^{\infty} B_n(\infty)$ by the definition of $\mathfrak{I}$, We get $\left[X - \bigcup_{n=1}^{\infty} B_n(\infty) \right]$ is a finite set.

Hence $X - \bigcup_{n=1}^{\infty} B_n(\infty) \neq X$ (Since X is uncountable) $i.e. \bigcap_{n=1}^{\infty} [X - B_n(\infty)] \neq X$.

Select $x \in X$ such that $x \notin \bigcap_{n=1}^{\infty} [X - B_n(\infty)]$. As $x \neq \infty$, $\infty \in X - \{x\}$.

As $X - \{x\}$ is an open set in X containing ∞, we get $\infty \in B_{n_0}(\infty) \subseteq X - \{x\}$

for some $n_0 \in N$.

But by the choice of x, $x \in B_{n_0}(\infty)$ implies $x \in X - \{x\}$; a contradiction.

Hence $\langle X, \mathfrak{J} \rangle$ is a non-first axiom space.

§2 Properties

Theorem 2.1: Let $\langle X, \mathfrak{J} \rangle$ be a first axiom space. Then $\exists$ a nested / monotone decreasing local base at each that $x \in X$.

Proof:- Let $x \in X$ and $\{B_n(x)\}_{n \in N}$ be a countable local base at x.

Define $B_1^*(x) = B_1(x)$.

$B_2^*(x) = B_1(x) \cap B_2(x)$.

$B_3^*(x) = B_1(x) \cap B_2(x) \cap B_3(x)$.

In general , $B_n^*(x) = \bigcap_{i=1}^{n} B_i(x)$.

Then $\{ B_n^*(x)\}_{n \in N}$ forms a monotone decreasing local base at x.

Theorem 2.2:- Being a first axiom space is a hereditary property.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a first axiom space and $\langle Y, \mathfrak{J}^* \rangle$ be its subspace. To prove that $\langle Y, \mathfrak{J}^* \rangle$ is a first axiom space. Select $y \in Y$. Then $y \in X$ and X is first axiom space

$\Longrightarrow$ $\exists$ a countable local base $\{B_n(y)\}_{n \in N}$ at y in $\langle X, \mathfrak{J} \rangle$.

Define $B_n^*(y) = B_n(y) \cap Y$ for each $x \in N$. Then $\{B_n^*(y)\}_{n \in N}$ forms a countable family of open sets in $\langle X^*, \mathfrak{J}^* \rangle$ and $y \in B_n^*(y)$ $\forall\, n \in N$.

Let $G^* \in \mathfrak{J}^*$ containing y.

Then $G^* = G \cap Y$ for some $G \in \mathfrak{J}$, As $y \in G$ and $G \in \mathfrak{J}$, $\exists$ $n_0 \in N$ such that y $\in B_{n_0}(y) \subseteq G$. But then $y \in B_{n_0}(y) \cap Y \subseteq G \cap Y$ will imply

y $\in B^*_{n_0}(y) \subseteq G^*$. Hence $\{B^*_n(y)\}_{n\in N}$ forms a countable local bas at y.

Hence $\langle Y, \mathfrak{I}^* \rangle$ is a first axiom space.

Theorem 2.3:- The property of being a first axiom space is a topological property.

Proof: Let $\langle X, \mathfrak{I} \rangle$ and $\langle Y, \mathfrak{I}^* \rangle$ be two topological spaces and let $f: X \to Y$ be a homeomorphism.

Assume that $\langle X, \mathfrak{I} \rangle$ is a f.a.s.

To prove that $\langle Y, \mathfrak{I}^* \rangle$ is a f.a.s.

Let $y \in Y$. As f is onto, $\exists\ x \in X$ such that $f(x) = y$. As X is a f.a.s. $\exists$ a countable local base say $\{B_n(x)\}_{n\in N}$ at x. As $B_n(x) \in \mathfrak{I}$, $\forall\ n \in \mathbb{N}$ we get $f[B_n(x)] \in \mathfrak{I}^*$ (since f is an open mapping).

Again, $x \in B_n(x) \Longrightarrow y \in f[B_n(x)]$, $\forall\ n \in \mathbb{N}$.

Claim that $\{f[B_n(x)]\}_{n\in\mathbb{N}}$ will form a countable local base at $y = f(x)$.

(i) $f[B_n(x)] \in \mathfrak{I}$, $\forall\ n \in \mathbb{N}$.

(ii) $y \in f[B_n(x)]$, $\forall\ n \in \mathbb{N}$.

(iii) Let $G^* \in \mathfrak{I}^*$ such that $y \in G^*$. Then $f^{-1}[G^*] \in \mathfrak{I}$ (since f is continuous) and $x \in f^{-1}[G^*]$.

Hence $\exists\ n_0 \in \mathbb{N}$ such that $B_{n_0}(x) \subseteq f^{-1}[G^*]$.

But then $f\left[B_{n_0}(x)\right] \subseteq f\left[f^{-1}[G^*]\right] \Longrightarrow f\left[B_{n_0}(x)\right] \subseteq G^*$.

From (i), (ii) and (iii) we get $\{f[B_n(x)]\}_{n\in\mathbb{N}}$ forms a local base at $y \in Y$.

Hence $\langle Y, \mathfrak{I}^* \rangle$ is a f.a.s.

Remark: Continuous image of a f.a.s. need not be a f.a.s.

For this consider the discrete topological space $\langle X, \mathfrak{I} \rangle$ and co-countable topological space $\langle X, \mathfrak{I}^* \rangle$, where X is an uncountable set. Then the identity map $i: X \to X$ is $\mathfrak{I} - \mathfrak{I}^*$ continuous and onto.

$\langle X, \mathfrak{I} \rangle$ is a f.a.s. but $\langle X, \mathfrak{I}^* \rangle$ is not a f.a.s. (see Example 1.2 (1) and Example 1.2 (5))

§3 Sequentially continuity and first axiom spaces

Definition 3.1: Let $\langle X, \mathfrak{I} \rangle$ and $\langle Y, \mathfrak{I}^* \rangle$ be two topological spaces and let $f: X \to Y$ be a function. f is said to be sequentially continuous on X if for every sequence $\{x_n\}$ converging to x in X, the image sequence $\{f(x_n)\}$ converges to $f(x)$ in Y.

Theorem3.2: Let $\langle X, \mathfrak{J} \rangle$ be a first axiom space and $\langle Y, \mathfrak{J}^* \rangle$ be any topological space. A function $f: X \to Y$ is continuous on X if and only if f is sequentially continuous.

Proof: Only if part –

Let f be a continuous on X. To prove that f is sequentially continuous.

Let $\{x_n\}$ be a sequence of points of X converging to $x \in X$. To prove that the image sequence $\{f(x_n)\}$ converges to $f(x)$ in Y.

Let $G^* \in \mathfrak{J}^*$ such that $f(x) \in G^*$.

$f: X \to Y$ is continuous $\Longrightarrow f^{-1}[G^*] \in \mathfrak{J}$.

$f(x) \in G^* \implies x \in f^{-1}[G^*]$.

As $x_n \to x$, $\exists\, m \in \mathbb{N}$ such that $x_n \in f^{-1}[G^*]$ for each $n \geq m$.

But then $f(x_n) \in G^*\ for\ m \geq n$, (as $\exists\ f^{-1}[G^*] \subseteq G^*$).

But this shows that $f(x_n) \to f(x)$ in Y.

If part –

Let $f: X \to Y$ be sequentially continuous.

To prove that f is continuous on X. Assume if possible, f be not continuous on X. Hence $\exists\, x \in X$ such that f is not continuous at x. Hence $\exists\, G^* \in \mathfrak{J}^*$ containing $f(x)$ such that $f(G) \nsubseteq G^*$ for any $G \in \mathfrak{J}$ containing x. As X is a f.a.s. $\exists$ countable monotone decreasing local base say $\{B_n(x)\}_{n \in \mathbb{N}}$ at x. As $x \in B_n(x)$ and $B_n(x) \in \mathfrak{J}$ for each n, we get $f\big(B_n(x)\big) \nsubseteq G^*$ for each x.

Select $f(x_n) \in f[B_n(x)] \cap [Y - G^*]$; $\forall\, n \in N$. Then $\{x_n\}$ is a sequence of points in X.

Claim 1: The sequence $\{x_n\}$ converges to x in X.

Let O be any open set in X containing x. Then $\exists\, m \in \mathbb{N}$ such that $B_m(x) \subseteq$ O (by definition of local base).

Hence $x_m \in$ O (by the choice of $\{x_n\}$, $x_m \in B_m(x)$).

As $\{B_m(x)\}$ is monotonically decreasing, we get $x_n \in B_m(x)$ for all $n \geq m$.

This shows that $x_n \to x$ in X.

Claim 2: The sequence $f(x_n)$ does not converge to $f(x)$ in Y.

We know that $G^* \in \mathfrak{J}^*$ containing $f(x)$ and $f(x_n) \notin G^*$ for any n (by the choice of $f(x_n)$).

Hence $f(x_n) \nrightarrow f(x)$. Thus $x_n \to x$ in X but $f(x_n) \nrightarrow f(x)$ in Y. This contradicts the data that f

is sequentially continuous on X. Hence, our assumption must wrong. Hence f is continuous on X.

Remark: Note that the property that X is a first axiom space is not used in the proof of only if part. Hence if $f: X \to Y$ continuous on X then $f: X \to Y$ is sequentially continuous, for any topological space X.

Theorem3.3: Let X be a first axiom space and $A \subseteq X$. Let $a \in X$. Then a is a limit point of A if and only if $\exists$ a sequence $\{a_n\}$ with $a_n \in A \cap [X - \{a\}]$ for all n, which converges to a.

Proof: Only if part –

Let a be a limit point of A.

Hence for any $G \in \mathfrak{J}$ containing A, $G \cap A - \{a\} \neq \emptyset$ _______________ [I]

As X is a f.a.s. $\exists$ a countable, monotonically decreasing local base at a, say $\{B_n(a)\}$. Hence by [I], $B_n(a) \cap A - \{a\} \neq \emptyset$; $\forall\, n \in \mathbb{N}$.

Select $a_n \in B_n(a) \cap A - \{a\}$; $\forall\, n \in \mathbb{N}$. Then $\{a_n\}$ is a sequence of limit points of $A \cap [X - \{a\}]$ and $a_n \to a$ (since $a_n \in B_n(a)$ and $\{B_n(a)\}$ is monotonically decreasing local base).

If part –

Let $\exists$ a sequence $a_n \in A \cap [X - \{a\}]$ such that $a_n \to a$. To prove that a is a limit point of A.

Let $G \in \mathfrak{J}$ containing a. Then as $a_n \to a$, $\exists\, N$ such that $a_n \in G$ for all $n \geq N$. Thus $a_N \in G \cap A - \{a\}$ …(by choice of the sequence $\{a_n\}$)

But this shows that for any open set G containing a, $G \cap A - \{a\} \neq \emptyset$.

Hence a is a limit point of A.

Remark: Note that the property that X is a f.a.s. is not used in the proof of 'If part' and hence the if part is true in any topological space.

Exercises

(I) State whether the following statements are true or false.

(1) Every compact space is a f.a.s.

(2) Every f.a.s. is compact.

(3) Every discrete topological space is a f.a.s.

(4) Every indiscrete topological space is a f.a.s.

(5) $p -$ exclusion topological space a f.a.s.

(6) $p -$ inclusion topological space a f.a.s.

(II) Prove or disprove the following statements.

(1) Let $\langle X, \mathfrak{J} \rangle$ be a first axiom space and $\langle Y, \mathfrak{J}^* \rangle$ be any topological space. $f: X \to Y$ is continuous if and only if f is sequentially continuous.

(2) Continuous image of a f.a.s. is a f.a.s.

(3) Homeomorphic image of a f.a.s. is a f.a.s.

(4) Subspace of a f.a.s. is a f.a.s.

Unit 9

Second axiom spaces

§1 Definition and properties of second axiom spaces.

§2 Sequentially compact spaces and second axiom spaces.

Unit 9: Second axiom spaces

§1 Definition and properties of second axiom spaces.

Definition 1.1: A topological space $\langle X, \mathfrak{J}\rangle$ is a second axiom space (s.a.s. in short) if it satisfies the following second axiom of countability.

(*) $\mathfrak{J}$ has a countable base $\{ B_n \}_{n \in N}$.

Examples 1.2:

1) $\langle \mathbb{R}, \mathfrak{J}_u\rangle$ is a second axiom space as $\mathfrak{B} = \{ (a, b) \mid a, b \in Q\}$ is a countable base for $\mathfrak{J}_u$.

2) Let $\langle \mathbb{R} , \mathfrak{J}\rangle$ be a discrete topological space with X-an uncountable set. Then $\langle X, \mathfrak{J}\rangle$ is a non-second axiom space.

Theorem 1.3: Every second axiom space is a first axiom space**.**

Proof: - Let $\langle X, \mathfrak{J}\rangle$ be a second axiom space. Let $\mathfrak{B} = \{ B_n \}_{n \in N}$ be a countable base for $\mathfrak{J}$. Fix up any $x \in X$. Consider those $B_n \in \mathfrak{B}$ for which $x \in B_n$ and denote this family by $\{B_n(x)\}$. As $\{B_n(x)\} \subseteq \{B_n\}$, the family $\{B_n(x)\}$ is a countable family of open sets. By selection, $x \in B_n(x),\ \forall x$. Let $G \in \mathfrak{J}$ such that $x \in G$. As $\mathcal{B}$ is a base for $\mathfrak{J}$, $\exists\ B_{n_0} \in \mathfrak{B}$ such that $x \in B_{n_0} \subseteq G$. But then $B_{n_0} \in \{B_n(x)\}$.

Hence $\{B_n(x)\}$ forms a countable local base at x. As this is true for any $x \in X, \langle X, \mathfrak{J}\rangle$ is a first axiom space.

Remark:- Converse of the Theorem 1.3 need not be true.

i.e. every first axiom space need not be a second axiom space. For this consider a discrete topological space $\langle X, \mathfrak{J}\rangle$ defined on an uncountable set X. This space is a first axiom space but it is not a second axiom space. As $\{\{x\}\}$ forms a countable local base at each $x \in X$, we get $\langle X, \mathfrak{J}\rangle$ is a first axiom space. Let if possible $\langle X, \mathfrak{J}\rangle$ be a second axiom space. Then there exists a countable base say $\mathfrak{B} = \{ B_n \}_{n \in N}$ for $\mathfrak{J}$. As $\{x\} \in \mathfrak{J}$ and $x \in \{x\}$, $\exists\ n_0 \in N$ such that $x \in B_{n_0} \subseteq \{x\}$. But Then $B_{n_0} = \{ x \}$.

Thus $\{\{x\} \mid x \in X\} \subseteq \{B_n \mid n \in N\} \Longrightarrow \{\{x\} \mid x \in X\}$ is a countable family; a contradiction, as X is uncountable. Hence $\langle X, \mathfrak{J}\rangle$ is not a second axiom space.

Theorem 1.4: The property of being a second axiom space is a hereditary property.

Proof:- Let $\langle X, \mathfrak{J}\rangle$ be a second axiom space and let $\langle Y, \mathfrak{J}^*\rangle$ be its subspace. To prove that $\langle Y, \mathfrak{J}^*\rangle$ is a second axiom space. By definition of subspace $Y \subseteq X\ (Y \neq \emptyset)\ and\ \mathfrak{J}^* = \{G \cap Y / G \in \mathfrak{J}\}$. Let $\mathfrak{B}$ be a countable base for $\langle X, \mathfrak{J}\rangle$. Then Define $\mathfrak{B}^* = \{B \cap Y \mid B \in \mathfrak{B}\}$ .Claim that $\mathfrak{B}^*$ is a countable base for $\mathfrak{J}^*$. Obviously, $\mathfrak{B} \subseteq \mathfrak{J} \Longrightarrow \mathfrak{B}^* \subseteq \mathfrak{J}^*$ and $\mathfrak{B}^*$ is countable set.

Let $G^* \in \mathfrak{J}^*\ and\ x \in G^*$. Then $x \in Y \subseteq X$ and $\exists\ G \in \mathfrak{J}$ such that $G^* = G \cap Y$. As $G \in \mathfrak{J}$ and $x \in G, \exists\ B \in$, such that $x \in B \subseteq G$.

Hence, $x \in B \cap Y \subseteq G \cap Y \Longrightarrow \exists\ B^* \in \mathfrak{B}^*$ such that $x \in B^* \subseteq G^*$. This shows that $\mathfrak{B}^*$ is a countable base for $\mathfrak{J}^*$. As a subspace of a second axiom space is a second axiom space, the property of being a second axiom space is a hereditary property.

Theorem 1.5: The property of being a second axiom space is a topological property.

Proof:- Let $\langle X, \mathfrak{J}\rangle$ be a second axiom space and let $\langle X^*, \mathfrak{J}^*\rangle$ be any topological space. Let $f: X \to X^*$ be a homeomorphism. To prove that $\langle X^*, \mathfrak{J}^*\rangle$ is a second axiom space. As $\langle X, \mathfrak{J}\rangle$ is a second axiom space, there exists a countable base say $\mathfrak{B}$ for $\mathfrak{J}$. As $\mathfrak{B} \subseteq \mathfrak{J}$, $f(B) \in \mathfrak{J}^*$ for each $B \in \mathfrak{B}$, being an open mapping. Define $\mathfrak{B}^* = \{f(B) \mid B \in \mathfrak{B}\}$.

Claim that $\mathfrak{B}^*$ is a countable base for $\mathfrak{J}^*$.

$\mathfrak{B}^*$ is a countable family of open sets in $\langle X^*, \mathfrak{J}^*\rangle$. Let $G^* \in \mathfrak{J}^*$ and $x^* \in G^*$. As f is onto, $\exists\ x \in X$ such that $f(x) = x^*. f^{-1}(G^*) \in \mathfrak{J}$ and $x \in f^{-1}(G^*)$

As $\mathfrak{B}$ is a base for the topology $\mathfrak{J}$, $\exists\ B \in \mathfrak{B}$ such that $x \in B \subseteq f^{-1}(G^*)$.

But then $f(x) = x^* \in B^* \subseteq G^*$ will imply that $\mathfrak{B}^*$ is a base for $\mathfrak{J}^*$.

Thus $\mathfrak{B}^* = \{f(B) \mid B \in \mathfrak{B}\}$ forms a countable base. Hence $\langle X^*, \mathfrak{J}^*\rangle$ is a second axiom space. Thus, homeomorphic image of a second axiom space is a second axiom space

Theorem 1.6: Any family of disjoint open sets in a second axiom space is countable.

Proof:- Let $\langle X, \mathfrak{J}\rangle$ be a second axiom space. Let $\mathcal{K}$ denote the family of disjoint open sets in X. To prove that $\mathcal{K}$ is countable. As X is a second axiom space, there exists a countable base say $\mathfrak{B} = \{B_n \mid n \in N\}$ for $\mathfrak{J}$. Let $A \in \mathcal{K}$. As $A \in \mathfrak{J}$, $\exists\ n_0 \in N$ such that $B_{n_0} \subseteq A$.

Let $m = \{ x \in N \mid B_n \subseteq A\}$ and let $m =$ the smallest member of M. As the Members of $\mathcal{K}$ are disjoints, the assignment of m to $A \in \mathcal{K}$ is unique. Now list the members of $\mathcal{K}$ according to the order of the associated integers to them. But this shows that $\mathcal{K}$ is countable.

Theorem 1.7: Let $\langle X, \mathfrak{J} \rangle$ be a second axiom space and let A be an uncountable subset of X. Then some point of A will be a limit of A.

Proof:- $\langle X, \mathfrak{J} \rangle$ is a second axiom space. Hence, $\exists$ a countable base say $\mathfrak{B} = \{ B_n \mid n \in N\}$ for $\mathfrak{J}$. If possible, assume that no point of A is its limit point. Hence for each $a \in A, \exists\, G_a \in \mathfrak{J}$ such that $a \in G_a$ and $G_a \cap A - \{a\} = \emptyset$. As A is uncountable, $G_a \cap A = \emptyset$ is not possible. Hence $G_a \cap A = \{a\}$ for each $a \in A$.

As $a \in G_a$ *and* $G_a \in \mathfrak{J}$, $\exists\, n_a \in N$ such that $a \in B_{n_a} \subseteq G_a$. Hence $B_{n_a} \cap A = \{a\}$.

Note that for $a \neq b$, $B_{n_a} \neq B_{n_b}$.

[$a \neq b \Longrightarrow \{a\} \neq \{b\} \Longrightarrow B_{n_a} \cap A \neq B_{n_b} \cap A$. $a \in B_{n_a}$ and $b \notin B_{n_a}$, $b \in B_{n_b}$ and $a \notin B_{n_b}$]

Thus $\exists$ a one-one, onto correspondence $a \to B_{n_a}$ from A to $\{ B_n \mid n \in N\}$. But this shows that A is a countable set a contradiction. Hence our assumption is wrong. This proves that A has a limit point in it.

Remark : The converse of Theorem 1.7 need not be true. For this consider the following example.

Let X be an uncountable set and let $\mathfrak{J}$ be the co-finite topology on X. Let A be any infinite subset of X. Claim that each $a \in A$ is its limit point. Fix up any $a \in A$. For any open set G containing a, X-G is finite. Hence G contains almost all points of A except finitely many points of A. But then $G_a \cap A \setminus \{ a \} \neq \emptyset$. This in turn shows that each $a \in A$ is its limit point. But $\langle X, \mathfrak{J} \rangle$ being non first axiom space (see Example …) we get X is not a second axiom space (Theorem 1).

Theorem 1.8: In a second axiom space $\langle X, \mathfrak{J} \rangle$, every open covering of X is reducible to a countable sub covering.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a second axiom space. Let $\mathcal{G} = \{G_\lambda \,/\, \lambda \in \Delta\}$ be an open covering of X. As X is a second axiom space $\exists$ a countable base say $\mathfrak{B} = \{\, B_n \mid n \in N\}$ for $\mathfrak{J}$.

Define $N(\mathcal{G}) = \{n \in N \;/\; B_n \subseteq G_\lambda$ for some $\lambda \in \Delta\}$ for each $n \in N(\mathcal{G})$ associate a set $G_n \in \mathcal{G}$ such that $B_n \subseteq G_n$.

Thus the set $\{G_n \mid n \in N(\mathcal{G})\}$ is a countable set and $\{G_n \mid n \in N(\mathcal{G})\} \subseteq \mathcal{G}$.

Claim that $\{G_n \mid n \in N(\mathcal{G})\}$ forms a cover for X.

Fix up any $x \in X$. As $X \subseteq \bigcup\{G_\lambda \,/\, \lambda \in \Delta\}$ we get $x \in G_\lambda$ for some $\lambda \in \Delta$. As $\mathfrak{B}$ is a base for $\mathfrak{J}$, $\exists\, n \in N$ such that $x \in B_n \subseteq G_n$.

This in turn shows that $X \subseteq \bigcup\{\, G_n \,/\, n \in N(\mathcal{G})\,\}$. Hence $\{G_n \mid n \in N(\mathcal{G})\}$ forms a cover for X. This shows that any arbitrary open cover of X is reducible to a countable sub-cover.

Remark: Converse of Theorem 1.8 need not be true.

i.e. every open covering of a topological space X is reducible to a countable sub covering need not imply X is a second axiom space.

For this consider the following example.

Let $\langle X, \mathfrak{J} \rangle$ be a Fort's space. As $\langle X, \mathfrak{J} \rangle$ is non-first axiom space, we get $\langle X, \mathfrak{J} \rangle$ is a non-second axiom space (see Theorem 1). As $\langle X, \mathfrak{J} \rangle$ is a compact space (see Example 1.4 (4) in Unit 6) every open covering of X is reducible to a countable sub covering.

§2 Sequentially compact spaces and second axiom spaces.

Definition 2.1: Let $\langle X, \mathfrak{J} \rangle$ be a T – space. A subset E of X is said to be sequentially compact if every sequence of points of E has a subsequence which converges to a point of E.

Theorem 2.2: Every sequentially compact space is countably compact.

Proof: Let E be an infinite subset of a sequentially compact space X. Select an infinite sequence $\{x_n\}$ of points of E. As X is sequentially compact, the sequence $\{x_n\}$ of points of X (since $E \subseteq X$) has a convergent subsequence say $\{x_{n_k}\}$. Then $\{x_{n_k}\}$ being a sequence of points of E, x is a limit point of (see Theorem …). This shows that the topological space X is countably compact.

Remark: Converse of Theorem 2.2 need not be true in general. But it is true if X is a f.a.s.

Theorem 2.3: Let $\langle X, \mathfrak{J} \rangle$ be a first countable, countably compact space. Then $\langle X, \mathfrak{J} \rangle$ is sequentially compact.

Proof: Let $\{x_n\}$ be an infinite sequence in X. Then A is an infinite set of a countably compact space X. Hence, it has a limit point say x in X. As X is a first countable space, $\exists$ a decreasing countable local base say $\{B_n(x)\}_{n\in\mathbb{N}}$ at x. As $x \in B_n(x)$ and $B_n(x) \in \mathfrak{J}$, $\exists\, k$ such that $x_n \in B_n(x)$ for all $n \geq k$. Fix up $x_{n_k} \in B_n(x)$, $\forall\, n \in \mathbb{N}$. Then obviously, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ will converge to x.

Thus for a sequence $\{x_n\}$ in X $\exists$ a convergent subsequence $\{x_{n_k}\}$. Hence X is sequentially compact.

Corollary 2.4: Let $\langle X, \mathfrak{J} \rangle$ be a f.a.s. Then X is countably compact if and only if X is sequentially compact.

Proof: Result follows by Theorem 1 and Theorem 2.

Exercises

Prove or disprove the following.

1) Fort's space is a second axiom space
2) Co- countable topological space defined on an uncountable set is a second axiom space.
3) Co- finite Co- countable topological space defined on an uncountable set is a second axiom space.
4) Discrete topological space defined on countable set is a second axiom space.
5) $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a second axiom space.
6) Every sequentially compact space is compact.
7) Every compact space is sequentially compact.
8) Every countably compact space is sequentially compact.
9) In any metric space compactness, sequentially compactness and countably compactness are equivalent.
10) A subspace of a second axiom space is a second axiom space.

Unit 10

Lindelof Spaces

§1 Definition and Examples.

§2 Properties.

§3 Solved examples.

Unit 10: Lindelof Spaces

§1 Definition and Examples

Definition 1.1: A topological space $\langle X, \mathfrak{J} \rangle$ is a Lindelof space if every open cover of X has a countable sub-cover.

Remarks:

1) Every compact space is a Lindelof space (Obviously, by the Definition).
2) Every second axiom space is a Lindelof space as in a second axiom space every open cover has a countable sub-cover (see Unit 9 Theorem 1.8).

Examples 1.2:

1) Let $\langle X, \mathfrak{J} \rangle$ is a discrete topological space with X as a infinite countable set.
This space is a Lindelof space. As $\{\{x\} \mid x \in X\}$ forms a countable family of open sets, every open cover of X will have a countable sub-cover.

2) $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a second axiom space as $\{(a\ , b) \mid a, b \in Q\}$ forms a countable base for $\mathfrak{J}_u$.
Hence $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a Lindelof space.

§2 Properties

Theorem 2.1: Closed subspace of a Lindelof space is a Lindelof space**.**

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a Lindelof space and let $\langle Y, \mathfrak{J}^* \rangle$ be a closed subspace of X i.e. Y is closed subset of X and $\mathfrak{J}^* = \{G \cap Y \mid G \in \mathfrak{J}\}$.

$Let\ \{G_\lambda^* \mid \lambda \in \Lambda\}$ be any open cover of Y in $\langle Y, \mathfrak{J}^* \rangle$.

$$Hence\ ,\ Y \subseteq \bigcup_{\lambda \in \Delta} G_\lambda^* \ \ and\ G_\lambda^* \in \mathfrak{J}\ ; \forall\ \lambda \in \Lambda.$$

$G_\lambda^* \in \mathfrak{J}^* \Longrightarrow\ G_\lambda^* = G_\lambda \cap Y$ for some $G_\lambda \in \mathfrak{J}$.

Thus $X = (X - Y) \cup Y$.

$$= (X - Y) \cup \left[\bigcup_{\lambda \in \Delta} G_\lambda^* \right]$$

$$= (X - Y) \cup \left[\bigcup_{\lambda \in \Delta} G_\lambda \cap Y \right]$$

$$= (X - Y) \cup \left[\bigcup_{\lambda \in \Delta} G_\lambda \right]$$

As Y is a closed subset of X, $X - Y$ is an open set in X.

Hence $\{G_\lambda \mid \lambda \in \Delta\} \cup \{X - Y\}$ forms an open cover for X. As X is a Lindelof space, this open cover has a countable sub-cover.

Let $\left[\bigcup_{i=1}^{\infty} G_{\lambda_i} \right] \cap (X - Y)$. But Then we have ,

$$Y \subseteq \left[\bigcup_{i=1}^{\infty} G_{\lambda_i} \right] \cap Y = \bigcup_{i=1}^{\infty} [G_{\lambda_i} \cap Y] = \bigcup_{i=1}^{\infty} G_{\lambda_i}^*$$

But this shows that the open cover $\{G_\lambda^* \mid \lambda \in \Delta\}$ of Y has a countable sub-cover. Hence Y is a Lindelof space.

Theorem 2.2: Being a Lindelof space is a topological space.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a Lindelof space. Let $\langle Y^*, \mathfrak{J}^* \rangle$ be any T – space and let $f: X \to X^*$ be a homeomorphism. To prove that X^* is a Lindelof space. Let $\{G_\lambda^*\}$ be any open cover of X^*.

Then $G_\lambda^* \in \mathfrak{J}^*$ and $f: X \to X^*$ is continuous

$$\Rightarrow f^{-1}[G_\lambda^*] \in \mathfrak{J} \text{ for each } \lambda \in \Lambda.$$

As $X^* = \bigcup_{\lambda \in \Delta} G_\lambda^*$ and f is onto, we get $X = f^{-1}\left[\bigcup_{\lambda \in \Delta} G_\lambda^* \right] = \bigcup_{\lambda \in \Delta} f^{-1}[G_\lambda^*]$

But this shows that $\{f^{-1}[G_\lambda^*]\}_{\lambda \in \Delta}$ forms an open cover for X^***.** As X is a Lindelof space the open cover $\{f^{-1}[G_\lambda^*]\}_{\lambda \in \Lambda}$ of X has a countable sub-cover.

Denote the countable sub-cover by $\{f^{-1}[G_{\lambda_i}^*]\}_{i \in N}$.

Then $X = \bigcup_{i \in N} f^{-1}[G_{\lambda_i}^*]$. But f is onto $\Rightarrow X^* = f(x) = f\left[\bigcup_{i \in N} f^{-1}[G_{\lambda_i}^*] \right]$

Hence $X^* = \bigcup_{i \in N} \left[f\left[f^{-1}[\, G^*_{\lambda_i}] \right] \right]$

$\Rightarrow \; X^* = \bigcup_{i \in N} G_{\lambda_i}$

This shows that any open cover $\{ G^*_\lambda \}_{\lambda \in \Delta}$ of X^* has a countable sub-cover $\{ G^*_{\lambda_i} \}_{i \in N}$ for X^*. Hence $\langle X^*, \mathfrak{J}^* \rangle$ is a Lindelof space. Thus homeomorphic image of a Lindelof space is a Lindelof space. Hence being a Lindelof space is a topological property.

§3 Solved examples

Problem 1:- Show by an example that every Lindelof space need not be a compact space.

Solution:- Let $\langle X, \mathfrak{J} \rangle$ be a discrete topological space with X as a infinite countable set. This space is a Lindelof space but it is not a compact space as the open cover $\{\{x\} \mid x \in X\}$ of X has no finite sub-cover.

Problem 2:- Show by an example that every Lindelof space need not be a second axiom space.

Solution:- Consider an uncountable set X. Let $\mathfrak{J}$ denote a co-finite topology on X. Then $\langle X, \mathfrak{J} \rangle$ is a compact space (see Unit (6) §1.4 example 2). Hence $\langle X, \mathfrak{J} \rangle$ is a Lindelof space. But $\langle X, \mathfrak{J} \rangle$ is not a first axiom space (see Unit (8) §1.2 example 4). We get $\langle X, \mathfrak{J} \rangle$ is not a second axiom space (see Unit (9) Theorem 1.3). Thus $\langle X, \mathfrak{J} \rangle$, the cofinite topological space defined on an uncountable set X is a Lindelof space but not a second axiom space (first axiom space).

Problem3: Show by an example that being a Lindelof space is not a hereditary property.

Solution: Let X be an uncountable set and $\mathfrak{J} = p -$ exclusion topology on X $(p \in X)$

i.e. $\mathfrak{J} = \{X\} \cup \{A \subseteq X \mid p \in A\}$. Then $\langle X, \mathfrak{J} \rangle$ be a T – space.

I] $\langle X, \mathfrak{J} \rangle$ is a Lindelof space.

As $\langle X, \mathfrak{J} \rangle$ is a compact space (see Unit (6) §1.4 example 5), we get $\langle X, \mathfrak{J} \rangle$ is a Lindelof space.

II] Let $Y = X - \{p\}$. Consider the subspace $\langle Y, \mathfrak{J}^* \rangle$. Then the relative topology $\mathfrak{J}^*$ is the discrete topology on Y. Hence $\langle Y, \mathfrak{J}^* \rangle$ is not a Lindelof space as the open cover $\{\{x\} \mid x \in Y\}$ of Y has no countable sub-cover.

Thus $\exists$ a subspace $\langle Y, \mathfrak{J}^* \rangle$ of Lindelof space $\langle X, \mathfrak{J} \rangle$ which is not a Lindelof space. Hence the result.

Problem4: Show that continuous image of a Lindelof space is a Lindelof space.

Solution: Let $\langle X, \mathfrak{J} \rangle$ be a Lindelof space. Let $\langle Y^*, \mathfrak{J}^* \rangle$ be any $T - space$ and let $f: X \to X^*$ be a homeomorphism. To prove that X^* is a Lindelof space. Let $\{G_\lambda^*\}$ be any open cover of X^*.

Then $as\ G_\lambda^* \in \mathfrak{J}^*$ and $f: X \to X^*$ is continuous ,we get $f^{-1}[G_\lambda^*] \in \mathfrak{J}$ for each $\lambda \in \Delta$.

As $X^* \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda^*$ and f is onto, we get $X \subseteq f^{-1}\left[\bigcup_{\lambda \in \Lambda} G_\lambda^*\right] = \bigcup_{\lambda \in \Lambda} f^{-1}[\,G_\lambda^*]$

But this shows that $\{f^{-1}[\,G_\lambda^*]\}_{\lambda \in \Lambda}$ forms an open cover for X^*. As X is a Lindelof space, the open cover $\{f^{-1}[\,G_\lambda^*]\}_{\lambda \in \Lambda}$ of X has a countable sub-cover.

Denote the countable sub-cover by $\{f^{-1}[\,G_{\lambda_i}^*]\}_{i \in N}$.

Then $X = \bigcup_{i \in N} f^{-1}[\,G_{\lambda_i}^*]$. But f is onto $\Rightarrow X^* = f(X) = f\left[\bigcup_{i \in N} f^{-1}[\,G_{\lambda_i}^*]\right]$

Hence $X^* = \bigcup_{i \in N} \left[f\left[f^{-1}[\,G_{\lambda_i}^*]\right]\right]$.

$\Rightarrow X^* = \bigcup_{i \in N} G_{\lambda_i}$

This shows that any open cover $\{\,G_\lambda^*\,\}_{\lambda \in \Delta}$ of X^* has a countable sub-cover $\{\,G_{\lambda_i}^*\}_{\,i \in N}$ for X^*. Hence $\langle X^*, \mathfrak{J}^* \rangle$ is a Lindelof space. This shows that continuous image of a Lindelof space is a Lindelof space.

Exercises

Prove or disprove the following statements.

1) Subspace of a Lindelof space is a Lindelof space
2) Every Lindelof space is a second axiom space.
3) Every second axiom space is a Lindelof space .
4) Every first axiom space is a Lindelof space
5) Every Lindelof space is a first axiom space.

6) Every Lindelof space is a space.
7) Every compact space is a Lindelof space .

Unit 11

Separable spaces

§1 Definition and Examples.

§2 Properties.

§3 Solved problems.

Unit 11: Separable spaces

§1 Definition and Examples

We know that a subset E of a topological space $\langle X, \mathfrak{I} \rangle$ is dense in X if $\bar{E} = X$. viz. the set Q of all rational numbers is dense in $\langle \mathbb{R}, \mathfrak{I}_u \rangle$.

Definition 1.1: A topological space $\langle X, \mathfrak{I} \rangle$ is called separable if there exists a countable dense subset of X.

Examples 1.2:

Separable spaces.

1) $\langle \mathbb{R}, \mathfrak{I} \rangle$ is a separable space as the set of all rational numbers Q is a countable dense subset of $\mathbb{R}$.
2) Let X be a countable set and $\langle X, \mathfrak{I} \rangle$ be discrete topological space. Then $\langle X, \mathfrak{I} \rangle$ is a separable spaces as $\bar{X} = X$ and X is a countable set.
3) Let $\langle X, \mathfrak{I} \rangle$ be a co finite topological space and let X be an uncountable set. For any countable set A of X, $\bar{A} = X$ (since the only closed set containing A is X). Hence $\langle X, \mathfrak{I} \rangle$ is a separable space.
4) Let X be an uncountable set and $\mathfrak{I}$ be the discrete topology on X. Then $\langle X, \mathfrak{I} \rangle$ is not a separable space as X is the only dense subset of X. In particularly, the discrete topological space $\langle \mathbb{R}, \mathfrak{I} \rangle$ is not a separable space.

   Note that the discrete topological space is separable if and only if X is a countable set.

Non separable spaces.

1) Discrete topological space defined on uncountable set X is a non-separable space.
2) Co-countable topological space $\langle X, \mathfrak{I} \rangle$ defined on an uncountable set X is not a separable space.
3) Let X be any uncountable set and $p \in X$.

   $\mathfrak{I} = p -$ exclusion topology on X i.e. $\mathfrak{I} = \{X\} \cup \{A \subseteq X \mid p \notin A\}$.

Let if possible X is a separable space.

Hence $\exists$ a countable set A such that $\bar{A} = X$. Select any $x \in X - A\,, x \neq p$. [This is possible as $X - A$ is an uncountable set]. Then $x \notin A$ and $x \in \bar{A}$ imply $x \in d(A)$.

As $\{x\}$ is an open set containing x, we get $\{x\} \cap A - \{x\} \neq \emptyset$. But as $x \in X - A$, we have $\{x\} \cap A = \emptyset$. Thus $\{x\} \cap A - \{x\} = \emptyset$; a contradiction.

Hence our assumption is wrong .Therefore $\langle X, \mathfrak{J} \rangle$ is not a separable.

§2 Properties

Theorem 2.1: Property of being a separable space is a topological property**.**

Proof: - Let $\langle X, \mathfrak{J} \rangle$ be a separable space. Let $\langle X^*, \mathfrak{J}^* \rangle$ be any topological space. Let $f\colon X \to X^*$ be a homeomorphism. To prove that $\langle X^*, \mathfrak{J}^* \rangle$ is a separable space.

Let $\langle X, \mathfrak{J} \rangle$ be a separable space. Hence $\exists$ a countable subset A of X such that $\bar{A} = X$. As f is onto, $f(X) = X^*$. $f\colon X \to X^*$ being continuos we get $f[\bar{A}] \subseteq \overline{f[A]}$ (see Theorem … Continuous function). Hence $f[X] \subseteq \overline{f[A]}$ implies $X^* \subseteq \overline{f[A]}$ $i.e. X^* = \overline{f[A]}$. Thus $f[A]$ is a countable dense subset of X^*. Hence X^* is a separable space. Thus homeomorphic image $\langle X^*, \mathfrak{J}^* \rangle$ of a separable space $\langle X, \mathfrak{J} \rangle$ is a separable space. Hence being a separable space is a topological property.

Theorem 2.2: Every second axiom space is a separable space.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a second axiom space. Hence there exists a countable base say $\mathfrak{B} = \{B_n \mid n \in \mathbb{N}\}$ for $\mathfrak{J}$. Define $A = \{\, b_n \in B_n \mid n \in \mathbb{N} \,\}$.

Then A is a countable subset of X.

Claim that $\bar{A} = X$.

Let $x \in X$ and G be any open set containing x. Hence by the definition of base, $\exists\; n \in N$ such that $x \in B_n \subseteq G$. Select $b_n \in B_n$ such that $b_n \neq x$. Then $b_n \in G \cap A - \{x\}$ implies x is a limit of A. Thus $X \subseteq \bar{A}$ implies $\bar{A} = X$. Thus there exists a countable dense subset A of X. Hence $\langle X, \mathfrak{J} \rangle$ is a separable space.

A property of a space is said to be hereditarily separable if each subspace of the space is separable.

Theorem 2.3: Every second axiom space is hereditarily separable**.**

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a second axiom space. Let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of $\langle X, \mathfrak{J} \rangle$. As every sub-space of a second axiom space is a second axiom space, we get $\langle Y, \mathfrak{J}^* \rangle$ is a second axiom space (see Unit (9) Theorem 1.4).

By Theorem 2.2 , $\langle Y, \mathfrak{J}^* \rangle$ is a separable space. Thus each subspace of a second axiom space is separable. Hence the result.

Theorem 2.4: Any topological space is a subspace of a separable space.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be any topological space and $\infty \notin X$. Define $X^* = X \cup \{\infty\}$ and $\mathfrak{J}^* = \{\, \emptyset \,\} \cup \{\, G \cup \{\infty\} \mid G \in \mathfrak{J}\}$.

I] To prove that $\mathfrak{J}^*$ is a topology on X^*.

(i) $\emptyset \in \mathfrak{J}^*$ and $X^* \in \mathfrak{J}^*$.

(ii) $A^*, B^* \in \mathfrak{J}^* \Longrightarrow A^* = A \cup \{\infty\}$ and $B^* = B \cup \{\infty\}$ for $A, B \in \mathfrak{J}$

$\Longrightarrow A^* \cap B^* = (A \cap B) \cup \{\infty\} \in \mathfrak{J}^*$ as $A \cap B \in \mathfrak{J}$.

(iii) Let $G_\lambda^* \in \mathfrak{J}^*$, $\lambda \in \Lambda$. Then $G_\lambda^* = G_\lambda \cup \{\infty\}$, where $G_\lambda \in \mathfrak{J}$; $\forall \lambda \in \Lambda$.

$$\text{Then } \bigcup_{\lambda \in \Lambda} G_\lambda^* = \bigcup_{\lambda \in \Lambda} [\, G_\lambda \cup \{\infty\}] = \left[\bigcup_{\lambda \in \Lambda} G_\lambda\right] \cup \{\infty\}.$$

$$\text{As } \bigcup_{\lambda \in \Lambda} G_\lambda \in \mathfrak{J}. \text{ We get } \bigcup_{\lambda \in \Lambda} G_\lambda^* \in \mathfrak{J}^*$$

From (i), (ii) and (iii) we get $\mathfrak{J}^*$ is a topology on X^*.

II] $\langle X, \mathfrak{J} \rangle$ is a subspace of $\langle X^*, \mathfrak{J}^* \rangle$ as $\mathfrak{J} = \{G^* \cap X \mid G^* \in \mathfrak{J}^*\}$ and $X \subseteq X^*$.

III] $\overline{\{\infty\}} = X^*$. Let $x \in X^*$ $(x \neq \infty)$ and $G^* \in \mathfrak{J}^*$ with $x \in G^*$.

$G^* \cap \{\infty\} - \{\, x \,\} = \{\infty\}$ implies x is a limit point of $\{\infty\}$.

Thus each $x \neq \infty \in X^*$ is a limit point of $\{\infty\}$. Hence $d(\{\, \infty \,\}) = X$.

As $\overline{\{\infty\}} = \{\, \infty \,\} \cup d(\{\infty\}) = \{\infty\} \cup X = X^*$.

IV] As $\{\infty\}$ is a countable dense set in X^*, we get $\langle X^*, \mathfrak{J}^* \rangle$ is a separable space.

Thus given any space $\langle X, \mathfrak{J} \rangle$ there exists a separable space $\langle X^*, \mathfrak{J}^* \rangle$ such that $\langle X, \mathfrak{J} \rangle$ is a subspace of $\langle X^*, \mathfrak{J}^* \rangle$.

Remark: As any topological space is a subspace of a separable space, subspace of a separable space need not be a separable space.

Theorem 2.5: A metric space is separable if and only if it is a second axiom space.

Proof:- As if part follows by Theorem 2.3 we prove 'Only if part' only.

Only if part –

Let $\langle X, d \rangle$ be a metric space and let $\mathfrak{J}$ denote the topology induced by d on X. Hence $\mathfrak{J} = \{ G \subseteq X \mid \forall\, x \in G\ \exists\, r > 0 \text{ such that } S(x, r) \subseteq G \}$.

Given $\langle X, \mathfrak{J} \rangle$ is a separable space. To prove that $\langle X, \mathfrak{J} \rangle$ is a second axiom space.

As X is separable, ∃ a countable dense set say A in X.

Let $A = \{ x_1, x_2, x_3, \ldots \ldots \}$. Then $\bar{A} = X$.

Define $\mathfrak{B} = \left\{ S\left(x_n, \frac{1}{m}\right) : x_n \in A,\ m, n \in N \right\}$. Then $\mathfrak{B}$ is a countable set and $\mathfrak{B} \subseteq \mathfrak{J}$.

To prove that $\mathfrak{B}$ is a base for .

(1) $\mathfrak{B} \subseteq \mathfrak{J}$.

(2) Let $G \in \mathfrak{J}$ and $x \in G$. By the definition of $\mathfrak{J}$, $\exists\, r > 0$ such that $x \in S(x, r) \subseteq G$.

Select $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{r}{2}$. Then $S\left(x, \frac{1}{m}\right) \subseteq S\left(x, \frac{r}{2}\right) \subseteq S(x, r)$.

As $\bar{A} = X$, x is a limit point of A. Hence $x \in S\left(x, \frac{1}{m}\right)$ and $S\left(x, \frac{1}{m}\right) \in \mathfrak{J}$ will imply

$S\left(x, \frac{1}{m}\right) \cap A - \{x\} \neq \emptyset$.

Let $x_n \in S\left(x, \frac{1}{m}\right) \cap A$. To prove that $S\left(x_n, \frac{1}{m}\right) \subseteq S(x, r)$.

Let $y \in S\left(x_n, \frac{1}{m}\right)$. $y \in S\left(x_n, \frac{1}{m}\right) \Rightarrow d(x_n, y) < \frac{1}{m}$.

Hence, $d(x, y) < d(x, x_n) + d(x_n, y)$

$\Rightarrow d(x, y) < \frac{1}{m} + \frac{1}{m}$ …… $\left[\text{since } x_n \in S\left(x, \frac{1}{m}\right)\right]$

$\Rightarrow d(x, y) < \frac{2}{m}$

$\Rightarrow d(x, y) < r$ …… $\left[\text{since } \frac{1}{m} < \frac{r}{2}\right]$

$\Longrightarrow y \in S(x,r)$.

This proves $S\left(x_n, \frac{1}{m}\right) \subseteq S\left(x, \frac{1}{m}\right)$.

As $S\left(x, \frac{1}{m}\right) \subseteq S(x,r) \subseteq G$, we get $S\left(x_n, \frac{1}{m}\right) \subseteq G$.

Thus given $G \in \mathfrak{J}$ and $x \in G$, $\exists\, S\left(x_n, \frac{1}{m}\right) \in \mathfrak{B}$ such that $S\left(x_n, \frac{1}{m}\right) \subseteq G$.

Hence, from (1) and(2) we get, the countable family $\mathfrak{B}$ forms a base for the topology $\mathfrak{J}$ .Hence, $\langle X, d\rangle$ is a second axiom space.

Theorem 2.6: In a separable space any countable family mutually disjoint open sets is countable.

Proof:- Let $\langle X, \mathfrak{J}\rangle$ be a separable space. As X is separable space there exists a countable dense set say D in X. Let $\mathcal{K}$ denote the family of mutually disjoint open sets in X.

To prove that $\mathcal{K}$ is countable.

For $G, H \in \mathcal{K}\ ,\ G \cap H = \emptyset\ ,\ if\ G \neq H.$

Case (1): $\emptyset \notin \mathcal{K}$.

$\overline{D} = X \implies G \cap D \neq \emptyset$ for any $G \in \mathcal{K}$. Select $x_G \in G \cap D, \forall\, G \in \mathcal{K}$.

Define $f: \mathcal{K} \to D$ by $f(G) = x_G$.

Obviously f is onto.

$f(G) = f(H) \implies x_G = x_H \implies G \cap H \neq \emptyset \implies G = H \ldots (by\ definition\ of\ \mathcal{K}).$

This shows that f is one-one.

As $f: \mathcal{K} \to D$ is one-one and onto and D is countable we get $\mathcal{K}$ is countable.

Case (2): $\emptyset \in \mathcal{K}$.

Applying the case (1) for the family $\mathcal{K} - \{\emptyset\}$ we get the family $\mathcal{K} - \{\emptyset\}$ is countable.

And hence $\mathcal{K} = (\mathcal{K} - \{\emptyset\}) \cup \{\emptyset\}$ is countable.

§3 Solved problems

Problem 1: Show that being a separable space is not a hereditary property.

Solution: Let X be an uncountable set and $p \in X$.

$\mathfrak{J} = p -$ inclusion topology on X.

I) $\langle X, \mathfrak{J}\rangle$ is a separable space.

Consider $A = \{p\}$. Claim that $\bar{A} = X$.

Select any $x \in X - \{p\}$. Then any open set G containing x must contain p. Hence $G \cap A = \{p\}$ and hence $G \cap A - \{x\}$
$= \{p\} - \{x\} = \emptyset$. But this shows that x is a limit point of A. Thus $X - \{p\} = d(A)$.
Hence $\bar{A} = A \cup d(A) = \{p\} \cup (X - \{p\}) = X$.
Thus A is dense in X. As A is a countable, dense subset of X, X is separable space.

II) Define $Y = X - \{p\}$. Then the subspace $\langle Y, \mathfrak{J}^* \rangle$ is the discrete topological space.
Claim: $\langle Y, \mathfrak{J}^* \rangle$ is not a separable.
Let if possible $\langle Y, \mathfrak{J}^* \rangle$ is separable space. Hence there exists a countable dense set say A in Y. But since $\langle Y, \mathfrak{J}^* \rangle$ is the discrete topological space, each subset of Y is closed in Y. Hence $\bar{A} = A$ (since $A \neq Y$ as A is countable and Y is uncountable). This shows that our assumption is wrong. Hence $\langle Y, \mathfrak{J}^* \rangle$ is not a separable space.

Hence being a separable space is not a hereditary property.

Problem 2: Give an example to show that every separable space need not be a Lindelof space.

Solution:- Let X be an uncountable set and $p \in X$.
$\mathfrak{J} = p -$ inclusion topology on X.

I) $\langle X, \mathfrak{J} \rangle$ is a separable space.(see Problem 1)

II) X is not a Lindelof space.
Consider the family $\{\{x, p\} \mid x \in X\}$. This family of open sets forms an open cover for X. But this open cover has no countable sub-cover for X, as X is an uncountable set. This shows that $\langle X, \mathfrak{J} \rangle$ is not a Lindelof space.

Hence every separable space need not be a Lindelof space.

Problem 3: Give an example to show that every Lindelof space need not be a separable space.

Solution: -Let X be any uncountable set and $p \in X$.
$\mathfrak{J} = p -$ exclusion topology on X i.e. $\mathfrak{J} = \{X\} \cup \{A \subseteq X \mid p \notin A\}$.

I) $\langle X, \mathfrak{J} \rangle$ is a compact space.
Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be any open cover of X.

Then $X = \bigcup_{\lambda \in \Lambda} G_\lambda$ and $p \in X$ will imply $p \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

But by definition of $\mathfrak{J}$, $G_{\lambda_0} = X$. Hence, the open cover $\{G_\lambda \mid \lambda \in \Lambda\}$ of X has a finite sub-cover $\{G_{\lambda_0}\}$ of X.

This shows that $\langle X, \mathfrak{J} \rangle$ is a compact space.

II) $\langle X, \mathfrak{J} \rangle$ is a Lindelof space.

As every compact space is a Lindelof space we get $\langle X, \mathfrak{J} \rangle$ is a Lindelof space.

III) $\langle X, \mathfrak{J} \rangle$ is not a separable space.

Let if possible X is a separable space.

Hence $\exists$ a countable set A such that $\bar{A} = X$. Select any $x \in X - A$, $x \neq p$. [This is possible as $X - A$ is an uncountable set]. Then $x \notin A$ and $x \in \bar{A}$ imply $x \in d(A)$.

As $\{x\}$ is an open set containing x, we get $\{x\} \cap A - \{x\} \neq \emptyset$. But as $x \in X - A$, we have $\{x\} \cap A = \emptyset$. Thus $\{x\} \cap A - \{x\} = \emptyset$; a contradiction.Hence our assumption is wrong.

Therefore $\langle X, \mathfrak{J} \rangle$ is not a separable. Thus there exists a Lindelof space which not a separable space.

Problem 4: Show that for a metric space $\langle X, d \rangle$, the following statements are equivalent:

(1) The metric space X is separable.

(2) The metric space X is a Lindelöf space.

(3) The metric space X is a second axiom space.

Solution:-We know a metric space is separable if and only if it is a second axiom space and a metric space is a Lindelöf if and only if it is a second axiom space.

Hence for a metric space $\langle X, d \rangle$, the given three statements are equivalent.

Problem 5: Show that every subspace of a separable metric space is separable.

Solution: -Let X be a separable metric space and let Y be its subspace. Then X is a second axiom space (see Theorem 2.5). As subspace of a second axiom space is a second axiom space we get Y is a second axiom space (see Unit (9) Theorem 1.4)

This shows that any subspace of a separable metric space is a separable metric space.

Problem 6: Show that the open subspace of a separable space is separable.

Solution: Let $\langle X, \mathfrak{J} \rangle$ be a separable space and let $\langle Y, \mathfrak{J}^* \rangle$ be its open subspace. Then as Y is open is open in X, $\mathfrak{J}^* \subseteq \mathfrak{J}$.

As X is a separable space, $\exists$ a countable dense set say A in X i.e. $\bar{A} = X$.

Define $B = A \cap Y$. Then B is countable subset of Y.

Claim: $\bar{B} = Y$.

Let $y \in Y$. Let G^* be any open set in Y. Then $G^* \in \mathfrak{J}$ and $y \in G^*$.

As $y \in \bar{A}$, we get, $G^* \cap A - \{y\} \neq \emptyset$

i.e. $(G^* \cap Y) \cap A - \{y\} \neq \emptyset$

i.e. $G^* \cap (Y \cap A) - \{y\} \neq \emptyset$

i.e. $G^* \cap B - \{y\} \neq \emptyset$.

But this shows that $y \in Y$ is a limit point of B. Hence $\bar{B} = Y$.

Problem 7: Show that continuous image of a separable space is a separable space.

Solution: - Let $\langle X, \mathfrak{J} \rangle$ be a separable space. Let $\langle X^*, \mathfrak{J}^* \rangle$ be any topological space.

Let $f: X \to X^*$ be an onto continuous mapping.

To prove that $\langle X^*, \mathfrak{J}^* \rangle$ is a separable space.

Let $\langle X, \mathfrak{J} \rangle$ be a separable space. Hence $\exists$ a countable subset A of X such that $\bar{A} = X$. As f is onto, $f(X) = X^*$. $f: X \to X^*$ being continuous we get $f[\bar{A}] \subseteq \overline{f[A]}$ (See Unit (5) Theorem 2.5). Hence $f[X] \subseteq \overline{f[A]}$ implies $X^* \subseteq \overline{f[A]}$ $i.e. X^* = \overline{f[A]}$. Thus $f[A]$ is a countable dense subset of X^*. Hence X^* is a separable space. Thus continuous image $\langle X^*, \mathfrak{J}^* \rangle$ of a separable space $\langle X, \mathfrak{J} \rangle$ is a separable space. Continuous image of a separable space is a separable space.

Exercises

State whether the following statements are true or false.

1) Every Lindelof space is a separable space.
2) Every separable space is a Lindelof space.
3) Every metric space is a separable space.
4) Every subspace of a separable space is a separable space.

5) Every discrete topological space is a separable space.
6) Co-finite topological space defined on an uncountable set is a separable space.

Unit 12

$\mathcal{T}_o$ - Spaces

§1 Definition and Examples.

§ 2 Characterizations and properties.

Unit 12: $\mathcal{T}_o$ - Spaces

§1 Definition and Examples

Definition 1.1: A topological space $\langle X, \mathfrak{I} \rangle$ is said to be a T_0 – space if it satisfies the following axiom of Kolomogrov:

"If x and y are two distinct points of X then there exists an open set which contains one of them but not the other."

Examples 1.2:

(1) Any discrete topological space $\langle X, \mathfrak{I} \rangle$ with $|X| \geq 2$ is a T_0 – space. For $x \neq y$, $\{x\} \in \mathfrak{I}$ such that $x \in \{x\}$ and $y \in \{x\}$.

(2) Any co-finite topological space $\langle X, \mathfrak{I} \rangle$ is T_0 – space.

Case (1): X is finite. In this case $\mathfrak{I} = \wp(X)$ and hence $\langle X, \mathfrak{I} \rangle$ is a T_0 – space.

Case (2): X is infinite. Given $x \neq y$ in X, $X - \{x\} \in \mathfrak{I}$ such that $y \in X - \{x\}$ and $x \notin X - \{x\}$

(3) Any co-countable topological space $\langle X, \mathfrak{I} \rangle$ is T_0 – space (proof as in (2)).

(4) Let $X = \mathbb{N}$ and $\mathfrak{I} = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{A_n \mid n = 1,2,3, \dots\}$ where $A_n = \{1,2, \dots, n\}$.

Then $\langle X, \mathfrak{I} \rangle$ is a T_0 – space.

For $m \neq n$ in X, either $m < n$ or $n < m$. Let $m < n$. Then $m \in A_m$ but $n \notin A_m$ and $A_m \in \mathfrak{I}$.

(5) Let $\langle X, \mathfrak{I} \rangle$ be p – inclusion topological space ($p \in X$).

Here $\mathfrak{I} = \{\emptyset\} \cup \{A \subseteq X \mid p \in A\}$. This space is a T_0 – space. For $p \neq x$ in X there does exist an open set $\{p\}$ containing p but not x and for $x \neq$y (both different from p) there does exist an open set $\{p, x\}$ containing x but not y.

(6) Let $\langle X, \mathfrak{I} \rangle$ be p – exclusion topological space ($p \in X$).

Here $\mathfrak{I} = \{X\} \cup \{A \subseteq X \mid p \notin A\}$. This space is a T_0 – space. For $p \neq x$ in X, $\{x\} \in \mathfrak{I}$ such that $x \in \{x\}$ and $p \notin \{x\}$. For $x \neq y$ (both different from p) then $\{x\} \in \mathfrak{I}$ such that $x \in \{x\}$ but $y \notin \{x\}$.

(7) Every metric space is a T_0 –space. Let $\langle X, d\rangle$ be a metric space and let $\Im$ be the topology on X induced by d. Let $x \neq y$ in X. Then $d(x, y) = r > 0$. Then $S\left(x, \frac{r}{3}\right) \in \Im$ such that $x \in S\left(x, \frac{r}{3}\right)$ but $y \notin S\left(x, \frac{r}{3}\right)$.

Hence $\langle X, d\rangle$ is a T_0 – space.

(8) $\langle \mathbb{R}, \Im_u\rangle$ is a T_0 – space. Let $x \neq y$ in X. Then $|x - y| > 0$. Take $|x - y| = r$. Then $\left(x - \frac{r}{3}, x + \frac{r}{3}\right) \in \Im_u$ such that $x \in \left(x - \frac{r}{3}, x + \frac{r}{3}\right)$ but $y \notin \left(x - \frac{r}{3}, x + \frac{r}{3}\right)$. Hence $\langle \mathbb{R}, \Im_u\rangle$ is a T_0 – space.

(9) Any indiscrete topological space $\langle X, \Im\rangle$ is not a T_0 – space.

Remark: As any T – topological space need not be a T_0 – topological space, the set of T_0 – topological spaces is a proper subset of all topological spaces.

§ 2 Characterizations and properties

Theorem 2.1: A topological space $\langle X, \Im\rangle$ is a T_0 – space if and only if the closures of distinct points of X are distinct. i.e. for $x \neq y$ in X, $\overline{\{x\}} \neq \overline{\{y\}}$.

Proof: Only if part.

Let $\langle X, \Im\rangle$ be a T_0 – space and $x \neq y$ in X. Hence by definition of a T_0 – space , *there exists* $G \in \Im$ such that $x \in G$ and $y \notin G$. $G \in \Im$. $X - G$ is closed set $\Rightarrow$ $\overline{(X - G)} = X - G$.

As $y \in X - G$, $\{y\} \subseteq X - G$. Hence $\overline{\{y\}} \subseteq \overline{(X - G)} = X - G$. As $x \notin X - G$ we get $x \notin \overline{\{y\}}$. Thus $x \in \overline{\{x\}}$ but $x \notin \overline{\{y\}}$ $\Rightarrow$ $\overline{\{x\}} \neq \overline{\{y\}}$.

If part.

Let $\langle X, \Im\rangle$ be a topological space such that for $x \neq y$ in X, $\overline{\{x\}} \neq \overline{\{y\}}$.

Without loss of generality assume that $\exists\ z \in X$ such that $z \in \overline{\{x\}}$ and $z \notin \overline{\{y\}}$.

Claim that $x \notin \overline{\{y\}}$.

If $x \notin \overline{\{y\}}$, then $\{x\} \subseteq \overline{\{y\}}$ will imply $\overline{\{x\}} \subseteq \overline{\overline{\{y\}}} = \overline{\{y\}}$. In this case as $z \in \overline{\{x\}}$ we get $z \in \overline{\{y\}}$; which is not true by the choice of z. Hence $x \notin \overline{\{y\}}$. Define $G = X - \overline{\{y\}}$. Then G being the complement of closed set, $G \in \Im$ and $x \in G$ and $y \notin G$ (since $y \in \overline{\{y\}}$). This shows that $\langle X, \Im\rangle$ is a T_0 – space.

Theorem 2.2: A topological space $\langle X, \mathfrak{I}\rangle$ is a T$_0$ – space if and only if whenever x and y are distinct points of X either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Proof: Only if part.

Let $\langle X, \mathfrak{I}\rangle$ be a T$_0$ – space and let $x \neq y$ in X. Hence by definition of a T$_0$ – space, for $x \neq y$ in X, $\exists$ $G \in \mathfrak{I}$ such that $x \in G$ and $y \notin G$.

$G \in \mathfrak{I} \implies X - G$ is closed set in X $\Longrightarrow \overline{(X-G)} = X - G$. As $y \in X - G$, $\{y\} \subseteq X - G$. Hence $\overline{\{y\}} \subseteq \overline{(X-G)} = X - G$. But $x \notin X - G$ *implies* $x \notin \overline{\{y\}}$. Thus for $x \neq y$ in X, we get $x \notin \overline{\{y\}}$

If part.

Assume that $\langle X, \mathfrak{I}\rangle$ is a topological space such that for any two distinct points x and y in X, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Assume that $x \notin \overline{\{y\}}$. Define $G = X - \overline{\{y\}}$. Then $G \in \mathfrak{I}$, $x \in G$ and $y \notin G$ (since $y \notin \overline{\{y\}}$). But this shows that for $x \neq y$, $\exists$ an open set containing one but not the other. Similarly if $y \notin \overline{\{x\}}$, then $\exists$ an open set $H = X - \overline{\{x\}}$ containing y but not x. Hence $\langle X, \mathfrak{I}\rangle$ is a T$_0$ – space.

Theorem 2.3: Being T$_0$ – space is a hereditary property.

Proof: Let $\langle X, \mathfrak{I}\rangle$ be a T$_0$ – space and let $\langle Y, \mathfrak{I}^*\rangle$ be a subspace of $\langle X, \mathfrak{I}\rangle$.

To prove that $\langle Y, \mathfrak{I}^*\rangle$ is a T$_0$ – space.

Here $\mathfrak{I}^* = \{G \cap Y \mid G \in \mathfrak{I}\}$ and $Y \subseteq X$. Let $y \neq z$ in Y. As $Y \subseteq X$, $y \neq z$ in X.

X being a T$_0$ – space, there exists an open set G in X such that $y \in G$ and $z \notin G$.

Define $G^* = G \cap Y$. Then $G^* \in \mathfrak{I}^*$and G^* contains y but not z. This shows that $\langle Y, \mathfrak{I}^*\rangle$ is a T$_0$ – space. As any subspace of a T$_0$ – space $\langle X, \mathfrak{I}\rangle$ is a T$_0$ – space, the result follows.

Theorem 2.4: Being a T$_0$ – space is a topological property.

Proof: Let $\langle X, \mathfrak{I}\rangle$ and $\langle Y, \mathfrak{I}^*\rangle$ be any two topological spaces and $f: X \longrightarrow Y$ be a homeomorphism. Let $\langle X, \mathfrak{I}\rangle$ be a T$_0$ – space. To prove that $\langle Y, \mathfrak{I}^*\rangle$ is a T$_0$ – space.

Let $y_1 \neq y_2$ in Y. $f: X \longrightarrow Y$ being onto, there exist x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. As f is one-one, $f(x_1) \neq f(x_2) \implies x_1 \neq x_2$. As X is a T$_0$ – space, for $x_1 \neq x_2$ in X there exists an open set G in X such that $x_1 \in G$ and $x_2 \notin G$. f being an open map,

$f(G) \in \mathfrak{J}^*$. Thus for $y_1 \neq y_2$ in Y, there exists $f(G) \in \mathfrak{J}^*$ such that $y_1 \in f(G)$ and $y_2 \notin f(G)$.

This shows that $\langle Y, \mathfrak{J}^* \rangle$ is a T_0 – space. Thus any homeomorphic image of a T_0 – space is a T_0 – space. Hence the result.

Corollary 2.5: The property of a space being T_0 – space is preserved by one-one, onto open maps.

Theorem 2.6: Let $\langle X, \mathfrak{J} \rangle$ be a T_0 – space and $\mathfrak{J}^* \geq \mathfrak{J}$. Then $\langle X, \mathfrak{J}^* \rangle$ is also T_0 – space.

Proof: To prove that $\langle X, \mathfrak{J}^* \rangle$ is T_0 – space.

Let $x \neq y$ in X. As $\langle X, \mathfrak{J} \rangle$ be a T_0 – space, for $x \neq y$ in X, there exists $G \in \mathfrak{J}$ such that $x \in G$ and $y \notin G$. As $\mathfrak{J}^* \geq \mathfrak{J}$, $G \in \mathfrak{J}^*$. Thus for $x \neq y$ in X, there exists $G \in \mathfrak{J}^*$ such that $x \in G$ and $y \notin G$. Hence $\langle X, \mathfrak{J}^* \rangle$ is a T_0 – space.

Let $\{x_n\}$ be a sequence of points in a topological space $\langle X, \mathfrak{J} \rangle$. The sequence $\{x_n\}$ is said to be convergent to $x \in X$, if for any open set G containing x $\exists$ N such that $x_n \in G$ for all $n \geq N$.

Remark: In a T_0 – space, a sequence may converge to more than one point (In fact it may converge to every point of the space).

For this, consider the following example:

Let $X = \mathbb{N}$ and let $\mathfrak{J} = \{\emptyset\} \cup \{A_n \mid n = 1,2, \dots\}$ where $A_n = \{n, n+1, n+2, \dots\}$. Then $\langle X, \mathfrak{J} \rangle$ is a T_0 – space. Let $\{x_n\}$ be any sequence in $\langle X, \mathfrak{J} \rangle$. Fix up any $x \in X$. The open sets containing t are $A_1, A_2, \dots, A_n$. For each A_r we get $x_n \in A_r$ for $n \geq r$. But this shows that $x_n \to t$. Thus, any $t \in X$ is a limit point of $\{x_n\}$.

Exercises

Prove or disprove the following statements

1) The set of T_0 – topological spaces is a proper subset of the set of all topological spaces.
2) The property of a space being a T_0 – space is preserved by continuous maps.
3) The property of a space being a T_0 – space is a hereditary property.
4) The property of a space being a T_0 – space is a topological property.
5) A topological space $\langle X, \mathfrak{J} \rangle$ is a T_0 – space if and only if whenever x and y are distinct points of X have distinct closures.

Unit 13

T_1 – Spaces

Unit 13: $\mathcal{T}_1$ – Spaces

§1 Definition and Examples.

Definition 1.1 : A topological space is a T_1 – space if it satisfies the following axiom of Frechet :

"If x and y are two distinct points of X, then there exist two open sets, one containing x but not y, and the other containing y but not x".

Remarks:

(1) Obviously , every T_1 – space is a T_0 – space (follows by the Definition).

(2) Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space and $\mathfrak{J}^* \geq \mathfrak{J}$. Then $\langle X, \mathfrak{J}^* \rangle$ is also T_1 – space.

Examples 1.2:

<u>T_1 – spaces:</u>

(1) Any discrete topological space $\langle X, \mathfrak{J} \rangle$ with $|X| \geq 2$ is a T_1 – space.
Let $x \neq y$ in X. Define $G = \{x\}$ and $H = \{y\}$. Then $G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Hence $\langle X, \mathfrak{J} \rangle$ is a T_1 – space.

(2) Any co-finite topological space $\langle X, \mathfrak{J} \rangle$ with X is an infinite set, is a T_1 – space.
Let $x \neq y$ in X. Define $G = X - \{y\}$ and $H = X - \{x\}$. Then $G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Hence $\langle X, \mathfrak{J} \rangle$ is a T_1 – space.

(3) Any co-countable topological space with X as an uncountable set is a T_1 – space (proof as in Example 2).

(4) Any metric space is a T_1 – space.

Let $\langle X, d\rangle$ be a metric space and let $\mathfrak{J}$ be induced topology on X. Let $x \neq y$ in X. Then $d(x,y) = r > 0$. Define $G = S(x, r/3)$ and $H = S(y, r/3)$. Then $G, H \in \mathfrak{J}$, $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Hence, any metric space is a T_1 – space.

(5) $\langle \mathbb{R}, \mathfrak{J}_u\rangle$ is a T_1 – space. Let $x \neq y$ in $\mathbb{R}$**.** Then $|x - y| = r > 0$. Define $G = \left(x - \frac{r}{3}, x + \frac{r}{3}\right)$ and $H = \left(y - \frac{r}{3}, y + \frac{r}{3}\right)$. Then $G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Hence $\langle \mathbb{R}, \mathfrak{J}_u\rangle$ is a T_1 – space.

<u>Non T_1 – spaces:</u>

(6) Any indiscrete topological space $\langle X, \mathfrak{J}\rangle$ is not a T_1 – space.

(7) Let $\langle X, \mathfrak{J}\rangle$ be a topological space. $\mathfrak{J}$ is p – exclusion topology $(p \in X)$. i.e. $\mathfrak{J} = \{A \subseteq X \mid p \notin A\} \cup \{X\}$. If $x \neq p$ then $\nexists$ any open sets $G, H \in \mathfrak{J}$ such that $x \in G$ but $p \notin G$ and $p \in H$ but $x \notin H$ (as X is the only open set containing p). Hence the topological space $\langle X, \mathfrak{J}\rangle$ is not a T_1 – space.

(8) Let $\langle X, \mathfrak{J}\rangle$ be a topological space where $\mathfrak{J} = \{A \subseteq X \mid p \in A\} \cup \{\emptyset\}$. $(p \in X)$. (i.e. is p inclusion topology). Then $\langle X, \mathfrak{J}\rangle$ is not a T_1 – space as for $x \neq p$, every open set A containing x contains p also.

Remark: Every T_1 – space is a T_0 – space but not conversely .

Every T_0 – space need not be T_1 – space. For this consider the following examples:

(1) Let $X = \mathbb{R}$ and $\mathfrak{J} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Then for $x \neq y$ in $\mathbb{R}$ if $x < y$ then $y \in (x, \infty)$ and $x \notin (x, \infty)$. This shows that $\langle \mathbb{R}, \mathfrak{J}\rangle$ is a T_0 – space. But $\langle \mathbb{R}, \mathfrak{J}\rangle$ is not a T_1 – space, as for $x \neq y$ with $x < y$, there does not exists an open set containing x but not y.

(2) Let $X = \mathbb{N}$ and $\mathfrak{J} = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{A_n \mid n = 1,2, \dots \}$ where $A_n = \{1,2,3, \dots, n\}$. Then $\langle X, \mathfrak{J}\rangle$ is a T_0 – space (see example in §). But $\langle X, \mathfrak{J}\rangle$ is not a T_1 – space. [Let $m \neq n$ in X. Assume $m < n$. Then any open set containing n contains m. Hence $\langle X, \mathfrak{J}\rangle$ is not a T_1 – space.]

Hence the set of T_1 – topological spaces is a proper subset of all T_0 – topological spaces .

§ 2 Characterizations and Properties

Theorem 2.1: A topological space $\langle X, \mathfrak{J}\rangle$ is a T_1 – space if and only if $\{x\}$ is a closed in X for each $x \in X$.

Proof: Only if part.

Let $\langle X, \mathfrak{J}\rangle$ be a T_1 – space and let $x \in X$. To prove that $\overline{\{x\}} = \{x\}$. Let $y \in \overline{\{x\}}$ such that $y \notin \{x\}$. As $y \neq x$ and X is a T_1 – space, there exist $G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

$y \in \overline{\{x\}} \implies y$ is a limit point of $\{x\}$

$\implies [G \cap \{x\}] - \{y\} \neq \emptyset$ …. (as $y \in G$ and $G \in \mathfrak{J}$)

$\implies \{x\} - \{y\} \neq \emptyset$ …. (since $x \in G$)

$\implies \emptyset \neq \emptyset$; a contradiction.

Hence $\overline{\{x\}} \subseteq \{x\}$. As always $\{x\} \subseteq \overline{\{x\}}$, we get $\overline{\{x\}} = \{x\}$.

If part .

Let $\langle X, \mathfrak{J}\rangle$ be a T – space such that $\{x\}$ is closed set for each $x \in X$. To prove that $\langle X, \mathfrak{J}\rangle$ is a T_1 – space. Let $y \neq x$ in X. As $\overline{\{x\}} = \{x\}$, we get $y \notin \{x\} = \overline{\{x\}}$. Hence $y \in X - \{x\} = X - \overline{\{x\}}$. Similarly $x \notin \{y\} = \overline{\{y\}}$ will imply $x \in X - \{y\} = X - \overline{\{y\}}$. Define $G = X - \overline{\{x\}} = X - \{x\}$ and $H = X - \overline{\{y\}} = X - \{y\}$. Then $G, H \in \mathfrak{J}$ such that $y \in G$ but $x \notin G$ and $x \in H$ but $y \notin H$. Hence, $\langle X, \mathfrak{J}\rangle$ be a T_1 – space.

Theorem 2.2: A Topological space $\langle X, \mathfrak{J}\rangle$ is a T_1 – space if and only if any finite subset of X is closed.

Proof: Only if part.

Let $\langle X, \mathfrak{J}\rangle$ be a T_1 – space. By Theorem 1, $\{x\}$ is a closed set in X for each $x \in X$. Let A be any finite subset of X. Let $A = \{x_1, x_2, \dots, x_n\}$. Then $A = \bigcup_{i=1}^{n} \{X_i\}$

$$\implies \qquad \bar{A} = \overline{\left[\bigcup_{i=1}^{n} \{x_i\}\right]} = \bigcup_{i=1}^{n} \overline{\{x_i\}} \qquad \text{(see Unit 3, Theorem 3.4)}$$

$$= \bigcup_{i=1}^{n} \{x_i\} \quad (\text{since } \overline{\{x\}} = \{x\} \ \forall\, i, 1 \le i \le n)$$
$$= A$$

Hence, any finite subset A of X is closed in X.

If part.

Let any finite subset A of X is closed in topological space $\langle X, \mathfrak{J} \rangle$. Then obviously $\{x\}$ is closed in X for every $x \in \{x\}$. Hence by Theorem 2.1, $\langle X, \mathfrak{J} \rangle$ is aT_1 – space.

Theorem 2.3: A topological space $\langle X, \mathfrak{J} \rangle$ is aT_1 – space if and only if the topology $\mathfrak{J}$ is stronger than co-finite topology on X.

Proof: Only if part.

Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space. Let $\mathfrak{J}^*$ denote the co-finite topology on X. To prove that $\mathfrak{J}^* \le \mathfrak{J}$. Let $G \in \mathfrak{J}^*$. By definition of $\mathfrak{J}^*$, $X - G$ is a finite set. As $\langle X, \mathfrak{J} \rangle$ is a T_1 – space $X - G$ is a closed set in $\langle X, \mathfrak{J} \rangle$ (see Theorem 2.2). Hence $G \in \mathfrak{J}$. This shows that $\mathfrak{J}^* \le \mathfrak{J}$.

If part.

Let $\langle X, \mathfrak{J} \rangle$ be a topological space and $\mathfrak{J}^*$ be a co-finite topology on X such that $\mathfrak{J}^* \le \mathfrak{J}$. To prove that $\langle X, \mathfrak{J} \rangle$ is a T_1 – space. Select $x \in X$. Then $X - \{x\} \in \mathfrak{J}^* \Longrightarrow X - \{x\} \in \mathfrak{J}$. But this shows that $\{x\}$ is a closed set in X. Hence by Theorem 2.1, $\langle X, \mathfrak{J} \rangle$ is a T_1 – space.

Theorem 2.4: Being a T_1 – space is a hereditary property.

Proof: Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space and let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of $\langle X, \mathfrak{J} \rangle$. Then $Y \subseteq X$ and $\mathfrak{J}^* = \{G \cap Y \mid G \in \mathfrak{J}\}$. Let $y \ne z$ in Y. Then $y \ne z$ in X (as $Y \subseteq X$). As X is a T_1 – space, $\exists$ $G, H \in \mathfrak{J}$ such that $y \in G$ but $z \notin G$ and $z \in H$ but $y \notin H$. Define $G^* = G \cap Y$ and $H^* = H \cap Y$. Then $G^*, H^* \in \mathfrak{J}^*$ such that $y \in G^*$ but $z \notin G^*$ and $z \in H^*$ but $y \notin H^*$. This shows that $\langle Y, \mathfrak{J}^* \rangle$ is a T_1 – space. Thus any subspace of a T_1 – space is a T_1 – space. Hence being a T_1 – space is a hereditary property.

Theorem 2.5 :- Let $\langle X^*, \mathfrak{J}^* \rangle$ be a one point compactification of $\langle X, \mathfrak{J} \rangle$. $\langle X, \mathfrak{J} \rangle$ is a T_1 – space if and only if $\langle X^*, \mathfrak{J}^* \rangle$ is a T_1 – space.

Proof: Only if part.

Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space. To prove that $\langle X^*, \mathfrak{J}^* \rangle$ is a T_1 – space. Here $X^* = X \cup \{\infty\}$, $\infty \notin X$ and $\mathfrak{J}^* = \{A \subseteq X^* \mid \infty \in A$ and $X^* - A$ is a closed compact subset of $X\}$. Let $x \neq y$ in X^*.

Case (1): $x \neq \infty$ and $y \neq \infty$. Then $x, y \in X$ and $x \neq y$ in X. As X is a T_1 – space there exists $G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. As $G, H \in \mathfrak{J}^*$, we get for $x \neq y$ in X^* with $x \neq \infty$ and $y \neq \infty$, $\exists\ G, H \in \mathfrak{J}^*$ such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

Case (2): $x \neq \infty \implies x \in X \implies \{x\}$ is closed in X (since X is a T_1 – space). Thus $\{x\}$ is a closed compact subset of X.

Define $A = X^* - \{x\}$. Then $\infty \in A$ and $X^* - A = \{x\}$ is a closed compact subset of X.

Hence $A \in \mathfrak{J}^*$. Thus for $x \neq \infty$ in X^*, there exists open sets X and A in X^* such that $x \in X$ but $\infty \notin X$ and $\infty \in A$ but $x \notin A$. From case (1) and case (2) it follows that $\langle X^*, \mathfrak{J}^* \rangle$ is a T_1 – space.

If part.

Let $\langle X^*, \mathfrak{J}^* \rangle$ be a T_1 – space. As $\langle X, \mathfrak{J} \rangle$ is a subspace of $\langle X^*, \mathfrak{J}^* \rangle$ (see §one point compactification), by Theorem 2.4, we get $\langle X, \mathfrak{J} \rangle$ is a T_1 – space.

Theorem 2.6: Being a T_1 – space is a topological property.

Proof: Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space. Let $\langle Y, \mathfrak{J}^* \rangle$ be a topological space and $f: X \longrightarrow Y$ be a homeomorphism. To prove that $\langle Y, \mathfrak{J}^* \rangle$ is a T_1 – space. Let $a \neq b$ in Y. f being onto, $\exists\ x, y \in X$ such that $f(x) = a$ and $f(y) = b$. As f is one-one $a \neq b \implies x \neq y$ in X. X being a T_1 – space, $\exists\ G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Define $G^* = f(G)$ and $H^* = f(H)$. f being an open map, $G^*, H^* \in \mathfrak{J}^*$. Further $a \in G^*$ but $b \notin G^*$ and $b \in H^*$ but $a \notin H^*$. This shows that $\langle Y, \mathfrak{J}^* \rangle$ is a T_1 – space. As homeomorphic image of a T_1 – space is a T_1 – space , the result follows.

Corollary 2.7: The property of a space being T_1 – space is preserved under bijective open mappings.

Remark: T_1 – space need not be preserved under continuous functions.

Continuous image of a T_1 – space need not be a T_1 – space . For this consider the following example .

Let $\langle X, \mathfrak{I}_1 \rangle$ be a discrete topological space and $\langle X, \mathfrak{I}_2 \rangle$ be an indiscrete topological space ($|X| \geq 2$). Consider the identity map $i: \langle X, \mathfrak{I}_1 \rangle \longrightarrow \langle X, \mathfrak{I}_2 \rangle$. Then i is continuous. $\langle X, \mathfrak{I}_1 \rangle$ is a T_1 - space but continuous image of $\langle X, \mathfrak{I}_1 \rangle$ i.e. $\langle X, \mathfrak{I}_2 \rangle$ is not a T_1 – space.

Theorem 2.8: For any set X there exists a unique smallest topology $\mathfrak{I}$ such that $\langle X, \mathfrak{I} \rangle$ is a T_1 – space.

Proof: Let $\mathfrak{I}$ be the co-finite topology defined on X. Then $\langle X, \mathfrak{I} \rangle$ is a T_1 – space (see Example 2). Further if $\mathfrak{I}^*$ is a topology on X such that $\langle X, \mathfrak{I}^* \rangle$ is a T_1 – space, then $\mathfrak{I}^* \geq \mathfrak{I}$ (see Theorem 2.3). This shows that there exists a unique smallest topology $\mathfrak{I}$ – the co-finite topology – such that $\langle X, \mathfrak{I} \rangle$ is a T_1 – space.

Theorem 2.9: Let $\langle X, \mathfrak{I} \rangle$ be a T_1 – space and let $A \subseteq X$. If a point $x \in X$ is a limit point of A, then any open set (neighbourhood) containing x contains infinitely many points of A.

Proof: Let $\langle X, \mathfrak{I} \rangle$ be a T_1 – space and $x \in X$ is a limit point of A. Let $G \in \mathfrak{I}$ such that $x \in G$. Then $G \cap A - \{x\} \neq \emptyset$. Define $F = G \cap A - \{x\}$. Assume that F is finite. $\langle X, \mathfrak{I} \rangle$ being a T_1 – space, F is closed set in $\langle X, \mathfrak{I} \rangle$ (by Theorem 2.2). Then $X - F$ is an open set containing x. Hence $G \cap (X - F)$ is an open set containing x and actually, $G \cap (X - F) \cap A = \{x\}$. Hence $[G \cap (X - F)] \cap A - \{x\} = \emptyset$, this contradicts the fact that x is a limit point of A. Hence $G \cap A$ must be an infinite set i.e. G contains infinite elements of A.

Theorem 2.10: Let $\langle X, \mathfrak{I} \rangle$ be a first axiom space, T_1 – space. $x \in X$ is a limit point of E if and only if there exists a sequence of distinct points in E converging to x.

Proof: If part .

Let $\{x_n\}$ denote a sequence of distinct points of E converging to x.

Claim: x is a limit point of E.

Let G be any open set containing x. As $x_n \longrightarrow x$, $\exists\, N$ such that $x_n \in G$ for $n \geq N$. As the point of the sequence are distinct, $x_n \neq x \;\; \forall\, n \geq N$. But then $G \cap E - \{x\} \neq \emptyset$. This shows that x is limit point of E.

Only if part.

Let x be a limit point of E. As X is a F.A.S. $\exists$ a monotonically decreasing countable local base, say $\{B_n(x)\}$ at x. As X is a T_1 – space and x is a limit point of E, any open set containing x

contains infinitely many points of E. As $B_n(x)$ is an open set containing x, $B_n(x)$ contains infinitely many points of E, $\forall\, n \in \mathbb{N}$.
Thus $B_n(x) \cap E - \{x\}$ must be infinite. Hence we can select points x_n different from previously selected x_k $(k < n)$ such that $x_n \in B_n(x) \cap E - \{x\}$, $\forall\, n \in \mathbb{N}$.
Claim: $x_n \longrightarrow x$.

Let G be any open set containing x. Then $\exists\, n \in \mathbb{N}$ such that $x \in B_n(x) \subseteq G$. By choice of x_n , $x_n \in B_n(x) \cap E - \{x\}$ i.e. $x_n \in G \cap E - \{x\}$ and hence $x_n \longrightarrow x$.
Note that for the proof of 'If part' is true in any topological space.

Remark: In a T_1 – space a sequence may converge to more than one limit. In fact it may converge to every point of the space.

For this consider the following example.
Let $\{x_n\}$ be any sequence in $\langle X, \mathfrak{I}\rangle$, where $\langle X, \mathfrak{I}\rangle$ is co-finite topological space with X an infinite set. Let $x \in X$. To prove that $x_n \longrightarrow x$. Let $G \in \mathfrak{I}$ such that $x \in G$. $G \in \mathfrak{I} \implies X - G$ is a finite set. Find the largest $n_0 \in \mathbb{N}$ such that $x_{n_0} \in X - G$. Then as $X = G \cup (X - G)$ we get $x_n \in G$ for all $n \geq n_0$. But this shows that $x_n \longrightarrow x$ in $\langle X, \mathfrak{I}\rangle$. As this is true for any $x \in X$, we get any sequence in $\langle X, \mathfrak{I}\rangle$ converges to each $x \in X$.

§3 T_1 – spaces and countably compact spaces

We know that a T – space $\langle X, \mathfrak{I}\rangle$ is countably compact if any infinite subset of X has a limit point. The countably compact T_1 – spaces have very important properties. The equivalent conditions countably compact to be aT_1 – spaces are mentioned in the following theorems.

Theorem 3.1: Let $\langle X, \mathfrak{I}\rangle$ be a T_1 – space. $\langle X, \mathfrak{I}\rangle$ is a countably compact if and only if every countable open cover of it has a finite sub-cover.

Proof: Only if part.

Let $\langle X, \mathfrak{I}\rangle$ be a countably compact space. To prove that any countable open subset of X has a finite sub-cover. Suppose this is not true. Hence there exists a countable open cover say $\{G_n\}_{n=1}$ of X, which has no finite sub-cover i.e. $X \neq \bigcup_{i=1}^{n} G_i$ for any finite n. For each

$n \in N$, define

$$F_n = X - \left[\bigcup_{i=1}^{n} G_i\right].$$

Then each F_n is a non-empty, closed set and $F_1 \supset F_2 \supset F_3 \supset \cdots$

Select from each F_n, a point x_n and define $E = \{x_n \mid n \in N,\ x_n \in F_n\}$.

Claim that E is not finite set. .

For if E is finite, then there exists some point say x_p, which will be in each F_n. But then

$$x_p \in \bigcap_{n=1}^{\infty} F_n \implies x_p \in X - \bigcup_{n=1}^{\infty} G_n = \emptyset \text{ as } X = \bigcup_{n=1}^{\infty} G_n \text{; a contradiction.}$$

As E is an infinite set and X is a countably compact space, E has a limit point say x in X. As X is a T_1 – space, any open set containing x must contain infinite points of E (see Theorem 2.9).

But this in turn will imply that x is a limit point of each set $E_n = \{x_n, x_{n+1}, \ldots\}$, $n \in N$.

$E_n \subseteq F_n \implies d(E_n) \subseteq d(F_n)$. As F_n is a closed set, $d(F_n) \subseteq F_n$.

Hence $x \in F_n\ \forall\, n \in N$.

Thus $x \in \bigcap_{n=1}^{\infty} F_n \implies x \in X - \bigcup_{n=1}^{\infty} G_n = \emptyset$ since $X = \bigcup_{n=1}^{\infty} G_n$; a contradiction.

Thus our assumption is wrong. Hence any countable open cover of X has a finite sub-cover.

If part.

Assume that any countable open cover of X has a finite sub-cover.

To prove that X is countably compact.

Suppose that X is not countably compact. Then there must exist an infinite subset say A of X such that E has no limit point in X. Hence $d(A) = \emptyset$ As A is infinite, select an infinite sequence $\{x_n\}_{n=1}$ of points of A. Define $B = \{x_n \mid n \in N\}$. As $B \subseteq A \implies d(B) \subseteq d(A)$.

As $d(A) = \emptyset$, we get $d(B) = \emptyset$.

i.e. B has no limit point in X. Hence, each x_n is not a limit point of B. But then there exists an open set G_n containing x_n such that $G_n \cap B = \{x_n\}\ : \forall\ n \in N$. As $d(B) = \emptyset$, we get B is closed set. Hence $X - B$ is open set in X.

Now $X = B \cup (X - B) \implies X = \left[\bigcup_{n=1}^{\infty} G_n\right] \cap (X - B)$

(since $G_n \cap B = \{x_n\}$ and $B = \{x_n \mid n \in N\}$). Hence by assumption, for countable open cover

$\{ G_n\}_{n=1} \cup (X - B)$ of X, there exists a finite sub-cover. But this is not possible since each G_n is required to cover the points x_n (since $G_n \cap B = \{ x_n\}$). Hence X must be a countably compact space.

Theorem3.2: Let $\langle X, \mathfrak{J}\rangle$ be a T_1 – space. $\langle X, \mathfrak{J}\rangle$ is countably compact if and only if every countable family of closed subsets of X, which has finite intersection property, has non-empty intersection.

Proof: Only if part .

Let X be a countably compact space. Let $\mathcal{K}$ denote family of closed sets having finite intersection property (f.i.p.). To prove

$$\bigcap_{F_i \in \mathcal{K}} F_i \neq \emptyset.$$

Let $\bigcap_{F_i \in \mathcal{K}} F_i = \emptyset$. Then $\left[\bigcap_{F_i \in \mathcal{K}} F_i\right]' = X \Longrightarrow \bigcup_{F_i \in \mathcal{K}} F_i' = X$ (here $F' = X - F$).

As F is closed, F' is an open set. Hence $\{F'\}_{F \in \mathcal{K}}$ forms a countable open cover for X.

By Theorem 3.1, this open cover has a finite sub-cover. Hence $X = \bigcup_{i=1}^{n} F_i'$.

But then $\bigcap_{i=1}^{n} F_i = \emptyset$, contradicts the fact that K satisfies f.i.p. Hence $\bigcap_{i=1}^{n} F_i \neq \emptyset$.

If part.

Assume that any countable family of closed sets satisfying f.i.p. has non-empty intersection in $\langle X, \mathfrak{J}\rangle$. To prove that $\langle X, \mathfrak{J}\rangle$ is countably compact space. Let $\{ G_i\}_{i=1}$ be any countable open cover of X. If this countable open cover, has no finite sub-cover, then the countable family $\{ G_i'\}_{i=1}$ of closed set will satisfy f.i.p.

Hence by assumption $\bigcap_{i=1}^{n} G_i' \neq \emptyset$.But then $X \neq \bigcup_{i=1}^{\infty} G_i$; a contradiction.

This shows that the countable open cover $\{ G_i\}_{i=1}$ of X has a finite sub-cover. Hence by Theorem 3.1,$\langle X, \mathfrak{J}\rangle$ is a countably compact space.

Theorem 3.3: A T_1 – space X is countably compact if and only if every infinite open covering of X has a proper sub-cover.

Proof: Only if part.

Let X be a T_1 countably compact space. To prove that any infinite open covering of X has a proper sub-cover.

Let this be not true i.e. ∃ an infinite open cover U of X that fails to have a proper sub-cover. But this means that each member of U contains a point, which does not, belong to any other member of U. Thus ∃ an infinite subset A of X such that $A \cap O$ is singleton set for each $O \in U$. As X is countably compact, the infinite set A has a limit point say x in X.

Now $x \in X$ and $X = \bigcup_{O \in U} O \Rightarrow x \in O$ for some $O \in U$.

x being a limit point of A, $A \cap O - \{x\} \neq \emptyset$. But by choice of A, $A \cap O$ is singleton set. As x is limit point of A and X is a T_1 – space, any open set containing x must contain infinite points of A (see Theorem 2.9); which is not true. Hence our assumption is wrong. This proves that any arbitrary open cover has a proper sub-cover.

If part.

Assume that any arbitrary open cover of X has a proper sub-cover and X is a T_1 – space. To prove that X is countably compact.

Let X be not countably compact then there exists an infinite subset A of X which has no limit point in X. As $d(A)=\emptyset$, $\bar{A} = A$. Therefore A is closed set in X. For any $a \in A$, as a is not a limit point of A, ∃ an open set G_a such that $a \in G_a$ and $G_a \cap A - \{a\} = \emptyset$.

As $a \in A$ and $a \in G_a$ we get $G_a \cap A = \{a\}$ (as $G_a \cap - \{a\} = \emptyset$).

Thus $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} G_a$.

$$X = A \cup (X - A) \subseteq \left[\bigcup_{a \in A} G_a\right] \cup (X - A).$$

This shows that $\{\{G_a\}_{a \in A} \cup (X - A)\}$ forms an open cover for X. But this open cover has no proper sub-cover for X ; a contradiction. Therefore, X is countably compact.

Theorem 3.4: Let X be a T_1 – space. X is countably compact if and only if every sequence in X has a limit point.

Proof: Only if part.

Let X be countably compact. Let $\{x_n\}$ be any sequence in X. Let $x_n = x$ for infinitely many n, then obviously x is a limit point of $\{x_n\}$. Let x_n be distinct points of X.

Then $A = \{x_n | \ n \in N\}$ is an infinite subset of X. As X is countably compact, A has a limit point in X. say x.

Claim that $x_n \to x$.

Suppose $\{x_n\}$ does not converge to x. Then there exists an open set G in X such that $x \in G$ and an integer m such that $x_n \notin G$ for $n \geq m$ i.e. $x_n \in X - G$ for $n \geq m$. Then the open set containing x will contain only finite number of points $x_1, x_2, \dots, x_{m-1}$ of A. _____________ (I)

As X is a T_1 – space and x is a limit point of A imply the open set G containing x must contain infinitely many points of A (see Theorem 2.9). ________________________________ (II)

As (I) and (II) contradicts each other our assumption is wrong. Therefore $\{x_n\}$ converges to x in X.

If part.

Assume that every sequence $\{x_n\}$ in X has a limit point in X.

To prove that X is countably compact. It is enough to prove that any countable open cover of X has finite sub-cover as X is a T_1 – space (see Theorem3.1)

Suppose there exists a countable sub-cover $\{G_n\}_{n \in N}$ of X, which has no finite sub-cover. Let

$V_n = \bigcup_{k=1}^{n} G_k$, $n \in N$.Then by assumption , $V_n \neq X \ : \ \forall \, n \in N. \ V_1 = G_1 \neq X.$

Fix up $x_1 \in G_1$. Let $n_1 = 1$. Select $x_2 \in V_2$ such that $x_2 \notin V_1$. Select $x_3 \in G_3$ such that $x_3 \notin G_1 \cup G_2$. Continuing in this way, there exists a sequence $\{x_n\}$ in X. By assumption $\{x_n\}$ has a limit point say x in X. As $x_n \to x, \exists \ n_0 \in N$ such that $x \in V_{n_0}$. But $V_{n_0} \in \mathfrak{I}, x \in V_{n_0}$ and V_{n_0} does not contains terms of $\{x_k\}$ for $k \geq n_0$, this contradicts the fact that $x_n \to x$.

Hence X must be countably compact.

Combining all the equivalent conditions we get

Theorem 3.5: For a T_1 – space X following statements are equivalent :

(1) X is countably compact.

(2) Every countable open cover of X has a finite sub-cover..

(3) Every countable family of closed subsets of X having f.i.p. will have a non-empty intersection.
(4) Every infinite open cover of X has a proper sub-cover.
(5) Every sequence of X has a limit point in X.

§4 T_1 – spaces and First axiom spaces (f.a.s.)

Let $\langle X, \mathfrak{J} \rangle$ be a T – space . Recall that X is a f.a.s. if there exists a countable local base $\{B_n(x)\}_{n=1}$ at each $x \; in \; X$.

Theorem 4.1: Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space and f.a.s. Let $E \subseteq X$. A point $x \in X$ is a limit point of E if and only if there exists a sequence of distinct points of E converging to x.

Proof: Only if part –

Let x be a limit point of E. As X is f.a.s., there exists a countable monotonically decreasing local base $\{B_n(x)\}_{n=1}$ at x. Now $x \in B_n(x)$ and $B_n(x) \in \mathfrak{J} \Longrightarrow B_n(x) \cap E - \{x\} \neq \emptyset$; $\forall \; n \in N$. Further as X is T1 – space and x is limit point of E, each $B_n(x)$ contains infinite number of distinct points of E (see Theorem 2.9).

Hence $B_n(x) \cap E - \{x\}$ must be an infinite set. Hence by induction select $x_n \in B_n(x) \cap E - \{x\}$ such that x_n is different from $x_1, x_2, \dots, x_{n-1}$.

As $\{B_n(x)\}$ is monotonically decreasing family, $x_n \to x$.

Thus if x is limit point of E ,then $\exists$ a sequence of distinct points of E converging to x.

If part –

Let $\{x_n\}$ be a sequence of distinct points of E converging to x. To prove that x is a limit point of E. Let G be an open set containing x. But $x_n \to x$ and $x \in G$, G is open in X $\Longrightarrow \exists \; N$ such that $x_n \in G$ for $n \geq N$. Since the points x_n are distinct points of X, $G \cap E - \{x\} \neq \emptyset$. Hence x is a limit point of E.

[Note that for the proof of **'If part',** X is T_1 – space and f.a.s., both are not used. Hence the if part is true in any T – space.]

Theorem 4.2:- For each x in first axiom, T_1 – space X,

$$\{x\} = \bigcap_{n=1}^{\infty} B_n(x) \ .$$

where $\{B_n(x)\}_{n\in N}$ is a countable local base at x.

Proof: As X is a first axiom space, $\exists$ a countable local base $\{B_n(x)\}$ at x.

Hence $\{x\} \subseteq \bigcap_{n\in\mathbb{N}} B_n(x)$. Let $y \in \bigcap_{n\in\mathbb{N}} B_n(x)$ such that $y \neq x$.

Then X being a T_1 – space, $\exists\ G \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$. As $x \in G$ and $G \in \mathfrak{J}$, $\exists\ n_0 \in \mathbb{N}$ such that $B_n(x) \subseteq G$. But then $y \in G$; a contradiction.

Hence $\bigcap_{n\in\mathbb{N}} B_n(x) \subseteq \{x\}$.

$$\text{Thus } \{x\} = \bigcap_{n\in\mathbb{N}} B_n(x).$$

Remarks:

(1) $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a first axiom, T_1 – space. Hence $\bigcap_{n=1}^{\infty}\left(x - \frac{1}{n}, x + \frac{1}{n}\right) = \{x\}$

since the family $\left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \mid x \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right),\ \forall\, n\right\}$ is a countable local base at x.

(2) Any metric space is first axiom, T_1- space. Hence $\bigcap_{n=1}^{\infty} S\left(x, \frac{1}{n}\right) = \{x\}$

since the family $\left\{S\left(x, \frac{1}{n}\right) \mid x \in S\left(x, \frac{1}{n}\right),\ \forall\, n\right\}$ is a countable local base at x.

§5 Solved problems

Problem1: Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space. If $\mathfrak{J}$ is closed for arbitrary intersection, then show that $\mathfrak{J}$ is the discrete topology on X.

Solution :-By data, $\mathfrak{J}$ is closed for arbitrary intersection and hence any union of closed sets in X is a closed set in X. Let $A \subseteq X$.

Then $A = \bigcup_{a\in A}\{a\} \implies A$ is a closed set in X (since $\{a\}$ is a closed set in X).

Thus, any subset of X is closed in X. Hence $\mathfrak{J}$ is the discrete topology on X.

Problem 2 :-Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space and $\mathfrak{B}(x)$ be a local base at $x \in X$. Show that for $y \neq x$ there exists $B \in \mathfrak{B}(x)$ such that $y \notin B$.

Solution: Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space. As $y \neq x$, there exist $G \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$.
As $x \in G$ and $G \in \mathfrak{J}$, by definition of local base, $\exists\ B \in \mathfrak{B}(x)$ such that $x \in B \subseteq G$. As $y \notin G$ we get $y \notin B$. Hence the result.

Problem 3: Let X be a T_1 – space and A is finite subset of X then no point of A is limit point of A.

Solution :-Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space and let $A = \{a_1, a_2, \dots, a_n\}$ be a finite subset of X. As X is a T_1 – space, by Theorem 2, A is a closed set. Hence $\bar{A} = A \cup d(A) = A$ i.e. $d(A) \subseteq A$. Hence, limit points of A must be members of A. Claim that $a_1 \in A$ is not a limit point of A.
Let $B = \{a_2, a_3, \dots, a_n\}$. B being a finite subset of a T_1 space, B is closed in $\langle X, \mathfrak{J} \rangle$. Hence $X - B$ is open in $\langle X, \mathfrak{J} \rangle$. As $a_1 \in X - B = \{a_1\}$ and $(X - B) \cap A - \{a_1\} = \{a_1\} - \{a_1\} = \emptyset$, we get a_1 is not a limit point of A. Similarly, we can prove that any $a_i \in A$ is not a limit point of A.
Hence $d(A) = \emptyset$.

Problem 4: Show that a finite topological space $\langle X, \mathfrak{J} \rangle$ is a T_1 – space if and only if $\langle X, \mathfrak{J} \rangle$ is a discrete space.

Solution :- Only if part.

Let $\langle X, \mathfrak{J} \rangle$ be a finite T_1 – space. To prove $\mathfrak{J} = \wp(X)$. Let $A \subseteq X$. Then $X - A$ being finite, $X - A$ is closed in $\langle X, \mathfrak{J} \rangle$ (see Theorem 2). Hence A is open in X i.e. $A \in \mathfrak{J}$.
Thus $A \in \wp(X) \Longrightarrow A \in \mathfrak{J}$. Hence $\mathfrak{J} = \wp(X)$. This shows that $\langle X, \mathfrak{J} \rangle$ is a discrete topological space.

If part.

Let $\langle X, \mathfrak{J} \rangle$ be a discrete topological space. Then for $x \neq y$ in X, define $G = \{x\}$ and $H = \{y\}$. Then $G, H \in \mathfrak{J}$, such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Hence $\langle X, \mathfrak{J} \rangle$ is a T_1 – space. [Note that $\langle X, \mathfrak{J} \rangle$ is a finite space – is not used]

Problem 5: Let a T_1 – topology $\mathfrak{J}$ on X be generated by finite family of subsets of X. Show that $\mathfrak{J}$ is the discrete topology on X.

Solution: As $\mathfrak{J}$ is generated by a finite family of subsets of X, $\mathfrak{J}$ must be finite i.e. there are only finite number of open sets in X. Hence there exists only finite number of closed sets in X. As X is a T_1 – space, singleton sets are closed in X. Thus the family of singleton sets in X is finite.

Hence $X = \bigcup_{i=1}^{n} \{x_i\} \implies X$ is a finite set. As $\mathfrak{J}$ is a T_1 topology defined on a finite set X,

$\mathfrak{J}$ must be the discrete topology on X.

Problem 6: Show that a topological space $\langle X, \mathfrak{J} \rangle$ is a T_1- space if and only if

$$\bigcap \{G \mid G \in \mathfrak{J}, x \in G\} = \{x\}.$$

Solution :-Only if part.

Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space. To prove that $\cap \{G \mid G \in \mathfrak{J}, x \in G\} = \{x\}$.

Let $y \in \cap \{G \mid G \in \mathfrak{J}, x \in G\}$ such that $y \neq x$. As $y \neq x, \exists\, G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$. By the choice of $y, x \in G, G \in \mathfrak{J} \implies y \in G$; a contradiction.

Hence $\bigcap \{G \mid G \in \mathfrak{J}, x \in G\} = \{x\}$.

If part.

Let $\bigcap \{G \mid G \in \mathfrak{J}, x \in G\} = \{x\}$ for any $x \in X$.

To prove that $\langle X, \mathfrak{J} \rangle$ is a T_1 – space. Let $a \neq b$ in X.

Then $\{a\} = \bigcap \{G \mid G \in \mathfrak{J}, a \in G\}$ and $\{b\} = \bigcap \{G \mid G \in \mathfrak{J}, b \in G\}$.

As $a \neq b$, $b \notin \{a\} \implies b \notin \bigcap \{G \mid G \in \mathfrak{J}, a \in G\}$.

Hence $\exists$ an open set G such that $a \in G$ but $b \notin G$.

Similarly as $a \notin \{b\} \implies a \notin \bigcap \{G \mid G \in \mathfrak{J}, b \in G\}$.

Hence $\exists$ an open set H such that $b \in H$ but $a \notin H$.

This shows that $\langle X, \mathfrak{J} \rangle$ is a T_1 – space.

Problem 7: Show that a topological space $\langle X, \mathfrak{J} \rangle$ is a T_1 – space if and only if the intersection of all neighbourhoods of any arbitrary point of X is a singleton set.

Solution :-Only if part .

Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space. To prove that $\cap\{N \mid N$ is a nbd. of $x , x \in G\} = \{x\}$.

Let $y \in \cap\{N \mid N$ is a nbd. of $x , x \in G\}$ such that $y \neq x$.

As $y \neq x, \exists\ G, H \in \mathfrak{J}$ such that $x \in G$ but $y \notin G$

By the choice of $y, x \in G, G \in \mathfrak{J} \Rightarrow y \in G$; a contradiction.

Hence $\cap\{N \mid N$ is a nbd.of $x , x \in G\} = \{x\}$.

If part.

Let $\bigcap\{N \mid N$ is a nbd. of $x , x \in G\} = \{x\}$ for any $x \in X$.

To prove that $\langle X, \mathfrak{J} \rangle$ is a T_1 – space. Let $a \neq b$ in X.

Then $\{a\} = \bigcap\{N \mid N$ is a nbd.of $a , a \in G\}$ and $\{b\} = \bigcap\{N \mid$N is a nbd.of $b , b \in G\}$.

As $a \neq b,\ \ b \notin \{a\} \ \Rightarrow \ b \notin \bigcap\{N \mid N$ is a nbd. of $x , x \in G\}$.

Hence $\exists$ a nbd. N of $a \in N$ such that $b \notin M$

Similarly as $a \notin \{b\} \ \Rightarrow \ a \notin \cap\{N \mid N$ is a nbd. of $x , x \in G\}$ hence $\exists$ nbd. M of b such that $a \notin M$. As N and M are neighbourhoods of a and b respectively, there exists open sets G and H such that $a \in G \subseteq N$ and $b \in H \subseteq M$. Hence $\exists$ an open set G such that $a \in G$ but $b \notin G$. *and* $\exists$ an open set H such that $b \in H$ but $a \notin H$.

This shows that $\langle X, \mathfrak{J} \rangle$ is a T_1 – space.

Exercises

1) Let $\langle X, \mathfrak{J} \rangle$ be a T_1 – space and let A be finite subset of X. Then no point of A is limit point of A.

2) A finite topological space $\langle X, \mathfrak{J} \rangle$ is a T_1 – space if and only if for two distinct points x and y of X, there exist a nbd. N of x not containing y, and a nbd. M of y not containing x.

3) Give an example of each of the following .
   a) A T_1 – space which is countably compact.
   b) A T_1 – space which is not countably compact.

c) A countably compact which is a T_1 – space.

d) A countably compact space which is not a T_1 – space.

e) A T_1 – space which is first countable.

f) A T_1 – space which is not first countable.

g) First countable space which is a T_1 – space.

h) First countable space which is not a T_1 – space.

Unit 14

$\mathcal{T}_2$ – Spaces

Unit 14: $\mathcal{T}_2$ – Spaces

§1 Definition and Examples

Definition1.1: A topological space $\langle X, \mathfrak{J} \rangle$ is a T_2 – space or Hausdorff space if it satisfies the following axiom of Hausdorff:

"If x and y are two distinct points of X, then there exist two disjoint open sets one containing x and the other containing y".

Remarks: (1) Obviously, every T_2 – space is a T_1 – space and hence a T_0 – space (follows by the Definition).

(2) Let $\langle X, \mathfrak{J} \rangle$ be a T_2 – space and $\mathfrak{J}^* \geq \mathfrak{J}$. Then $\langle X, \mathfrak{J}^* \rangle$ is also T_2 – space.

Examples 1.2:

T_2 – spaces.

(1) Any discrete topological space $\langle X, \mathfrak{J} \rangle$ with $|X| \geq 2$ is a T_2 – space.

For $x \neq y$ in X, $\{x\}$ and $\{y\}$ are two disjoint open sets containing x and y respectively.

(2) Any metric space is a T_2 – space.

Let $\langle X, d \rangle$ be a metric space and let $\mathfrak{J}$ be the induced topology on X by the metric d.

Let $x \neq y$ in X. Then $d(x, y) = r > 0$. Then $S\left(x, \frac{r}{3}\right)$ and $S\left(y, \frac{r}{3}\right)$ are two disjoint open sets containing x and y respectively. Hence $\langle X, d \rangle$ is a T_2 – space.

(3) $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a T_2 – space.

Let $x \neq y$ in $\mathbb{R}$. Then $|x - y| = r > 0$. $\left(x - \frac{r}{3}, x + \frac{r}{3}\right)$ and $\left(y - \frac{r}{3}, y + \frac{r}{3}\right)$ are disjoint open sets in $\mathbb{R}$ containing x and y respectively. Hence $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a T_2 – space.

(4) Fort's space is a T_2 – space.

Let X be any uncountable set and let ∞ be a fixed point of X .

Let $\mathfrak{J} = \{G \subseteq X \mid \infty \notin G\} \cup \{G \subseteq X \mid \infty \in G$ and $X - G$ is finite$\}$. Then $\mathfrak{J}$ is a topology on X. Define $\mathfrak{J}_1 = \{G \subseteq X \mid \infty \notin G\}$ and $\mathfrak{J}_2 = \{G \subseteq X \mid \infty \in G$ and $X - G$ is finite$\}$ then $\mathfrak{J} = \mathfrak{J}_1 \cup \mathfrak{J}_2$ is a topology on X . This Fort's space $\langle X, \mathfrak{J} \rangle$ is a T_2 – space.

Let $x \neq y$ in X.

Case 1: Let x and y both are different from ∞ . Then $G = \{x\}$ and $H = \{y\}$ are disjoint open sets containing x and y respectively.

Case 2: Let $y = \infty$. Then $G = X - \{\infty\}$ and $H = \{\infty\}$ are disjoint open sets containing x and ∞ respectively. Thus from both the cases we get the Fort's space $\langle X, \mathfrak{J} \rangle$ is a T_2 – space.

Non T_2 – spaces.

(1) Any indiscrete topological space $\langle X, \mathfrak{J} \rangle$ is not a T_2 – space.

(2) Any co-finite topological space $\langle X, \mathfrak{J} \rangle$ with X an infinite set is not a T_2 – space.
[Note that if X is finite, then co-finite topology $\mathfrak{J}$ on X is discrete topology on X and hence in this case $\langle X, \mathfrak{J} \rangle$ is a T_2 – space.]
Let $G, H \in \mathfrak{J}$ such that $G \cap H = \emptyset$. G is open $\Longrightarrow X - G$ is finite set.
H is open $\Longrightarrow X - H$ is finite set. $G \cap H = \emptyset \Longrightarrow G \subseteq X - H \Longrightarrow G$ is a finite set.
As $X = G \cup (X - G)$, we get X is a finite set; a contradiction. This shows that no two open sets in X are disjoint. Hence $\langle X, \mathfrak{J} \rangle$ is not a T_2 – space.

(3) Let $\langle X, \mathfrak{J} \rangle$ be a p-inclusion topology ($p \in X$). $\mathfrak{J} = \{\emptyset\} \cup \{A \subseteq X \mid p \in A\}$. $\langle X, \mathfrak{J} \rangle$ is not a T_2 – space. For $p \neq x$ in X we cannot find two disjoint open sets one containing p and other containing x.

(4) Let $\langle X, \mathfrak{J} \rangle$ be a p-exclusion topology ($p \in X$). $\mathfrak{J} = \{X\} \cup \{A \subseteq X \mid p \notin A\}$.
Case (1): $x \neq y$ ($x \neq p$ and $y \neq p$). $\{x\}, \{y\} \in \mathfrak{J}$ such that $x \in \{x\}$, $y \in \{y\}$ and $\{x\} \cap \{y\} = \emptyset$.
Case (2): $x \neq p$. As X is the only open set containing p, in this case we cannot find disjoint open sets one containing p and other containing x.
Hence $\langle X, \mathfrak{J} \rangle$ is not a T_2 – space.

(5) Let $X = \mathbb{N}$ and $\mathfrak{J} = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{A_n \mid n = 1,2,3, \dots\}$ where $A_n = \{1,2, \dots, n\}$.
Let $m \neq n$ in X. Assume $m < n$. Then by definition of $\mathfrak{J}$ any open set containing n must contain m. Hence $\langle X, \mathfrak{J} \rangle$ is not a T_2 – space.

(6) Let $X = \mathbb{N}$ and $\mathfrak{J} = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{A_n \mid n = 1,2,3, \dots\}$ where $A_n = \{n, n + 1, n + 2, \dots\}$.
Let $m \neq n$ in X. Then if $m < n$, then every open set containing m must contain n.
Hence $\langle X, \mathfrak{J} \rangle$ is not a T_2 – space.

Remark: Every T_1 – space need not be a T_2 – space .

We know that co-finite topological space is a T_1 – space (see Example in §) But . any co-finite topological space $\langle X, \mathfrak{J} \rangle$ with X an infinite set is not a T_2 – space. Hence we get every T_1 – space need not be a T_2 – space. Thus the collection of all T_2 – spaces is a proper subset of the collection of all T_1 – spaces. As every T_2 – space is a T_1 – space, all the properties of T_1 – space hold for T_2 – space.

e.g. (1) Every finite subset of a T_2 – space is closed in it.

(2) Finite T_2 – space is the discrete topological space.

(3)Distinct points have distinct closures in a T_2 – space.

§2 Characterizations and Properties

Theorem 2.1: Let $\langle X, \mathfrak{J} \rangle$ be a T– space.Then $\langle X, \mathfrak{J} \rangle$ is a T_2 – space if and only if the intersection of all closed neighbourhoods of a point x in X is $\{x\}$.

Proof: -Only if part –

Let $\langle X, \mathfrak{J} \rangle$ be a T_2 – space. To prove that $\cap \{N \mid N$ is a closed nbd. of $x\} = \{x\}$.

Let $y \in \cap$ { N | N is a closed nbd. of x } such that y $\neq x$. As $\langle X, \mathfrak{J} \rangle$ be a T_2 – space and $x \neq y$ in X. Hence $\exists\ G , H \in \mathfrak{J}$ such that $x \in G, y \in H$ and $\cap\ H = \emptyset$.

Hence $x \in G \subseteq X - H$. But this shows that $X - H$ is a closed nbd. of x .But by the choice of y, $y \in X - H$; a contradiction. Hence $\cap \{N \mid N$ is a closed nbd. of $x\} = \{x\}$.

If part –

Let $\cap \{N \mid N$ is a closed nbd. of $x\} = \{x\}$ for each x in X. To prove that X is a T_2 – space.

Let $x \neq y$ in X. Hence $y \notin \{x\} = \cap \{N \mid N$ is a closed nbd. of $x\}$ implies y $\notin N$, for some closed nbd. N of x. As N is a nbd.of x there exists an open set G such that $x \in G \subseteq N$. Thus for $x \neq y$ in X there exist disjoint open sets G and $X - N$ such that $x \in G$ and $y \in X - N$.

This shows that X is a T_2 – space.

Theorem 2.2: A T – space $\langle X, \mathfrak{J} \rangle$ is a T_2 – space if and only if for any $x \neq y$ in X, $\exists$ basic open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Proof: Only if part –

Let $\langle X, \mathfrak{J} \rangle$ be a T$_2$ – space and $x \neq y$ in X. Hence $\exists\ G, H \in \mathfrak{J}$ such that $x \in G, y \in H$ and $G \cap H = \emptyset$. Let $\mathfrak{B}$ be a base for $\mathfrak{J}$. Then for $G, H \in \mathfrak{J}$ $\exists\ B_1, B_2 \in \mathfrak{B}$ such that $x \in B_1 \subseteq G$ and $x \in B_2 \subseteq H$ (by definition of base). Thus there exists basic open sets B_1 and B_2 such that $x \in B_1, y \in B_2$ and $B_1 \cap B_2 = \emptyset$.

If part –

To prove $\langle X, \mathfrak{J} \rangle$ that is a T$_2$ – space.

Let $x \neq y$ in X . By assumption, $\exists$ basic open sets $B_1, B_2 \in \mathfrak{B}$ such that $x \in B_1, y \in B_2$ and $B_1 \cap B_2 = \emptyset$. As $\mathfrak{B} \subseteq \mathfrak{J}$, there exist disjoint sets B_1, B_2 in $\mathfrak{J}$ such that $x \in B_1, y \in B_2$.

Hence $\langle X, \mathfrak{J} \rangle$ is a T$_2$ – space.

Theorem 2.3: Being T$_2$ – space is a hereditary property.

Proof: Let $\langle X, \mathfrak{J} \rangle$ be a T$_2$ – space and let $\langle Y, \mathfrak{J}^* \rangle$ be a subspace of $\langle X, \mathfrak{J} \rangle$.

Then $Y \subseteq X$ and $\mathfrak{J}^* = \{G \cap Y \mid G \in \mathfrak{J}\}$. Let $y \neq z$ in Y. As $Y \subseteq X$, $y \neq z$ in X.

X being a T$_2$ – space, there exist $G, H \in \mathfrak{J}$ such that $y \in G$ and $z \in H$ and $G \cap H = \emptyset$.

Define $G^* = G \cap Y$ and $H^* = H \cap Y$. Then $G^*, H^* \in \mathfrak{J}^*$ such that $y \in G^*$, $z \in H^*$ and $G^* \cap H^* = \emptyset$. But this shows that $\langle Y, \mathfrak{J}^* \rangle$ is a T$_2$ – space. As any subspace of a T$_2$ – space $\langle X, \mathfrak{J} \rangle$ is a T$_2$ – space, the result follows.

Theorem2.4: Being a T$_2$ – space is a topological property.

Proof: Let $\langle X, \mathfrak{J} \rangle$ and $\langle Y, \mathfrak{J}^* \rangle$ be any two topological spaces and $f: X \longrightarrow Y$ be a homeomorphism. Let $\langle X, \mathfrak{J} \rangle$ be a T$_2$ – space. To prove that $\langle Y, \mathfrak{J}^* \rangle$ is a T$_2$ – space.

Let $y_1 \neq y_2$ in Y. $f: X \longrightarrow Y$ being one-one and onto, there exist $x_1 \neq x_2$ in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. As X is a T$_2$ – space, there exist $G, H \in \mathfrak{J}$ such that $x_1 \in G$ and $x_2 \in H$ and $G \cap H = \emptyset$. f being an open map, $f(G), f(H) \in \mathfrak{J}^*$. Thus $y_1 \in f(G)$, $y_2 \in f(H)$ and $f(G) \cap f(H) = \emptyset$. But this in turns shows that $\langle Y, \mathfrak{J}^* \rangle$ is a T$_2$ – space. Thus any homeomorphic image of a T$_2$ – space is a T$_2$ – space. Hence the result.

Corollary 2.5: The property of a space being T$_2$ – space is preserved by one-one, onto open maps.

Theorem 2.6: Let $\langle X, \mathfrak{J} \rangle$ be a topological space and $\langle Y, \mathfrak{J}^* \rangle$ be a Hausdorff space. Let $f: X \to Y$ and $g: X \to Y$ be continuous mappings. The set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

Proof: Let $A = \{x \in X \mid f(x) = g(x)\}$. Select any $t \in X - A$. Then $f(t) \neq g(t)$ as $t \notin A$.
As $f(t) \neq g(t)$ in Y and Y is a T_2 –space, $\exists$ $G^*, H^* \in \mathfrak{J}^*$ such that $f(t) \in G^*, g(t) \in H^*$ and $G^* \cap H^* = \emptyset$.
$f: X \to Y$ is continuous and $g: X \to Y$ is continuous $\Longrightarrow f^{-1}(G^*)$ and $g^{-1}(H^*) \in \mathfrak{J}$.
Hence $O = f^{-1}(G^*) \cap g^{-1}(H^*) \in \mathfrak{J}$ ________________________ (I)
$f(t) \in G^* \ \Longrightarrow t \in f^{-1}(G^*)$
$g(t) \in H^* \ \Longrightarrow t \in g^{-1}(H^*)$.
Thus $t \in f^{-1}(G^*) \cap g^{-1}(H^*) = O$ _________________________ (II)
Again for any $x \in O$, we get $x \in f^{-1}(G^*) \cap g^{-1}(H^*)$ i.e. $f(x) \in G^*$ and $g(x) \in H^*$.
As $G^* \cap H^* = \emptyset$, we must have $f(x) \neq g(x)$. Thus $x \in O \ \Longrightarrow x \notin A \ \Longrightarrow x \in X - A$.
Hence $O \subseteq X - A$ ________________________________ (III)
Thus for $t \in X - A$, $\exists$ $O \in \mathfrak{J}$ such that $t \in O \subseteq X - A$ (from (I), (II) and (III)).
But this shows that each point of $X - A$ is its interior point. Hence $X - A$ is an open set.
This proves that A is a closed set.

Theorem 2.7: Le-t $\langle X, \mathfrak{J} \rangle$ be any topological space and let $\langle Y, \mathfrak{J}^* \rangle$ be a T_2 – space. Let f and g be continuous mappings of X into Y. If f and g agree on a dense subset of X, then $f = g$ on the whole X.

Proof: Let D= $\{x \in X \mid f(x) = g(x)\}$ .Then D is dense subset of X .
To prove that $f(x) = g(x)$, $\forall\, x \in X$.
By Theorem 2.6, $\overline{D} = D$. D being dense in X, $\overline{D} = X$. Thus $D = X$.
Therefore, $f(x) = g(x)$ for all $x \in X$.

Theorem2.8: Let $\langle X, \mathfrak{J} \rangle$ be a T_2 – space and $f: X \to X$ be a continuous map. Then $A = \{x \in X \mid f(x) = x\}$ is a closed set in X.

Proof: $f: X \to X$ be a continuous map. Let $I: X \to X$ be the identity map. Then I is a continuous map. Hence by Theorem 2.7, the set $A = \{x \in X \mid f(x) = I(x)\}$ is closed in X.
But as $I(x) = x$ we get the set $\{x \in X \mid f(x) = x\}$ is closed set in X.

§3 T_2 – spaces and compact spaces

Theorem 3.1: Let $\langle X, \mathfrak{J}\rangle$ be a Hausdorff – space and $\langle X, \mathfrak{J}^*\rangle$ be a compact space such that $\mathfrak{J} \subseteq \mathfrak{J}^*$. Then $\mathfrak{J} = \mathfrak{J}^*$.

Proof: $\mathfrak{J} \subseteq \mathfrak{J}^*$. Hence only to prove that $\mathfrak{J}^* \subseteq \mathfrak{J}$. Let $G^* \in \mathfrak{J}^*$. $G^* \in \mathfrak{J}^*$ implies $X - G^*$ is closed in $\langle X, \mathfrak{J}^*\rangle$. Hence $X - G^*$ is compact in $\langle X, \mathfrak{J}^*\rangle$ (see Theorem …)

As $X - G^*$ is a compact subset of a T_2 - space $\langle X, \mathfrak{J}\rangle$ we get $X - G^*$ is a closed subset of $\langle X, \mathfrak{J}\rangle$ (see Theorem 3.1). Hence G^* is an open set in $\langle X, \mathfrak{J}\rangle$ Thus $G^* \in \mathfrak{J}^* \implies G^* \in \mathfrak{J}$.

Hence $\mathfrak{J}^* \subseteq \mathfrak{J}$. As $\mathfrak{J} \subseteq \mathfrak{J}^*$ and $\mathfrak{J}^* \subseteq \mathfrak{J}$ we get $\mathfrak{J} = \mathfrak{J}^*$.

Theorem 3.2: Any compact subset of a T_2 – space is closed.

Proof: Let $\langle X, \mathfrak{J}\rangle$ be a T_2 – space and let F be any compact subset of X. To prove that F is closed. Fix up any $x \in X - F$. Then for each $y \in F$ we get $x \neq y$. As X is a T_2 – space, $\exists$ disjoint open sets G_x and G_y such that $x \in G_x$ and $y \in G_y$. As this is true for any $y \in F$,

$F = \bigcup_{y \in F} \{y\} \subseteq \bigcup_{y \in F} G_y \implies \{G_y\}_{y \in F}$ forms an open cover for F. As F is compact, $F \subseteq \bigcup_{i=1}^{n} G_{y_i}$.

Find corresponding $G_{x_i} \in \mathfrak{J}$ such that such that $x \in G_{x_i}$ and $G_{x_i} \cap G_{y_i} = \emptyset$, $\forall\, i,\ 1 \le i \le n$.

Define $H = \bigcup_{i=1}^{n} G_{y_i}$ and $G = \bigcap_{i=1}^{n} G_{x_i}$. Then $G, H \in \mathfrak{J}$, $G \cap H = \emptyset$, $x \in G$ and $F \subseteq H$.

As $G \cap H = \emptyset$, we get $G \cap F = \emptyset$. But then $G \subseteq X - F$. Thus for given $x \in X - F$, $\exists$ an open set G in X such that $x \in G \subseteq X - F$. This shows that each point of $X - F$ is its interior point. Hence $X - F$ is an open set. This proves that F is a closed set.

Remark: Compact subset of any T – space need not be closed set.

For this consider an indiscrete topological space $\langle X, \mathfrak{J}\rangle$. Let A be any subset of X. Then A is compact in X, but A is not closed in $\langle X, \mathfrak{J}\rangle$.

Theorem 3.3: A T – space $\langle X, \mathfrak{J}\rangle$ is a Hausdorff space if and only if any two disjoint compact subsets of X can be separated by disjoint open sets.

Proof: Only if part –

Let $\langle X, \mathfrak{I} \rangle$ be T_2 – space and let A, B be disjoint compact sets in X.

Fix up any $a \in A$. Then for each $x \in B$ we get $a \neq x$ $(A \cap B = \emptyset)$. As X is a T_2 – space, $\exists$ G_x , $H_x \in \mathfrak{I}$ such that $a \in G_x, x \in H_x$ and $G_x \cap H_x = \emptyset$. As $B = \bigcup_{x \in B} \{x\} \subseteq \bigcup_{x \in B} H_x$ we get $\{H_x\}_{x \in B}$ form an open cover of B. B being compact, this open cover has finite sub-cover for B say $\{H_{x_i}\}_{i=1}^{n}$.

Define $H = \bigcup_{i=1}^{n} H_{x_i}$. Then $H \in \mathfrak{I}$ and $B \subseteq H$ .For the corresponding sets $G_{x_i} \in \mathfrak{I}$, define $G_a = \bigcap_{i=1}^{n} G_{x_i}$ .Then $G_a \in \mathfrak{I}$, $a \in G_a$ and $G_a \cap H = \emptyset$. As $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} G_a$, we get $\{G_a\}_{a \in A}$ forms an open cover for A. As A is compact , $A \subseteq \bigcup_{i=1}^{m} G_{a_i}$.

Define $G = \bigcup_{i=1}^{m} G_{a_i}$ .Then $G \in \mathfrak{I}$ .Further as $G_a \cap H = \emptyset \ \forall a$, we get $G \cap H = \emptyset$.

Thus for the disjoint compact sets A and B $\exists$ G , $H \in \mathfrak{I}$ such that $A \subseteq G$, $B \subseteq H$ and $G \cap H = \emptyset$.

If part –

Assume that any two disjoint compact subsets of X can be separated by disjoint open sets. To prove that X is a T_2 – space. Let $x \neq y$ in X. Then $\{x\}$ and $\{y\}$ are compact disjoint subsets of X. By assumption $\exists$ G , $H \in \mathfrak{I}$ such that $\{x\} \subseteq G$, $\{y\} \subseteq H$ and $G \cap H = \emptyset$. But this in turns shows that $\exists$ open sets G and H such that $x \in G$ and $y \in H$. Hence X is a T_2 – space.

Corollary 3.4: Let $\langle X, \mathfrak{I} \rangle$ be a T– space . $\langle X, \mathfrak{I} \rangle$ is a Hausdorff space if and only if for any compact set F and for any $x \notin F$ $\exists$ $G, H \in \mathfrak{I}$ such that $x \in G$, $F \subseteq H$ and $G \cap H = \emptyset$.

Proof :-The proof follows from Theorem 3.3 and the fact that $\{x\}$ and F are disjoint compact sets in X .

Corollary 3.5: Let $\langle X, \mathfrak{I} \rangle$ be a compact, T_2 – space. If F is closed set in X and $a \notin F$ $(a \in X)$, then there exists $G, H \in \mathfrak{I}$ such that $a \in G$, $F \subseteq H$ and $G \cap H = \emptyset$.

Proof: We know that any closed subset of a compact space is compact (see). Hence F is compact subset of a T_2 – space X and $a \notin F$. Therefore by Corollary 3.4 $\exists\ G, H \in \mathfrak{J}$ such that $a \in G$, $F \subseteq H$ and $G \cap H = \emptyset$.

Corollary 3.6: Let $\langle X, \mathfrak{J} \rangle$ be a compact, T_2 – space. Let F_1, F_2 be two disjoint closed subsets of X. Then there exists two disjoint open sets G and H in X such that $F_1 \subseteq G$ and $F_2 \subseteq H$.

Proof: Fix up any point $x \in F_2$. Then $x \notin F_1$ and F_1 is closed subset of compact T_2 – space X. By Corollary 3.4 there exist open sets G_x and H_x in X such that $x \in H_x$, $F_1 \subseteq G_x$ and $G_x \cap H_x = \emptyset$.

As $F_2 = \displaystyle\bigcup_{x \in F_2} \{x\} \subseteq \bigcup_{x \in F_2} H_x$ we get an open cover $\{H_x\}_{x \in F_2}$ for the set F_2.

As F_2 is closed subset of a compact space X, F_2 itself is a compact. Hence the open cover $\{H_x\}_{x \in F_2}$ has a finite sub-cover. Let $F_2 = \displaystyle\bigcup_{i=1}^{n} H_{x_i}$ .Find corresponding sets $G_{x_i} \in \mathfrak{J}$ such that $F_1 \subseteq G_{x_i}\ \forall\, i \quad 1 \le i \le n$, $G_{x_i} \cap H_{x_i} = \emptyset$.

Define $G = \displaystyle\bigcap_{i=1}^{n} G_{x_i}$ and $H = \displaystyle\bigcup_{i=1}^{n} H_{x_i}$ .Then $G, H \in \mathfrak{J}$, $F_1 \subseteq G$, $F_2 \subseteq H$ and $G \cap H = \emptyset$.

Hence the result.

Theorem 3.7: Every continuous mapping of a compact space into Hausdorff space is closed.

Proof: Let $\langle X, \mathfrak{J} \rangle$ be compact space and let $\langle Y, \mathfrak{J}^* \rangle$ be a Hausdorff space. Let $f: X \to Y$ be a continuous map. To show that f is a closed map. Let F be any closed set in X. To prove that $f(F)$ is closed in Y. F is closed in X and X is compact $\Longrightarrow$ F is compact (see …). $f: X \to Y$ is continuous and F is compact in X $\Longrightarrow$ $f(F)$ is compact subset (see …) of Y. As Y is a T_2 – space, $f(F)$ is closed subset of Y (see Theorem 3.2). This in turns shows that f is closed map.

Corollary 3.8: Every bijective continuous mapping of a compact space onto a Hausdorff space is a homeomorphism.

Corollary 3.9: If f is a one-one continuous mapping of the compact space X onto the T_2 – space Y. Then f is an open map and hence f is a homeomorphism.

Proof: To prove f is an open map. Let G be an open set in X. Then $F = X - G$ is closed in X. As F is closed subset of a compact space X, F is a compact subset of X (see …). Continuous image of compact space being compact (see …), $f(F)$ is compact in Y. But as Y is a T_2 – space, $f(F)$ is closed in Y (see Theorem 3.2).

Hence $Y - f(F) = Y - f(X - G)$ is an open set in X.

Now $Y - f(X - G) = f(G)$ (since f is one-one and onto $\Longrightarrow$ $f(X) = Y$)

$\Longrightarrow f(G)$ is open in Y.

This shows that f is an open mapping. As f is continuous, bijective and open, f is a homeomorphism.

Recall that a topological space $\langle X, \mathfrak{I} \rangle$ is locally compact if each point of X is contained in a compact neighbourhood .

Theorem 3.10: Let $\langle X, \mathfrak{I} \rangle$ be a locally compact, Hausdorff space. Then the one point compactification $\langle X^*, \mathfrak{I}^* \rangle$ of $\langle X, \mathfrak{I} \rangle$ is a Hausdorff space.

Proof: Let $x \neq y$ in X^*.

Case 1: $x, y \in X$.

As X is a Hausdorff space, $\exists\ G, H \in \mathfrak{I}$ such that $x \in G,\ y \in H$ and $G \cap H = \emptyset$. As $\mathfrak{I} \subseteq \mathfrak{I}^*$ we get $G, H \in \mathfrak{I}^*$ such that $x \in G,\ y \in H$ and $G \cap H = \emptyset$.

Case 2: $x = \infty, x \neq y$.

Then $y \in X$. As X is locally compact, $\exists$ a compact neighbourhood, say N of y in X.

As X is a T_2 – space, N is closed subset of X (see …). Hence $X^* - N \in \mathfrak{I}^*$ and $\infty \in X^* - N$.

Thus for $\infty \neq y$ $\exists$ disjoint open sets G and $X^* - N$ in X^* such that $y \in G,\ \infty \in X^* - N$ and $G \cap (X^* - N) = \emptyset$ … [N is nbd of y in $\langle X, \mathfrak{I} \rangle$ $\Longrightarrow \exists\ G \in \mathfrak{I}$ such that $y \in G \subseteq N$. But then $G \cap (X^* - N) = \emptyset$].

Thus from both cases $\langle X^*, \mathfrak{I}^* \rangle$ is a Hausdorff space.

Remark: If $\langle X, \mathfrak{I} \rangle$ is a locally compact, the one point compactification $\langle X^*, \mathfrak{I}^* \rangle$ of $\langle X, \mathfrak{I} \rangle$ is a compact, Hausdorff space.

§4 Convergent sequences in T_2 – spaces.

Let $\langle X, \mathfrak{J} \rangle$ be a T_2 – space. Let $\{x_n\}$ be a sequence of points of X. We say that the sequence $\{x_n\}$ converges to a point x in X if for any open set G containing x there exists N such that $x_n \in G \ \forall n \geq N$.

Remark: Convergent sequence in a topological space need not converge to unique limit.
In a co-finite topological space defined on an infinite set any sequence converges to each point of the space. But in T_2 – space convergent sequence has a unique limit.

Theorem 4.1: Let $\langle X, \mathfrak{J} \rangle$ be a T_2 – space. Any convergent sequence in X converges to a unique point in X.

Proof: Let $\{x_n\}$ be a sequence of points of a T_2 – space X and let it converge to two distinct points say x and y in X. As X is a T_2 – space, for $x \neq y$ in X, there exist $G, H \in \mathfrak{J}$ such that $x \in G$ and $y \in H$ and $G \cap H = \emptyset$. As $x_n \longrightarrow x$ and $x \in G$ there exists N_1 such that $x_n \in G \ \forall n \geq N_1$. Similarly $x_n \longrightarrow y$ and $y \in H$ there exists N_2 such that $x_n \in H \ \forall n \geq N_2$. Define $N = \max(N_1, N_2)$ then $x_N \in G \cap H = \emptyset$; a contradiction. Hence there does not exists any convergent sequence in T_2 – space, converging to two distinct points in it.

Theorem 4.2: Let $\langle X, \mathfrak{J} \rangle$ be a first axiom space. Then $\langle X, \mathfrak{J} \rangle$ is a T_2 – space if and only if every convergent sequence in X has a unique limit.

Proof: Only if part –

As X is a T_2 – space, every convergent sequence in X has a unique limit (see Theorem 4.1).
[Note that for the proof of **'Only if part'** the property that X is a F.A.S. is not used].

If part –

Let $\langle X, \mathfrak{J} \rangle$ be a first axiom space such that every convergent sequence in X has a unique limit. To prove that X is a T_2 – space.

Let if possible $\langle X, \mathfrak{J} \rangle$ is not a T_2 – space. Then $\exists \ x \neq y$ in X such that for any open sets G, H in X with $x \in G, y \in H, G \cap H \neq \emptyset$.

X is a F.A.S. $\Longrightarrow$ $\exists$ a countable decreasing local base, say $\{B_n(x)\}$ at x and $\exists$ a countable decreasing local base, say $\{B_n(y)\}$ at y .

By assumption $B_n(x) \cap B_n(y) \neq \emptyset, \ \forall n \in \mathbb{N}$. Select $x_n \in B_n(x) \cap B_n(y), \ \forall n \in \mathbb{N}$.

Consider the sequence $\{x_n\}$ in X. Claim that $x_n \to x$.
Let G be any open set in X such that $x \in G$. As $\{B_n(x)\}$ is a countable local base at x, $\exists\ N$ such that $B_N(x) \subseteq G$. But then $x_n \in G$ for $n \geq N$ ($\{B_n(x)\}$ being decreasing local base).
This shows that $x_n \to x$. Similarly, we can prove that $x_n \to y$. Thus $\exists$ a convergent sequence $\{x_n\}$ in X converging to two distinct points x and y in X ; a contradiction. Hence X must be a T_2 – space.

Remarks:

(1) The converse of the Theorem 4.1 need not be true.

i.e**. Every convergent sequence in a topological space $\langle X, \mathfrak{J}\rangle$ may converge to a unique point in X . But this need not imply that $\langle X, \mathfrak{J}\rangle$ is a T_2 – space.**

For this consider the following topological spaces .

(a) co-countable topological space $\langle X, \mathfrak{J}\rangle$ (X is uncountable set).
$\langle X, \mathfrak{J}\rangle$ is not a T_2 – space, since no two open sets in $\langle X, \mathfrak{J}\rangle$ will be disjoint (see example …) Let $\{x_n\}$ be any convergent sequence in X. Then $\{x_n\}$ converges to the unique point in X.

(b) Consider the indiscrete topological space $\langle X, \mathfrak{J}\rangle$ with $|X| \geq 2$. Then X is not a T_2 – space though each convergent sequence in $\langle X, \mathfrak{J}\rangle$ converge to unique point in X.

(2) The converse of the Theorem 4.1 is true if $\langle X, \mathfrak{J}\rangle$ is a first axiom space (see the proof of Theorem 4.2).

Exercises

(I) Let $\langle X, \mathfrak{J}\rangle$ be a compact, Hausdorff space. Show that

(1) X is not a compact with any topology larger than and different from $\mathfrak{J}$.

(2) X is not a Hausdorff space with any topology smaller than and different from $\mathfrak{J}$

(II) Show that the property of a space being T_2 – space is not preserved by continuous maps.

(III) Prove or disprove the following statements.

(1) Any compact subset of a compact space is closed.

(2) Any closed subset of a compact space is compact..

(3) Continuous mapping of the compact space into a T_2 – space is a closed mapping.

(4) Continuous mapping of the Hausdorff space into a compact space is a closed mapping.

(5) Continuous mapping of the compact space into any T – space is a closed mapping .

(6) Continuous mapping of the compact space into a compact space is a closed mapping.

(7) Continuous mapping of the compact space into a T_2 – space is a homeomorphism.

(8) Convergent sequence in a topological space converges to a unique limit.

(9) Convergent sequence in a Hausdorff topological space converges to a unique limit.

(10) If every convergent sequence in a topological space converges to a unique limit then the space is a Hausdorff space .

(IV) Let $\langle X, \mathfrak{J} \rangle$ be a T– space. Show that the following statements are equivalent.

(1) $\langle X, \mathfrak{J} \rangle$ be a T_2 – space.

(2) The intersection of all closed neighbourhoods of a point x in X is $\{x\}$.

(3) Given finite number of distinct points $x_1, x_2, \dots, x_n$ there exist neighbourhoods $N_1, N_2, \dots, N_n$ of points $x_1, x_2, \dots, x_n$ respectively, which are pair wise disjoint**.**

(V) State whether the following statements are true or false.

(1) Every discrete T – space is a T_2 – space.

(2) Every indiscrete T – space is a T_2 – space.

(3) In a T_2 – space $\overline{\{x\}} = \{x\}, \ \forall\, x \in X$.

(4) In a T_2 – space $x \neq y \implies \overline{\{x\}} \neq \overline{\{y\}}$.

(5) In a T_2 – space $\cap \{\overline{N} \mid N \text{ is a nhd.of } x\} = \{x\}$.

(6) If every convergent sequence in topological space X converges to unique limit, then X is a T_2 – space.

(7) In a T_2 – space every convergent sequence converges to a unique limit.

(8) Every f.a.s. is a T_2 – space.

(9) Every T_2 – space is a f.a.s.

(10) Every T_2 – space is compact.

(11) Every T_2 – space is a T_0 – space.

(12) Every T_0 – space is a T_2 – space.

(13) Every T_2 – space is a T_1 – space.

(14) Every T_1 – space is a T_2 – space.

(15) Every T – space is a T_2 – space.

(16) Every subspace of T_2 – space is a T_1 – space.

(17) Being a T_2 – space is a topological property.

(18) Homeomorphic image of a T_2 – space is a T – space.

(19) Continuous image of a T_2 – space is a T_1 – space.

Unit 15

Regular spaces and T_3 – spaces

Unit 15: Regular spaces and T_3 – spaces

§1 Definition and Examples.

Definition 1.1:- A topological space $\langle X, \mathfrak{I} \rangle$ is said to be regular if it satisfies the following axiom of Vietoris:

"If F is closed set in X and if p is a point of X not in F, then there exist disjoint open sets G and H such that $p \in G \ \ and \ F \ \subseteq H$".

Examples 1.2:

1) Every discrete T-space $\langle x, \mathfrak{I} \rangle$ with $|X| \geq 1$ is a regular space.

2) Let $X = \{a, b, c\} \ and \ \mathfrak{I} = \{ \emptyset \ , X \ , \{ \ a \ \}, \{ \ b \ , c \}\}$ The topological space $\langle x, \mathfrak{I} \rangle$ is a regular space.

The family of closed sets in $\langle x, \mathfrak{I} \rangle$ is $\mathcal{K} = \{ \emptyset \ , X \ , \{ \ a \ \}, \{ \ b \ , c \}\}$.

Case 1: $a \ \notin \{ \ b \ , c \ \}$. Then take $G = \{ \ a \ \}$ and $H = \{ \ b, c \}$. $G, H \in \ \mathfrak{I}$. $G \cap H = \emptyset, a \in G$ and $\{b, c\} \subseteq H$.

Case 2: $b \ \notin \{ \ a \ \}$. Take $G = \{b \ , c\}$ and $H = \{a\}$. Then $G, H \in \ \mathfrak{I}$. $G \cap H = \emptyset$, $b \in \{b, c \ \} \subseteq G$ and $\{a\} \subseteq F$.

Case 3:- $c \ \notin \{ \ a \ \}$. Take $G = \{ \ b \ , c \ \} \ and \ H = \{ \ a \ \}. Then \ G, H \in \ \mathfrak{I}$. $c \ \in G \ , \{ \ a \ \} \ \subseteq \ H$ and $G \cap H = \emptyset$.

Thus given a closed set F and a point $p \ \notin F$ there exist disjoint open sets one containing p and the other containing F.

This shows that the T-space $\langle X, \mathfrak{I} \rangle$ is a regular space.

3) Every metric space is a regular space.

Let $\langle X, d \rangle$ be a metric space and $\mathfrak{I}$ denote the topology induced by d on X. Let F be any closed set and $p \ \notin F$, $(\ p \in X \ )$. As $p \ \notin \ \bar{F} = F \ , d(\ p \ , F) \neq 0$. Let $r = d(\ p \ , F)$.

Define $G_1 = S\big(\ p \ , \ {}^{r}/_{2}\big)$ and $G_2 = \displaystyle\bigcup_{y \in F} \mathrm{S} \left(y \ , \ {}^{\mathrm{r}}/_{4}\right)$.

Then $G_1, G_2 \in \mathfrak{J}$, $p \in G_1$ and $F \subseteq G_2$ only to prove that $G_1 \cap G_2 = \emptyset$.

Let $\mathfrak{z} \in G_1 \cap G_2$. Then $d(p, \mathfrak{z}) < {}^{r}/_{4}$ for some $x \in F$

$$d(p, x) \leq d(p, \mathfrak{z}) + d(\mathfrak{z}, x)$$
$$< {}^{r}/_{4} + {}^{r}/_{4} = {}^{r}/_{2}$$

Hence $d(p, x) < r \implies x \in S(p, r)$

But then $x \in S(p, r) \cap F = \emptyset$ (Since $r = d(p, F)$) ; a contradiction.

Hence, $G_1 \cap G_2 = \emptyset$. Thus given $p \notin F$, F is a closed set in $\langle X, d\rangle \implies \exists$ disjoint open sets one containing p and other containing F. Hence, the metric space $\langle X, d\rangle$ is a regular space.

4) $\langle \mathbb{R}, \mathfrak{J}_u\rangle$ is a regular space.

Proof:- Let $d(x, y) = |x - y|$ and $S(x, r) = (x - r, x + r)$.

Then the topology $\mathfrak{J}_u$ is induced by the metric d on R and hence by Example 3, $\langle \mathbb{R}, \mathfrak{J}_u\rangle$ is a regular space.

5) Let $X = \mathbb{R}$ and Let $\mathfrak{J}$ denote the topology on $\mathbb{R}$ having the open intervals and the set Q of rational Numbers as a sub-basis.

Proof:- Define $F = \mathbb{R} - Q$. Then F is closed set in $\langle \mathbb{R}, \mathfrak{J}\rangle$. $1 \notin F$ and there does not exist only two disjoint sets one containing 1 and other containing F. Hence $\langle \mathbb{R}, \mathfrak{J}\rangle$is not a regular space.

§2 Characterizations and Properties of regular spaces

Theorem 2.1: A topological space $\langle X, \mathfrak{J}\rangle$ is regular if and only if for any point $x \in X$ and any open set G containing x, there exists an open set H such that $x \in H$ and $\overline{H} \subseteq G$ $(or\ x \subseteq H \subseteq \overline{H} \subseteq G)$.

Proof:- Only if part

Let X be a regular space and $x \in G$, G is an open set in X $\implies X - G$ is a closed set and $x \notin X - G$. As X is a regular space $\exists$ $H, K \in \mathfrak{J}$ such that $x \in H$, $X - G \subseteq K$ and $H \cap K = \emptyset$. Now $X - G \subseteq K \implies G \supseteq X - K$.

$H \cap K = \emptyset \implies H \subseteq X - K \implies \overline{H} \subseteq \overline{X - K} = X - K$ (Since $X - K$ is a closed set).

Thus $x \in H \subseteq X - K \subseteq G$. Thus $\exists$ an open set H such that $x \in H$ and $\overline{H} \subseteq G$

If Part

Let F be a closed set in X and Let $x \notin F$. Then $X - F$ is an open set and $x \in X - F$. By assumption, $\exists$ an open set H such that $x \in H$ *and* $\overline{H} \subseteq X - F$.

Define $K = X - \overline{H}$ .Then $K \in \mathfrak{J}$ and $H \subseteq \overline{H}$

$$\Longrightarrow \quad H \cap K = H \cap (X - \overline{H}) = \emptyset \quad (\because \; X - \overline{H} \subseteq X - H).$$

Thus for $x \notin F$, $\exists$ disjoint open sets H and K such that $x \in H$ *and* $F \subseteq K$. Hence $\langle X, \mathfrak{J} \rangle$ is a regular space.

Theorem 2.2: Let $\langle X, \mathfrak{J} \rangle$ be a topological space. The following statements equivalent.

1) X is a regular space.
2) For any sub-basic open set G containing a point x, there exists an open set H such that $x \in H$ and $\overline{H} \subseteq G$.

Proof: 1) $\Longrightarrow$ 2)

G is a sub-basic open set $\Longrightarrow G \in \mathfrak{J}$. Hence, by Theorem 2.1, the implication follows.

2) $\Longrightarrow$ 1)

Let G be any open set and $x \in G$. By the definition of the sub-base there exist members of $\mathfrak{J}$ say $U_1, U_2, \dots \dots, U_n$ of the sub base such that

$$x \in \bigcap_{i=1}^{n} U_i \subseteq G$$

As $U_i \in \mathfrak{J} \; \forall \; i$, $1 \leq i \leq n$ and $x \in U_i$, by Theorem 2.1 , $\exists$ an open set H_i such that $x \in H_i$ and $\overline{H_i} \subseteq U_i$ for each i , $1 \leq i \leq n$.

Thus we get $\quad x \in \bigcap_{i=1}^{n} U_i \subseteq \bigcap_{i=1}^{n} \overline{H_i} \subseteq \bigcap_{i=1}^{n} U_i \subseteq G$

Define $H = \bigcap_{i=1}^{n} H_i$ .Then $x \in H$ and $\overline{H} \subseteq G$ ... $\left(\because \; \overline{H} = \overline{\left(\bigcap_{i=1}^{n} H_i \right)} \subseteq \bigcap_{i=1}^{n} \overline{H_i} \right)$

Hence, by the Theorem 2.1, X is a regular space.

Theorem 2.3: Let $\langle X, \mathfrak{J} \rangle$ be a regular space. The following statements are equivalent:

1) X is a regular space.
2) For each $x \in X$ and a nbd. U of x , $\exists$ a nbd. V of x such that $\bar{V} \subseteq U$.
3) For each $x \in X$ and a closed set F not containing x, there exists a nbd. V of x such that

$\bar{V} \cap F = \emptyset$.

Proof: 1) $\Longrightarrow$ 2)

Let U be a nbd.of $x \in X$. Hence $\exists$ an open set G such that $x \in H \subseteq \bar{H} \subseteq G$.

Thus $\exists$ a nbd. H of x such that $\bar{H} \subseteq U$.

2) $\Longrightarrow$ 3)

Let F be a closed set such that $x \notin F$. But then $X - F$ is a nbd of x. By (2), there exists a nbd. V of x such that $\bar{V} \subseteq X - F$. Hence $\exists$ a nbd. V of x such that $\bar{V} \cap F = \emptyset$.

3) $\Longrightarrow$ 1)

Let F be a closed set and let $x \notin F$. By (3), $\exists$ a nbd. V of x such that $\bar{V} \cap F = \emptyset$.

$\bar{V} \cap F = \emptyset \Longrightarrow F \subseteq X - \bar{V}$.

V is a nbd of $x \Longrightarrow \exists$ an open set H such that $x \in H \subseteq V$.

Define $K = X - \bar{V}$. Then $K \in \mathfrak{I}$ and $H \cap K = \emptyset$.

Thus $\exists$ $H, K \in \mathfrak{I}$ such that $x \in H$, $F \subseteq K$ and $H \cap K = \emptyset$.

Hence, $\langle X, \mathfrak{I} \rangle$ is a regular space.

Theorem 2.4: A topological space $\langle X, \mathfrak{I} \rangle$ is a regular space if and only if the family of closed nbds of any point of X forms a local base at that point.

Proof:- Only if part

Let $\langle X, \mathfrak{I} \rangle$ be a regular space. Let N be any nbd of a point x, Hence $\exists$ $G \in \mathfrak{I}$ such that $x \in G \subseteq N$. As X is a regular space, $\exists$ an open set H in X such that $x \in H$ and $\bar{H} \subseteq G$ (By Theorem 2.1). But this shows that $x \in \bar{H} \subseteq N$. Hence the family of closed nbds of x forms a local base at x.

If part

Let the family of closed nbds of any point of X forms a local base at that point. To prove that X is a regular space. Let $x \in X$ and let F be a closed set such that $x \notin F$. Then $X - F$ is a nbd of x. By assumption $\exists$ a closed nbd. K of x such that $x \in K \subseteq X - F$.

Define $G =$ interior of K and $H = X - K$. Then $G, H \in \mathfrak{I}$, $x \in G$, $F \subseteq H$ and $G \cap H = \emptyset$.

Hence $\langle X, \mathfrak{I} \rangle$ is a regular space.

Theorem 2.5: Being a regular space is a hereditary property.

Proof:- Let $\langle X, \mathfrak{I} \rangle$ be a regular space and let $\langle Y, \mathfrak{I}^* \rangle$ be its subspace.

Then $\mathfrak{I}^* = \{G \cap Y \mid G \in \mathfrak{I}\}$ and $Y \subseteq X$.

Let F^* be any closed set in $\langle Y, \mathfrak{I}^* \rangle$ and $Y \notin F^*$ ($y \in Y$).

F^* is a closed set in $\langle Y, \mathfrak{I}^* \rangle \implies F^* = F \cap Y$ for some closed set F in $\langle X, \mathfrak{I} \rangle$.

As $\langle X, \mathfrak{I} \rangle$ is a regular space, $\exists\ H$, $K \in \mathfrak{I}$ such that $y \in H$, $F \subseteq K$ and $H \cap K = \emptyset$.

Define $H^* = H \cap Y$ and $K^* = K \cap Y$.

Then H^* , $K^* \in \mathfrak{I}^*$, $y \in H^*$, $F^* \subseteq K^*$ and $H^* \cap K^* = \emptyset$.

This shows that $\langle Y, \mathfrak{I}^* \rangle$ is a regular space.

As any sub-space of a regular space is a regular space, the result follows.

Theorem 2.6: Being a regular space is a topological property.

Proof:- Let $\langle X, \mathfrak{I} \rangle$ be a regular space. let $\langle Y, \mathfrak{I}^* \rangle$ be any T-space and $f : X \to Y$ be homeomorphism.

To prove $\langle Y, \mathfrak{I}^* \rangle$ is a regular space. Let F^* be a closed set in Y and $y \notin F^*$ $(y \in Y)$.

As F is onto, $\exists\ x \in X$ such that $y = f(x)$. F^* is a closed set and $f : X \to Y$ is continuous

$\implies f^{-1}[F^*]$ is a closed set in X.

As $y \notin F^*$, $x \notin f^{-1}[F^*]$. $\langle X, \mathfrak{I} \rangle$ is a regular space. Hence there exist $G, H \in \mathfrak{I}$ such that $x \in G$, $f^{-1}[F^*] \subseteq H$ and $G \cap H = \emptyset$.

Thus $f(x) \in f(G)$, $f\big[f^{-1}[F^*]\big] \subseteq f(H)$ and $f(G) \cap f(H) = \emptyset$.

Define $G^* = f(G)$ and $H^* = f(H)$.

Then G^*, $H^* \in \mathfrak{I}^*$, $y \in G^*$ and $F^* \subseteq H^*$ and $G^* \cap H^* = \emptyset$.

This shows that $\langle Y, \mathfrak{I}^* \rangle$ is a regular space. As homeomorphic image of a regular space is a regular space, the result follows.

Theorem 2.7: Let A be a compact subset of a regular space $\langle X, \mathfrak{I} \rangle$. For any open set G containing A, there exists a closed set F such that $A \subseteq F \subseteq G$.

Proof: We know that**,** $a \in A \implies a \in G$ and G is open in X .

As X is a regular space, by Theorem 2.1, $\exists$ an open set G_a such that $a \in G_a \subseteq \overline{G_a} \subseteq G$.

Thus $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} G_a$ shows that $\{G_a\}_{a \in A}$ forms an open cover for a compact set A .

Hence $A \subseteq \bigcup_{i=1}^{n} G_{a_i}$. Now $\overline{G_{a_i}} \subseteq G \;\; \forall\, i\,,\; 1 \le i \le n \;\; \Longrightarrow \bigcup_{i=1}^{n} G_{a_i} \subseteq G$.

Define $F = \bigcup_{i=1}^{n} \overline{G_{a_i}}$. Then F is a closed set such that $A \subseteq F \subseteq G$.

Hence the proof.

Theorem 2.8 : Let $\langle X, \Im \rangle$ be a regular space. Let A and B be disjoint subsets of X such that A is closed and B is compact in X. Then $\exists$ disjoint open sets in X one containing A and the other containing B.

Proof:- $A \cap B = \emptyset \;\; \Longrightarrow \;\; b \notin A$ for any $b \in B$.

As X is a regular space, $\exists$ disjoint open sets G_b and H_b in X such that $b \in G_b$ and $A \subseteq H_b$ for each $b \in B$. Thus $B = \bigcup_{b \in B} \{b\} \subseteq \bigcup_{b \in B} G_b \;\; \Longrightarrow \;\; \{G_b\}_{b \in B}$ forms an open cover for B.

B being compact, $B \subseteq \bigcup_{i=1}^{n} G_{b_i}$. Define $G = \bigcup_{i=1}^{n} G_{b_i}$. Then $G \in \Im$.

Find corresponding $H_{b_i}\ \forall\, i$. Then $A \subseteq \bigcap_{i=1}^{n} H_{b_i}$. Define $H = \bigcap_{i=1}^{n} H_{b_i}$

Then $H \in \Im$ and $A \subseteq H$.

Thus $\exists$ open sets G and H in X such that $B \subseteq G$, $A \subseteq H$ and $G \cap H = \emptyset$.

Theorem 2.9: Closure of a compact subset of a regular space is compact.

Proof:- Let $\langle X, \Im \rangle$ be a regular space and let A be a compact subset of X. Let $\{G_\propto\}$ be any cover for $\bar{A}$. Then $\{G_\propto\}$ is also an open cover for A (Since $A \subseteq \bar{A}$) . As A is compact,

$A \subseteq \bigcup_{i=1}^{n} G_{\propto_i}$. Define $G = \bigcup_{i=1}^{n} G_{\propto_i}$. Then $G \in \Im$ and $A \subseteq G$. By Theorem 2.7, $\exists$ a closed set F such that $A \subseteq F \subseteq G$. As $\bar{A} \subseteq \bar{F} = F$. We get $\bar{A} \subseteq G$ $i.e.$ $\bar{A} \subseteq \bigcup_{i=1}^{n} G_{\propto_i}$.

But this implies that the open cover $\{G_\propto\}$ of A has a finite sub cover. Hence $\bar{A}$ is compact.

§3 Definition and Examples of T_3 spaces.

Definition 3.1: Every regular, T_1 – space is said to be T_3 – space.

Examples 3. 2:

1) Every discrete T – space $\langle x, \mathfrak{I} \rangle$ with $|X| \geq 1$ is a regular, T_1 – space and hence a T_3 – space.

2) Let $X = \{a, b, c\}$ *and* $\mathfrak{I} = \{ \emptyset , X , \{ a \}, \{ b , c\}\}$. The T-space $\langle X, \mathfrak{I} \rangle$ is a regular space, but not a T_1 – space**.** Hence this space is not a T_3 – space.

3) Every metric space is a T_3 – space.

Metric space $\langle X, d \rangle$ is a regular space (see Example 1.2 (3)). We also know that any metric space $\langle X, d \rangle$ is a T_1 – space (see Unit 13). Hence, any metric space is a T_3 – space.

4) $\langle \mathbb{R}, \mathfrak{I}_u \rangle$ is a T_3 – space.

5) Let $X = \mathbb{R}$ and let $\mathfrak{I}$ denote the topology on $\mathbb{R}$ having the open intervals and the set Q of rational numbers as a sub-basis.

$\langle \mathbb{R} , \mathfrak{I} \rangle$ is a T_1 – space but not a regular space and hence $\langle \mathbb{R} , \mathfrak{I} \rangle$ is not a T_3 space.

Remarks:-

(1) $\mathfrak{I}_u < \mathfrak{I}$ and $\langle \mathbb{R} , \mathfrak{I}_u \rangle$ is a T_2 – space will imply $\langle \mathbb{R} , \mathfrak{I} \rangle$ is also a T_2 – space.

(See Unit 14).

(2) This example **5)** shows that

1) $\langle \mathbb{R} , \mathfrak{I} \rangle$ is a $T_1 - space$ but not a regular space.

2) $\mathfrak{I}_u < \mathfrak{I}$ and $\langle \mathbb{R}, \mathfrak{I}_u \rangle$ is a regular space but $\langle \mathbb{R} , \mathfrak{I} \rangle$ is not a regular space.

Hence a topology finer than a regular topology on X need not be a regular topology on X.

§4 Properties of T_3 spaces

Theorem 4.1: Every T_3 – space is a T_2 – space.

Proof:- Let $\langle X, \mathfrak{I} \rangle$ be a T_3 – space. Let $x \neq y$ in X . As X is a T_1 – space $\{y\}$ is a closed set in X . As $x \neq y$ we get $x \notin \{y\}$. As X is regular space, $\exists$ an open set such that $x \in G$, $F \subseteq H$ and $G \cap H = \emptyset$. But this shows that $\langle X, \mathfrak{I} \rangle$ is a T_2 – space.

Theorem 4.2: Being a T_3 – space is a hereditary property.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a T_3 – space and let $\langle Y, \mathfrak{J}^* \rangle$ be its subspace.

$\langle X, \mathfrak{J} \rangle$ be a regular space $\Rightarrow$ $\langle Y, \mathfrak{J}^* \rangle$ is a regular space (By Theorem 2.5).

$\langle X, \mathfrak{J} \rangle$ is a T_1 – space $\Rightarrow$ $\langle Y, \mathfrak{J}^* \rangle$ is a T_1 – space (see Unit 13).

Hence, $\langle Y, \mathfrak{J}^* \rangle$ is T_3 – space.

Theorem 4.3: Being a T_3 – space is topological property.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a T_3 – space. Let $\langle Y, \mathfrak{J}^* \rangle$ be any T-space and $f : X \rightarrow Y$ be a homeomorphism.

$\langle X, \mathfrak{J} \rangle$ be a regular space $\Rightarrow$ $\langle Y, \mathfrak{J}^* \rangle$ is a regular space (By Theorem 3) .

$\langle X, \mathfrak{J} \rangle$ is a T_1 – space $\Rightarrow$ $\langle Y, \mathfrak{J}^* \rangle$ is a T_1 – space (See Unit 13).

Hence $\langle Y, \mathfrak{J}^* \rangle$ is T_3 – space. As homeomorphic image of a T_3 – space is a T_3 – space, the result follows.

Theorem 4.4: Every regular, T_0 – space is a T_3 – space.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a regular T_0 – space. Let $x \neq y$ in X.

As X is a T_0 – space $\overline{\{x\}} \neq \overline{\{y\}}$ (See Theorem 2.1 Unit 12).

Let $\exists$ $\mathfrak{z} \in \overline{\{x\}}$ such that $\mathfrak{z} \notin \overline{\{y\}}$.

Claim: $x \notin \overline{\{y\}}$.

Let $x \in \overline{\{y\}}$ such that $\{x\} \subseteq \overline{\{y\}}$

$\Rightarrow$ $\overline{\{x\}} \subseteq \overline{\overline{\{y\}}} = \overline{\{y\}}$

$\Rightarrow$ $z \in \overline{\{y\}}$; a contradiction.

Hence $x \notin \overline{\{y\}}$.

As $\langle X, \mathfrak{J} \rangle$ is a regular space, $\exists$ $G, H \in \mathfrak{J}$ such that $x \in G$, $\overline{\{y\}} \subseteq H$ and $G \cap H = \emptyset$.

But then for $x \neq y$, $\exists$ $G, H \in \mathfrak{J}$ such that $x \in G, y \in H$ and $G \cap H = \emptyset$.

This shows that X is a T_2 – space. As every T_2 – space is a T_1 – space, we get $\langle X, \mathfrak{J} \rangle$ is a T_3 – space.

Theorem 4.5: A compact, T_2 – space is a regular space and hence T_3 – space.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a compact space, T_2 – space. Let F be a closed set and $x \notin F$.Then F is a compact subset of X (see Unit 7). Hence$\exists$ an open set such that $x \in G, F \subseteq H$ and $G \cap H = \emptyset$ (See Unit 14).

Hence $\langle X, \mathfrak{J} \rangle$ is a regular space. As every T_2 – space is a T_1 – space, $\langle X, \mathfrak{J} \rangle$ is a T_3 – space.

§5 Solved problems

Problem 1: Give an example of T_2 – space which is not a T_3 – space.

Solution: Consider a topology $\mathfrak{J}$ on $\mathbb{R}$ defined as follows.

The $\mathfrak{J}$ nbhds of any non-zero point in $\mathbb{R}$ are as in usual topology for $\mathbb{R}$ but $\mathfrak{J}$ nbhds of 0 have the form $\mathbb{N} - A$ where A is nbhd of 0 in the usual topology and

$$A = \left\{ \frac{1}{n} : n = 1, 2, \dots, n \right\}.$$

Then $\mathbb{R}$ with this topology is a Hausdorff space since this topology on $\mathbb{R}$ is finer than the usual topology which is Hausdorff.

But A is $\mathfrak{J}$ - closed and cannot be separated from 0 by disjoint open sets, and so $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a T_3 – space (by definition).

Problem 2: A T_2 – space need not be regular.

Solution: Let $\mathfrak{J}_u$ denote the usual topology on $\mathbb{R}$.

Let $\mathfrak{J}$ denotes the smallest topology on $\mathbb{R}$ containing $\mathfrak{J}_u \cup \{\mathbb{R} - A\}$ where $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

Then $\mathfrak{J}_u \leq \mathfrak{J}$

I) Claim: $\langle \mathbb{R}, \mathfrak{J} \rangle$ is a T_2 – space.

$\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a T_2 – space $\Longrightarrow$ $\langle \mathbb{R}, \mathfrak{J} \rangle$ is a T_2 – space (see Unit 14).

II) Claim: $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a regular space.

A is a closed set in $\langle \mathbb{R}, \mathfrak{J} \rangle$ (since $\mathbb{R} - A \in \mathfrak{J}$) and $0 \notin A$. As $\nexists$ any open set containing 0 and disjoint with A (as any open set in $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ contains a member of A), $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a regular space.

Hence, $\langle \mathbb{R}, \mathfrak{J} \rangle$ is a T_2 - space but $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a regular space.

Exercises

Prove or disprove the following statements.

1) Every T_2 – space is a regular space.
2) Every T_2 – space is a T_3- space.
3) Homeomorphic image of a regular space is a regular space.
4) Sub-space of a regular space is a regular space.
5) A compact T_2 – space is a regular space.
6) A compact, T_2 – space is a T_3 – space
7) Every regular, T_0 – space is a T_3 – space.
8) $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a T_3 – space.
9) Being a T_3 space is a hereditary property.
10) Co-finite topological space is a regular space.

Unit 16

Normal spaces and T_4 – spaces

§1 Definition and Examples of normal spaces.

§2 Characterizations and properties of normal spaces.

§3 Definition and Properties of T_4 spaces.

§4 Solved examples.

Unit 16: *Normal spaces and T_4 – spaces*

§1 Definition and Examples of normal spaces

Definition 1.1: A topological space $\langle X, \mathfrak{J}\rangle$ is said to be normal if for any two disjoint closed sets F_1 and F_2 in X, $\exists$ two disjoint open sets G_1 and G_2 in X such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.

Example 1.2:

Normal spaces.

1) $\langle \mathbb{R}, \mathfrak{J}_u\rangle$ is a normal space.

Let F_1 and F_2 be any two disjoint closed sets in $\langle \mathbb{R}, \mathfrak{J}_u\rangle$.

$F_1 \cap F_2 = \emptyset \implies F_1 \subseteq X - F_2$. Thus for each $x \in F_1$, $\exists\ r > 0$ such that $(x - r\ ,\ x + r) \subseteq X - F_2$ (since $X - F_2$ is an open set in X).

Hence $\left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right) \cap F_2 = \emptyset$.

Let $G = \bigcup_{x \in F1} \left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right)$.

Then $G \in \mathfrak{J}_u$ and $F_1 \subseteq G$ __________________________ (1)

Similarly for each $x \in F_2$, $\exists\ r > 0$ such that $\left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right) \subseteq X - F_1$.

Hence $\left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right) \cap F_1 = \emptyset$. Let $H = \bigcup_{x \in F2} \left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right)$.

Then $H \in \mathfrak{J}$ and $F_2 \subseteq H$ ____________________________ (2)

Only to prove that $G \cap H = \emptyset$.

Let $a \in G \cap H$. $a \in$ G $\implies a \in \left(x - \frac{r}{2}\ ,\ x + \frac{r}{2}\right)$, for some $x \in F_1$.

$a \in H \implies a \in \left(y - \frac{\epsilon}{2}\ , y + \frac{\epsilon}{2}\right)$ For some $y \in F_2$ but then

$|x - a| < \frac{r}{2}$ and $|y - a| < \frac{\epsilon}{2}$.

Hence $|x - y| = |x - a + a - y| \leq |x - a| + |y - a|$

As r and ϵ are real numbers, they are comparable.

Case 1: Let $r \leq \epsilon$.

Then $|x - y| < \epsilon$. Hence $x \in (y - \epsilon, y + \epsilon)$ By the choice of ϵ, $(y - \epsilon, y + \epsilon) \subseteq X - F_1$. Hence $x \in (y - \epsilon, y + \epsilon) \Rightarrow x \in X - F_1 \Rightarrow x \notin F_1$; which is contradiction.

Case 2. Let $\epsilon \leq r$.

Then $|x - y| < r$. Hence $y \in (x - r, x + r)$.

By the choice of r, $(x - r, x + r) \subseteq X - F_2$, will imply $y \in X - F_2$ i.e. $y \notin F_2$; which is contradiction.

Hence $G \cap H = \emptyset$.

Thus given two disjoint closed sets F_1 and F_2 in $\mathbb{R}$, $\exists$ two disjoint open sets G and H such that $F_1 \subseteq G$ and $F_2 \subseteq H$. Hence $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a normal space.

2) Let $X = \{a, b, c\}$ and $\mathfrak{J} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ Then $\langle X, \mathfrak{J} \rangle$ is a topological space.

The Family of closed sets $\mathcal{K}$ is given by $\mathcal{K} = \{\emptyset, X, \{b, c\}, \{c, a\}, \{c\}\}$

Each pair of disjoint closed sets contains $\emptyset$. Hence the space is $\langle X, \mathfrak{J} \rangle$ a normal space.

Non – normal spaces.

1) Let X$= \{(x, y) \mid x, y \in \mathbb{R}, y > 0\}$.*For each* $(p, a) \in X$ *define*

$$N_\in(p, a) = \left\{(x, y) \mid \sqrt{(x - p)^2 + (y - a)^2} < \epsilon \text{ where } \epsilon < q, if\ a > 0\right\}$$

$$N_\in(p, 0) = \left\{(x, y) \mid \sqrt{(x - p)^2 + (y - a)^2} < \epsilon \text{ where } \epsilon > 0\right\}$$

Define for each $(p, a) \in X$,

$$N_\in(p, a) = \left\{(x, y) \in X \mid \sqrt{(x - p)^2 + (y - a)^2} < \epsilon\right\}, \ if\ q > 0$$

$$N_\in(p, a) = \left\{(x, y) \in X \mid \sqrt{(x - p)^2 + (y - a)^2} < \epsilon\right\} \cup \{(p, 0)\}$$

Define

$$N(p, q) = \{N_\in(p, a) \mid \epsilon < q \text{ and } q > 0\} \cup \{N_\in(p, 0) \mid \epsilon > 0\}$$

Define $\mathcal{B} = \{N(p, q) / (p, a) \in X\}$. Then $\mathcal{B}$ forms a base for some topology $\mathfrak{J}$ on X.

Let $F = \{(p, 0) \mid p \text{ is a rational}\}$ and $K = \{(p, 0) \mid p \text{ is a irrational}\}$.

Then F and K are disjoint closed sets in X.As there do not exist two open sets G and H such that $F \subseteq G$, $K \subseteq H$ and $G \cap H = \emptyset$, we get $\langle X, \mathfrak{J} \rangle$ is not a normal space.

§2 Characterizations and properties of normal spaces

Theorem 2.1: A topological space X is normal if and only if for any closed set F and an open set G containing F, there exists an open set H such that F $\subseteq H \subseteq \overline{H} \subseteq G$.

Proof: Only if part

Let X be a normal space. Let F be a closed set and G be an open set such that F $\subseteq G$. Then X $- G$ is a closed set and F $\cap (X - G) = \emptyset$. As X is normal, $\exists$ open sets H and K in X such that F $\subseteq H$, $X - G \subseteq K$ and $H \cap K = \emptyset$.

$H \cap K = \emptyset \implies H \subseteq X - K$ and $X - G \subseteq K \implies X - K \subseteq G$.

Hence $H \subseteq X - K \subseteq G$.

Therefore $H \subseteq X - K$ and $X - K$ is a closed set $\implies \overline{H} \subseteq \overline{X - K} = X - K$.

Hence H $\subseteq G$ or $X - K \subseteq G$. Thus $\exists$ open sets H in X such that

F $\subseteq H \subseteq \overline{H} \subseteq G$.

If part

Assume that for any closed set F and an open set G containing F, there exists an open set H such that F $\subseteq H \subseteq \overline{H} \subseteq G$.

To prove that $\langle X_2 , \mathfrak{J}_2 \rangle$ is normal. Let F and K be disjoint closed sets in X.

$F \cap K = \emptyset \implies F \subseteq X - K$.

As F is a closed set and $X - K$ is an open set , by assumption $\exists$ open sets H in X such that,

F $\subseteq H \subseteq \overline{H} \subseteq X - K$. We get $K \subseteq X - \overline{H}$.

Define $G = X - \overline{H}$. Thus $\exists$ $G, H \in \mathfrak{J}$ such that $F \subseteq H, K \subseteq G$ and $H \cap G = \emptyset$. But this shows that X is a normal space.

Theorem 2.2: A topological space $\langle X, \mathfrak{J} \rangle$ is normal if and only if for any closed set F and an open set G containing F, there exist an open set H and a closed set K such that $F \subseteq H \subseteq K \subseteq G$.

Proof: Only if part

Let $\langle X, \mathfrak{J} \rangle$ be a normal space. Let F be a closed set and G be an open set in X such that $F \subseteq G$. By Theorem 2.1, there exists an open set H such that

$F \subseteq H \subseteq \overline{H} \subseteq G$. Define $K = \overline{H}$. Then K is closed set in X. Thus $\exists$ open sets H and a closed set K such that $F \subseteq H \subseteq K \subseteq G$

If part

Assume that for any closed set F and an open set G containing F, there exists an open set H and a closed set K such that $F \subseteq H \subseteq K \subseteq G$.

To prove that $\langle x, \mathfrak{J} \rangle$ be a normal space. Let F_1 and F_2 be any two disjoint closed sets in X.

Then $F_1 \cap F_2 = \emptyset \implies F_1 \subseteq X - F_2$

As F_1 is a closed and $X - F_2$ is an open set, by assumption, $\exists$ open set H and a closed set K such that $F_1 \subseteq H \subseteq K \subseteq X - F_2$. Then $F_1 \subseteq H$ and $F_2 \subseteq X - K$.

Define $G = X - K$. Then $G, H \in \mathfrak{J}$ such that $F_1 \subseteq H$ and $F_2 \subseteq G$ and $G \cap H = \emptyset$. This shows that $\langle X, \mathfrak{J} \rangle$ is a normal space.

Theorem 2.3: Urysohn's Lemma

A topological space $\langle X, \mathfrak{J} \rangle$ is normal if and only if for every two disjoint closed sets F_1 and F_2 of X and closed interval $[a, b]$, there exists a continuous mapping $f : X \to [a, b]$ such that $f[F_1] = \{a\}$ and $f[F_2] = \{b\}$.

Proof: Only if part:

Let $\langle X, \mathfrak{J} \rangle$ be a normal space and F_1, F_2 be two disjoint closed sets in X.

I] First we prove that $\exists$ a continuous function $f : X \to [0, 1]$ such that $f[F_1] = \{0\}$ and $f[F_2] = \{1\}$. $F_1 \cap F_2 = \emptyset \implies F_1 \subseteq X - F_2$

As F_2 is closed in X , $X - F_2$ is open. Hence by Theorem 2.1, $\exists$ an open set $G_{1/2}$ in X such that

$$F_1 \subseteq G_{1/2} \subseteq \overline{G_{1/2}} \subseteq X - F_2 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (1)$$

Again $F_1 \subseteq G_{1/2}$, F_1 is closed and $G_{1/2}$ is open and X is normal will imply, $\exists$ an open set say $G_{1/4}$ such that ,

$$F_1 \subseteq G_{1/4} \subseteq \overline{G_{1/4}} \subseteq G_{1/2} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (2)$$

Further $\overline{G_{1/2}} \subseteq X - F_2$, $\overline{G_{1/2}}$ is closed and $X - F_2$ is open and X is normal will imply $\exists$ an open set say $G_{3/4}$ such that

$$\overline{G_{1/2}} \subseteq G_{3/4} \subseteq \overline{G_{3/4}} \subseteq X - F_2 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (3)$$

From (1), (2) and (3) we get,

$F_1 \subseteq \; G_{1/4} \subseteq \overline{G_{1/4}} \subseteq \; G_{1/2} \; \subseteq \; \overline{G_{1/2}} \; \subseteq \; G_{3/4} \subseteq \overline{G_{3/4}} \subseteq X - F_2$

Continue this process.

Define $D = \left\{\frac{m}{2^n} \mid m = 1, 2, \ldots;\; n = 1, 2, \ldots\right\}$. Then D is a countable set and

for $r, s \in D$ with $r < s$ we get,

$F \subseteq \; G_r \subseteq \overline{G_r} \subseteq \; G_s \; \subseteq \; \overline{G_s} \; \subseteq X - F_2$

Define $f : X \to [0, 1]$ by

$$\begin{aligned} f(x) &= 1 \quad \text{if } x \in F_2 \\ &= inf\ \{r \in D \,/\, x \in \; G_r\}\ \ if\ x \notin F_2 \end{aligned}$$

Obviously, $f\,[F_2] = \{1\}$.

If $x \in F_1$, then $f(x) = inf\ \{r \,/\, r \in D\} = 0$ (Since $x \in \; G_r, \forall\ r \in D$ in this case)

Hence $f\,[F_1] = \{0\}$.

To prove that f is continuous.

Let $(c, d) \subseteq [0,1]$. To prove that $f^{-1}\big((c, d)\big) \in s$.

$$\begin{aligned} \text{Now} \quad x \in f^{-1}\big((c, d)\big) \; &\Rightarrow \; f(x) \in (c, d) \\ &\Rightarrow \; c < f(x) < d \\ &\Rightarrow \; 0 < c < f(x) < d < 1 \end{aligned}$$

Find $p, q \in D$ such that, $0 < c < p < f(x) < q < d < 1$

Now $p \in D$ and $p < f(x) \;\; \Rightarrow \;\; x \notin \overline{G_p}$.

Further $q \in D$ and $f(x) < q \;\; \Rightarrow \;\; x \in \; G_q$

Thus when $p < x < q$, $x \in G_q \; - \; \overline{G_p}$. This shows that $f^{-1}[(c, d)] \subseteq G_q - \overline{G_p}$.

Similarly if $x \in G_q$, $x \in \overline{G_p}$, then $x \in G_q \cap \overline{G_p}$.

Therefore, $b < f(x) < q \;\; \Rightarrow \;\; f(x) \in (c, d)$.

Hence $G_q \cap \overline{G_p} \;\subseteq\; f^{-1}[(c, d)]$.

Combining both the inclusions we get $f^{-1}\,[(c, d)] = \; G_q \; \cap \overline{G_p}$.

As $G_q - \overline{G_p} = G_q \cap (X - \overline{G_p})$ is an open set in X .

But this in turn shows that f is a continuous map. Thus for disjoint closed sets F_1 and F_2 in X, $\exists$ a continuous function $f : X \to [0, 1]$ such that $f\,[F_1] = \{0\}$ and $f\,[F_2] = \{1\}$.

II] To prove that $\exists$ a continuous function $h : X \to [a, b]$ such that

$h\,[F_1] = \{a\}$ and $h\,[F_2] = \{b\}$.

We know that g: $[\,0,1\,] \rightarrow [a, b]$ defined by $g(x) = a + (b - a)x$ is a continuous function. As composition of two continuous functions is a continuous function ,
$h = gof : X \rightarrow [a, b]$ is a continuous function. Further $h(x) = g[\,f(x)]\,; \forall\ x \in X$.
If $x \in F_1$, then $h(x) = g[\,f(x)] = h(x) = g[\,0] = a + (b - a).0 = a$.
Thus $h\,[F_1] = \{a\}$.
If $x \in F_2$, then $h(x) = g[\,f(x)] = h(x) = g[\,1] = a + (b - a).1 = b$.
Thus $h[F_2] = \{b\}$.

If part:

Let $\langle X, \mathfrak{I} \rangle$ be a topological space such that for any two disjoint closed sets F_1 *and* F_2 in X, $\exists$ a continuous function $f: X \rightarrow [0, 1]$ such that $f\,[F_1] = \{a\}$ and $f\,[F_2] = \{b\}$.
To prove that $\langle X, \mathfrak{I} \rangle$ is normal.
Let F_1 and F_2 be any two closed sets. By assumption $\exists$ a continuous function $f : X \rightarrow [0\,, 1]$ such that $f\,[F_1] = \{a\}$ and $f\,[F_2] = \{b\}$.
As $\left[\,\text{a}\,, \frac{b-a}{2}\right)$ and $\left(\frac{b-a}{2}\,, \text{b}\right]$ are open in [a, b] and f is continuous, we get
$f^{-1}\left[\left[\,\text{a}\,, \frac{b-a}{a}\right)\right]$ and $f^{-1}\left[\left(\frac{b-a}{a}\,, \text{a}\right]\right]$ are open in X. Further they are disjoint and
$F_1 \subseteq f^{-1}\left[\left[\,\text{a}\,, \frac{b-a}{2}\right)\right]$ and $F_2 \subseteq \left[\left(\frac{b-a}{2}\,, \text{b}\right]\right]$.
This shows that $\langle X, \mathfrak{I} \rangle$ is normal.

Theorem 2.4: Any compact, regular space is normal.

Proof:-Let $\langle X, \mathfrak{I} \rangle$ be a compact, regular space. Let F and K be any two disjoint sets in X. As X is compact, K is a compact subset of X (see Unit 6). Thus F is a closed subset of X and K is a compact subset of X with $F \cap K = \emptyset$. As X is regular space, $\exists$ disjoint open sets G and H such that $F \subseteq G$ and $K \subseteq H$ (see Unit 15). But this shows that $\langle X, \mathfrak{I} \rangle$ is a normal space.

Remark: Every regular space need not be normal but a compact, regular space is a normal space.

Theorem 2.5: Compact, Hausdorff space is normal.

Proof:-Let $\langle X, \mathfrak{J} \rangle$ be a compact, T_2 – space. Let F and K be any two disjoint closed sets in X. X being T_2 , $\exists$ two disjoint open sets G and H in X such that $F \subseteq G$, $K \subseteq H$ and $G \cap H = \emptyset$ (see Unit 14). Hence X is normal.

Remark: A compact space need not be normal but a compact, Hausdorff space is normal and a compact regular space is normal.

Theorem 2.6: Every regular, Lindelof space is normal.

Proof: Let $\langle X, \mathfrak{J} \rangle$ be a regular, Lindelof space. Let F and K be any two disjoint closed subsets of X. Fix up any $x \in F$. Then $x \notin K \implies x \in X - K$ and $X - K$ is an open set in X.

As X is a regular space, $\exists$ open set G_x in X such that $x \in G_x \subseteq \overline{G_x} \subseteq X - K$.

Hence $F = \bigcup_{x \in F} \{x\} \subseteq \bigcup_{x \in F} G_x$.

This shows that $\{G_x\}_{x \in F}$ forms an open cover for F . As X is a Lindelof space, and F is a closed subset of X, we get F is a Lindelof space.

Hence the open cover $\{G_x\}_{x \in F}$ of F contains a countable sub cover.

Let $F \subseteq \bigcup_{n=1}^{\infty} G_x$.

In the same way we can find a countable cover $\{H_i\}_{i \in N}$ of K.

Define $U_n = G_n - \bigcup_{i=1}^{n} (\overline{H_i})$ and $V_n = H_n - \bigcup_{i=1}^{n} (\overline{G_i})$

Then U_n and V_n are open sets in X for each n.

Define $U = \bigcup_{n=1}^{\infty} U_n$ and $V = \bigcup_{n=1}^{\infty} V_n$

Then U and V are open sets in X such that $F \subseteq U$ and $K \subseteq V$ and $U \cap V = \emptyset$.

Hence $\langle X, \mathfrak{J} \rangle$ is a normal space.

Corollary 2.7: Every regular, second axiom space is normal.

Proof:- Every second axiom space is a Lindelof space. Hence by Theorem 2.6 we get every regular second axiom space is a normal space.

Theorem 2.8: Being a normal space is a topological property.

Proof:-Let $\langle X, \Im \rangle$ be a normal space. Let $\langle Y, \Im^* \rangle$ be any T-space and Let $f : X \to Y$ be a homeomorphism.

To prove that Y is normal.

Let F^* and K^* be any two disjoint closed sets in $\langle Y, \Im^* \rangle$. $f : X \to Y$ being continuous, $f^{-1}[F^*]$ and $f^{-1}[K^*]$ are two disjoint closed sets in $\langle X, \Im \rangle$. As X is a normal space ,∃ two disjoint open sets G and H in X such that $f^{-1}[F^*] \subseteq G$ and $f^{-1}[K^*] \subseteq H$. As f is an open map. $f(G)$ and $f(H)$ are open sets in Y. Further $F^* \subseteq f(G)$ and $K^* \subseteq f(H)$ and $f(G) \cap f(H) = \mathrm{f(G \cap H)} = f(\emptyset) = \emptyset$. Thus any two disjoint closed sets F and K in Y can be separated by disjoint open sets in Y. Hence $\langle Y, \Im^* \rangle$ is a normal space.

Corollary 2.9: Every closed continuous image of a normal space is normal.

Theorem 2.10: Closed subspace of a normal space is normal**.**

Proof:- Let $\langle X , \Im \rangle$ be a normal space. Let $\langle Y, \Im^* \rangle$ be a closed subspace of $\langle X, \Im \rangle$. To prove that $\langle Y, \Im^* \rangle$ is normal. Let F^* and K^* be any two disjoint closed sets in $\langle Y, \Im^* \rangle$.

Then $F^* = F \cap Y$ and $K^* = K \cap Y$ for some closed sets F and K in X (see Unit (4)). As Y is a closed subset of X, $F^* = F \cap Y$ is a closed subset of X. Similarly, K^* is a closed subset of X. Further $F^* \cap K^* = \emptyset$ and $\langle X, \Im \rangle$ is a normal space will imply the existence of two disjoint open sets G and H in X such that $F^* \subseteq G$ and $K^* \subseteq H$.

But then $G^* = G \cap Y$ and $H^* = H \cap Y$ are disjoint open sets in $\langle Y, \Im^* \rangle$ such that $F \subseteq G^*$ and $K \subseteq H^*$. Hence $\langle Y, \Im^* \rangle$ is a normal space.

Remark:- Subspace of a normal space need not be normal. Hence being a normal space is not a hereditary property. But by Theorem 2.10, being a normal space is closed hereditary.

§3 Definition and Properties of T_4 spaces

Definition 3.1: A topological space which is both normal and T_1 is called a T_4-space.

Theorem 3.2: Every Compact, Hausdorff space is a T_4 – space.

Proof:- Every Compact, Hausdorff space is a normal space (see Theorem 2.5) and every Hausdorff space is a T1 – space (see Unit 14). Hence Every Compact, Hausdorff space is a T_4 – space.

Theorem 3.3: Every T_4 – space is a T_3 – space.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ is a T_4 – space. To prove that $\langle X, \mathfrak{J} \rangle$ is a regular space. Let F be any closed set in X and $x \notin F$. As $\langle X, \mathfrak{J} \rangle$ is a T_1 – space, {x} is a closed set in $\langle X, \mathfrak{J} \rangle$.

$x \notin F \implies \{x\} \cap F = \emptyset$

As X is a normal space, $\exists\ G, H \in \mathfrak{J}$ such that $\{x\} \subseteq G,\ F \subseteq H$ and $G \cap H = \emptyset$.

But this in turn implies that X is a regular space. As X is a regular, T_1 – space.

We get X is a T_3– space.

Corollary 3.4: Every T_4 space is a regular space.

Theorem 3.5: Being a T_4 - space is a topological property.

Proof: We know that the property of being a T_1 – space is a topological property and the property of a space being a normal space is also a topological property. Hence the property of being a T_4 – space is a topological property.

§4 Solved examples

1) Show that any metric space is normal.

Solution: Let $\langle X, d \rangle$ be a metric space and let $\mathfrak{J}$ be the topology induced by d on X. To prove that $\langle X, \mathfrak{J} \rangle$ is a normal space.Let F be any closed set and G be any open set in X such that $F \subseteq G$ .As G is open, for each $x \in F\ \exists\ r_x > 0$ such that S$(r, r_x) \subseteq$ G.

Define $H = \bigcup_{x \in F} \mathrm{S}\,(\mathrm{x} + \mathrm{r}_{x/2})$

Then H is an open set in X and $F \subseteq H$.Further,

$$\overline{H} = \overline{\bigcup_{x \in F} \mathrm{S}\,(\mathrm{x} + \mathrm{r}_{x/2})}$$

$$= \bigcup_{x \in F} \overline{S\,(x + r_{x/2})}$$

$$\subseteq \bigcup_{x \in F} S\,(x + r_x)$$

$\subseteq G$ (By Construction)

Thus given a closed set F and an open set G with $F \subseteq G$, $\exists$ an open set H such that $F \subseteq H \subseteq \overline{H} \subseteq G$. Hence $\langle X , \mathfrak{I} \rangle$ is a normal space(see theorem 2.1).

2) Give an example of a normal space which is not regular.

Solution:-Let X={a,b,c} and $\mathfrak{I} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ Then $< x, \mathfrak{I} >$ is a topological space.

I) $\langle X , \mathfrak{I} \rangle$ is not regular.

The Family of closed sets $\varkappa = \{\emptyset, X, \{b, c\}, \{c, a\}, \{c\}\}$ For $a \notin \{b, c\}$,there do not exist two disjoint open sets one containing a and other containing {b,c}.

Hence $< x, \mathfrak{I} >$ is not a regular space.

II) $\langle X , \mathfrak{I} \rangle$ is a normal space.

The pair of disjoint closed sets contains $\emptyset$.Hence the space is a normal space.

Thus every normal space need not be a regular space.

Remark: This space $\langle X , \mathfrak{I} \rangle$ is normal but it is not a T_1-space (Since {a} is not a closed set in X).Hence this space is not a T_4-space.Thus this example also shows that every normal space need not be a T_4-space.

3) Give an example to show that subspace of a normal space need not be normal.

Solution: Let $\langle X , \mathfrak{I} \rangle$ be any discrete T-space with X as an uncountable set.

Let $X_1^* = X_1 \cup \{\propto\}$ be any point compactification of X_1.

Let $\langle X , \mathfrak{I} \rangle$ be any discrete topological space with X_2 as an infinite set.

Let $X_2^* = X_2 \cup \{\beta\}$ be one point compactification on of X_2.

Define $X = X_1^* \ X \ X_2^*$. Let $\mathfrak{I}$ denote the product topology on X. The product space $\langle X , \mathfrak{I} \rangle$ is a compact space as X_1^* and X_2^* both are compact spaces. The product space $\langle X , \mathfrak{I} \rangle$ is a T_2-space. Hence the product space is a normal space (see Theorem 2.5)

Define $Y = X - \{\alpha, \beta\}$. Consider the subspace $\langle Y, \mathfrak{I}^* \rangle$ of $\langle X, \mathfrak{I} \rangle$.

Then $A = \{(\alpha, y) \mid y \in X_2\}$ and $B = \{ (x, \delta) \mid x \in X_1\}$ are disjoint closed sets in X.

As these disjoint closed closed sets are not contained in any disjoint open sets in $\langle Y, \mathfrak{I}^* \rangle$ we get $\langle Y, \mathfrak{I}^* \rangle$ is not a normal space. Thus this example shows that subspace of a normal space need not be normal.

4) Give an example of a regular space which is not normal.

Solution: Niemytzki's space is a regular space but not normal.

5) Show that every T_3 space need not be a T_4 space.

Solution: Let X= $\{(x, y) \mid \ x, y \in \ \mathbb{R}\,, \ \ y > 0\}$

For each $(p, a) \in X$ define,

$$N_{\in}(p, a) = \left\{(x, y) \mid \sqrt{(x-p)^2 + (y-a)^2} < \epsilon \ \ \text{where } \ \epsilon < q, if\ a > 0\right\}$$

$$N_{\in}(p, 0) = \left\{(x, y) \mid \sqrt{(x-p)^2 + (y-a)^2} < \epsilon \, where\ \epsilon > 0\right\}$$

Define for each $(p, a) \in X$,

$$N_{\in}(p, a) = \left\{(x, y) \mid \ \in X \sqrt{(x-p)^2 + (y-a)^2} < \epsilon \right\},\ if\ q > 0$$

$$N_{\in}(p, a) = \left\{(x, y) \in X \ \mid \ \sqrt{(x-p)^2 + (y-a)^2} < \epsilon \right\} \cup \{(p, 0)\}$$

Define

$$N(p, q) = \{N_{\in}(p, a) \mid \epsilon < q \ and\ q > 0 \}$$
$$\cup \{N_{\in}(p, 0) \mid \epsilon > 0\}$$

Define $\mathcal{B} = \{N(p, q) \mid (p, a) \in X\}$

Then $\mathcal{B}$ forms a base for some topology $\mathfrak{I}$ on X.

$\langle X , \mathfrak{I} \rangle$ is a T_3 space .

Let $F = \{(p, 0) \mid p$ is a rational$\}$ and $K = \{(p, 0) \mid p$ is a irrational$\}$

Then F and K are disjoint closed sets in X. As there do not exists two open sets G and H such that $F \in G \,, K \in H \ and\ G \cap H = \emptyset$, we get $\langle X , \mathfrak{I} \rangle$ is not a normal space and hence $\langle X, \mathfrak{J} \rangle$ is not a T_4 space. Thus every T_3 space need not be a T_4 space. T_4 space This space is called Niemytzki space.

Remark:- This example also shows that

1.Subspace of a T_4-space need not be a T_4-space.

2.Subspace of a T_4-space need not be a normal space.

6) Give an example to show that every normal space need not be a T_4-space.

   Solution:-Let X={a,b,c} and $\mathfrak{J} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\langle X, \mathfrak{J} \rangle$ is a topological space.

   $\langle X, \mathfrak{J} \rangle$ is a normal space as the pair of disjoint closed sets contains $\emptyset$ but it is not a T_1-space (Since {a} is not a closed set in X). Hence this space is not a T_4-space.

   This example shows that every normal space need not be a T_4-space.

Exercises

I) Prove or disprove the following statements.

1) Being a normal space is a hereditary property.
2) Being a normal space is a closed hereditary property.
3) Every normal space is a T_4-space.
4) Every regular space is a T_4-space
5) Every T_3 space is a T_4-space
6) Subspace of a T_4-space is a normal space.
7) Subspace of a T_4-space is a T_4 space.
8) Every normal space is regular.
9) Every regular space is normal.
10) Being a normal space is a topological property.
11) Being a T_4 space is a topological property.
12) Every T_4 space is a regular space.
13) Every Compact, Hausdorff space is a T_4 – space.
14) Every Compact, Hausdorff space is a normal space.
15) Every Compact space is a T_4 – space.

16) Every Hausdorff space is a normal space.
17) Every closed continuous image of a normal space is normal.
18) Every continuous image of a normal space is normal.
19) Every Compact, regular space is a normal space.
20) Every second axiom regular space is a normal space.

II) Show by an counter example that a subspace of a normal space need not be normal

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Unit 17

Completely normal spaces and T_5 – spaces

§1 Definition and Examples.

§2 Properties and characterizations.

§3 T_5 – spaces.

Unit 17: Completely normal spaces and T_5 – Spaces

§1 Definition and Examples

Definition 1.1: A topological space $\langle X, \mathfrak{I} \rangle$ is said to be completely normal if it satisfies the following axiom.

If A and B are two separated subsets of X, then there exists two disjoint open sets, one containing A and the other containing B.

Examples 1.2:

1) Any discrete topological space is completely normal.
2) Every metric space is completely normal.

Let $\langle X, \mathfrak{I} \rangle$ be a metric space and Let $\mathfrak{I}$ be the topology introduced by d. To prove that $\langle X, \mathfrak{I} \rangle$ is completely normal. Let A and B be separated sets in X.

Therefore $A \cap \bar{B} = \emptyset$ and $B \cap \bar{A} = \emptyset$.

Hence $a \in A \implies a \notin \bar{B} \implies \exists\, r_a > 0$ such that $S(a, r_a) \cap B = \emptyset$.

Similarly $b \in B \implies b \notin \bar{A} \implies \exists\, r_b > 0$ such that $S(b, r_b) \cap A = \emptyset$.

Define $G = \cup \left\{ S\left(a, {r_a}/_2 \right) \middle| a \in A \right\}$ and $H = \cup \left\{ S\left(b, {r_b}/_2 \right) \middle| b \in B \right\}$

By the definition of $\mathfrak{I}$, $S(a, r) \in \mathfrak{I}\ \forall\ x \in X$ and $r > 0$. We get G, H are open sets.

Further $a \in S\left(a, {r_a}/_2 \right)\ \forall\, a \in A \implies A = \bigcup_{a \in A} \{a\} \subseteq \bigcup \left\{ S\left(a, {r_a}/_2 \right) / a \in A \right\}$

$$\implies A \subseteq G.$$

Similarly $B \subseteq H$.

Claim that $G \cap H = \emptyset$.

Let $G \cap H \neq \emptyset$. Hence $\exists\ z \in G \cap H$. $z \in G \implies z \in S\left(a, {r_a}/_2 \right)$ for some $a \in A$.

$z \in H \implies z \in S\left(b, {r_b}/_2 \right)$ for some $b \in B$.

Hence $d(a, z) < {r_a}/_2$ and $d\left(b, r_{b/2} \right) < {r_b}/_2$.

Hence $d(a\,,\,b) \leq \; d(a\,,z) + \; d(z\,,b)$

$< \; r_{a/2} \; + r_{b/2}.$

$< \; r_a$

But $d(a,b) < r_a \quad \Longrightarrow \quad b \in S(a\,,r_a)$

$\Longrightarrow \quad b \in S(a\,,r_a) \cap B = \emptyset$; a contradiction.

Hence $G \cap H = \emptyset$.

Thus given two separated sets A and B in X, $\exists$ disjoint sets G and H such that $A \subseteq G$, and $B \subseteq H$.

Hence X is a completely normal space.

§2 Properties and characterizations

Theorem 2.1: Every completely normal space is normal.

Proof: Let F_1 , F_2 be any two disjoint closed sets in a completely normal space $\langle X, \mathfrak{J}\rangle$.

As $\overline{F_1} = F_1$ and $\overline{F_2} = F_2$ we get $F_1 \cap F_2 = \emptyset \;\; \Longrightarrow \;\; F_1 \cap \; \overline{F_2} = \emptyset$ and $\overline{F_1} \; \cap \; F_2 = \; \emptyset$.

Thus F_1 and F_2 are separated sets in X. Hence by definition, $\exists$ disjoint open sets G and H such that $F_1 \; \subseteq G$ and $F_2 \; \subseteq H$. This shows that $\langle X, \mathfrak{J}\rangle$ is normal.

Remark: Converse of the Theorem 2.1 need not be true.

i.e. Normal space need not be completely normal.

For this consider the following example.

Let $X = \{a, b, c, d\}$ and $\mathfrak{J} = \{\emptyset\,, \{a\}\,, \{a, b\}\,, \{a, c\}\,, \{a, b, c\}\,, X\}$. The family of closed sets is given by, $\mathcal{K} = \{\emptyset, \{b, c, d\}, \{c, d\}, \{b, d\}, \{d\}, X\}$.

(I) To prove that $\langle \mathrm{X}\,, \mathfrak{J}\rangle$ is normal.

Let A and B be any two disjoint closed sets in X. Then one of them must be empty.

Let $A = \emptyset$. Take $G = \emptyset$ and $H = X$. Then $G, H \in \mathfrak{J}\,,\; G \cap H = \emptyset\,,\; A \subseteq G$ and $B \subseteq H$.

Hence $\langle X\,, \mathfrak{J}\rangle$ is a normal space.

(II) To prove that $\langle X\,, \mathfrak{J}\rangle$ is not completely normal.

By Theorem 2.3, it is sufficient to prove that there exists a subspace $\langle Y\,, \mathfrak{J}^*\rangle$ of $\langle X\,, \mathfrak{J}\rangle$ such that $\langle Y\,, \mathfrak{J}^*\rangle$ is not normal.

Define $Y = \{a, b, c\}$. The relative topology $\mathfrak{J}^*$ on Y is given by,

$\mathfrak{J}^* = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, Y\}$

Here, $\mathcal{K}^* = \{\emptyset, \{b,c\}, \{c\}, \{b\}, Y\}$ denotes the family of closed sets in $\langle Y, \mathfrak{J}^* \rangle$

$\{b\}$ and $\{c\}$ are disjoint closed sets in Y which are not separated by disjoint open sets in $\langle Y, \mathfrak{J}^* \rangle$. Hence $\langle Y, \mathfrak{J}^* \rangle$ is not normal.

As there exists a subspace $\langle Y, \mathfrak{J}^* \rangle$ of $\langle X, \mathfrak{J} \rangle$ such that $\langle Y, \mathfrak{J}^* \rangle$ is not normal.

Hence $\langle X, \mathfrak{J} \rangle$ is not completely normal. [A normal space is completely normal if and only if each subspace of it is normal]

This example also shows that any T4 – space need not be completely normal.

Theorem 2.2: Being a completely normal space is a hereditary property.

Proof:- Let $\langle X, \mathfrak{J} \rangle$ be a complete normal space and Let $\langle X^*, \mathfrak{J}^* \rangle$ be its subspace. To prove that $\langle X^*, \mathfrak{J}^* \rangle$ is a completely normal space.

Let A and B be any two separated sets in $\langle X^*, \mathfrak{J}^* \rangle$.

Claim that A and B be are separated subsets of $\langle X, \mathfrak{J} \rangle$. A and B are separated in $\langle X^*, \mathfrak{J}^* \rangle$.

$\Longrightarrow$ $A \cap C^*(B) = \emptyset$ and $B \cap C^*(A) = \emptyset$.

$\Longrightarrow$ $A \cap [\, C(B) \cap X^*] = \emptyset$ and $B \cap [\, C(A) \cap X^*] = \emptyset$.

$\Longrightarrow$ $[A \cap X^*] \cap C(B) = \emptyset$ and $[B \cap X^*] \cap C(A) = \emptyset$.

$\Longrightarrow$ A and B are separated sets in X.

As X is a completely normal space, $\exists\ G, H \in \mathfrak{J}$ such that $A \subseteq G$, $B \subseteq H$ and $G \cap H = \emptyset$.

Define $G^* = A \cap X^*$ and $H^* = H \cap X^*$. Then $G^*, H^* \in \mathfrak{J}^*$.

$G^* \cap H^* = G \cap H \cap X^* = \emptyset$, $A \subseteq G^*$ and $B \subseteq H^*$. This shows that $\langle X^*, \mathfrak{J}^* \rangle$ is a completely normal space.

Thus as any subspace of a completely normal space is completely normal, the result follows.

Theorem 2.3: A topological space $\langle X, \mathfrak{J} \rangle$ is a completely normal space if and only if every subspace of X is normal.

Proof: -‘ Only if part’ follows by Theorem 2.1 and Theorem 2.2.Hence to prove ‘if part’ only

Let $\langle X, \mathfrak{J} \rangle$ be a topological space such that each subspace of X is normal. To prove $\langle X, \mathfrak{J} \rangle$ is completely normal.

Let A and B be any two separated sets in X. Then $A \cap \bar{B} = \emptyset$ and $B \cap \bar{A} = \emptyset$.

Define $X^* = X - [\bar{A} \cap \bar{B}] = (X - \bar{A}) \cap (X - \bar{B})$.

Let $\mathfrak{J}^* = \{ G \cap X^* \mid G \in \mathfrak{J}\}$ be the relative topology on X^*. As $\bar{A}$ is closed in X, $\bar{A} \cap X^*$ is closed in X^*. Similarly, $\bar{B} \cap X^*$ is closed in X^*.

Further $(\bar{A} \cap X^*) \cap (\bar{B} \cap X^*) = (\bar{A} \cap \bar{B}) \cap X^*$

$$= (\bar{A} \cap \bar{B}) \cap [X - (\bar{A} \cap \bar{B})] = \emptyset .$$

Thus $\bar{A} \cap X^*$ and $\bar{B} \cap X^*$ are disjoint closed subsets of X^*. As X^* is normal,

$\exists\ G^*, H^* \in \mathfrak{J}^*$ such that $(\bar{A} \cap X^*) \subseteq G^*$ and $(\bar{B} \cap X^*) \subseteq H^*$ and $G^* \cap H^* = \emptyset$.

As G^*, $H^* \in \mathfrak{J}^* \implies G^* = G \cap X^*$ and $H^* = H \cap X^*$ for some $G, H \in \mathfrak{J}$.

Claim that $A \subseteq G$ *and* $B \subseteq G$.

1] $A \cap \bar{B} = \emptyset \implies A \subseteq (X - \bar{B})$

$\bar{A} \cap \bar{B} \subseteq \bar{B} \implies X - [\bar{A} \cap \bar{B}] \supseteq (X - \bar{B})$.

$\implies X^* \supseteq (X - \bar{B}) \supseteq A$

Thus $A \subseteq X^*$, Similarly we get $B \subseteq X^*$.

2] $A \cap X^* = A \cap [X - (\bar{A} \cap \bar{B})]$

$= A \cap [(X - \bar{A}) \cup (X - \bar{B})]$

$= [A \cap [X - \bar{A}]] \cup [A \cap [X - \bar{B}]]$

$= \emptyset \cup A = A$

[$A \subseteq \bar{A} \implies A \cap [X - \bar{A}] = \emptyset$.

$A \cap \bar{B} = \emptyset \implies A \subseteq (X - \bar{B}) \implies A(X - \bar{B}) = A$]

Thus $A \cap X^* = A$, Similarly we get $B \cap X^* = B$.

3] $A \subseteq X^* \implies A = A \cap X^* \subseteq \bar{A} \cap X^* \subseteq G^* \subseteq G$

Thus $A \subseteq G$, similarly we get $B \subseteq H$.

Thus for given any pair of separated sets in A and B in $\langle X, \mathfrak{J}\rangle \ \exists\ G, H \in \mathfrak{J}$ such that $A \subseteq G$ and $B \subseteq H$ and $G \cap H = \emptyset$.

Hence $\langle X , \mathfrak{J}\rangle$ is a completely normal space.

Example: Let X = { a, b, c, d } and $\mathfrak{J} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$.

Take $X^* = \{a, b, c\}$ then the relative topology $\mathfrak{J}^*$ and X^* is given by as follows,

$\mathfrak{J}^* = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X^*\}$. $\{b\}$ and $\{c\}$ are disjoint closed sets in $\langle X^*, \mathfrak{J}^*\rangle$.

As there are no disjoint open sets G and H in $\langle X^*, \mathfrak{I}^* \rangle$ such that $\{b\} \in G$ and $\{c\} \in H$. We get $\langle X^*, \mathfrak{I}^* \rangle$ is not a normal space. Hence by the Theorem2.3 , $\langle X, \mathfrak{I} \rangle$ is not a completely normal space.

Theorem 2.4: Being a completely normal space is a topological property.

Proof:- Let $\langle X, \mathfrak{I} \rangle$ be a complete normal space. Let $\langle X^*, \mathfrak{I}^* \rangle$ be any topological space and let $f: X \longrightarrow X^*$ be a homomorphism. To prove that X^* is a completely normal space.

Let A and B be any two separated sets in X^*. Hence $A \cap \bar{B} = \emptyset$ and $B \cap \bar{A}$.

Since f is continuous $\overline{f^{-1}[A]} \subseteq f^{-1}[\bar{A}]$ and $\overline{f^{-1}[B]} \subseteq f^{-1}[\bar{B}]$.

Hence $f^{-1}[A] \cap f^{-1}[B] \subseteq f^{-1}[\bar{A}] \cap f^{-1}[\bar{A}] = f^{-1}[A \cap \bar{B}] = f^{-1}[\emptyset] = \emptyset$.

Therefore, $f^{-1}[A] \cap f^{-1}[B] = \emptyset$ ________________________________ (1)

Similarly we can show that $f^{-1}[A] \cap f^{-1}[B] = \emptyset$ __________________ (2)

From(1) and (2) we get $f^{-1}[A]$ and $f^{-1}[B]$ are separated sets in X. As X is a completely normal space $\exists$ disjoint open sets say G, H in X such that $f^{-1}[A] \subseteq G$ and $f^{-1}[B] \subseteq H$.

As f is onto, $A = f\left[f^{-1}[A]\right]$ and $B = f\left[f^{-1}[B]\right]$.

Hence $A = f\left[f^{-1}[A]\right] \subseteq f(G)$ and $B = f\left[f^{-1}[B]\right] \subseteq f(H)$.

Further $f(G) \cap f(H) = f(G \cap H) = \emptyset$ (Since f is one $-$ one).

Further f is an open map $\Longrightarrow f(G), f(H) \in \mathfrak{I}^*$.

Thus for two disjoint separated sets A and B in X^* there exist $f(G), f(H) \in \mathfrak{I}^*$ one containing A and the other containing B.

Hence $\langle X^*, \mathfrak{I}^* \rangle$ is a completely normal space.

As homeomorphic image of a completely normal space is completely normal, the result follows.

Theorem 2.5: Every regular, second axiom space is completely normal.

Proof: Let $\langle X, \mathfrak{I} \rangle$ be a regular, s.a.s. To prove that $\langle X, \mathfrak{I} \rangle$ is completely normal. Let A and B be separated sets in X. i.e. $A \cap \bar{B} = \emptyset$ and $B \cap \bar{A} = \emptyset$.

$x \in A \implies x \notin \bar{B} \implies x \in X - \bar{B} \implies \exists\, G_x \in \mathfrak{I}$.

such that $x \in G_x \subseteq \overline{G_x} \subseteq X - \bar{B}$.

Hence $\{G_x \mid x \in A\}$ will form an open cover for A. As X is s.a.s. the open covering $\{G_x \mid x \in A\}$ of A can be reducible to a countable sub-covering (See Unit (9) Theorem 1.8) Denote it by $\{ G_i\}_{i \in N}$. Similarly for B $\exists$ a countable sub-covering $\{ H_i\}_{i \in N}$.

Define $G = \bigcup_{n \in N} \left[G_n - \bigcup_{i \le n} \overline{H_\iota} \right]$ and $H = \bigcup_{n \in N} \left[H_n - \bigcup_{i \le n} \overline{G_\iota} \right]$

Define $U_n = G_n - \bigcup_{i \le n} \overline{H_\iota} \quad \forall\, n \in N$ and $V_n = H_n - \bigcup_{i \le n} \overline{G_\iota} \quad \forall\, n \in N$

Then for each n , U_n and V_n are open sets. Hence $\bigcup_{i \in n} U_n = G$ and $\bigcup_{i \in n} V_n = H$ are open.

Further $U_n \cap V_k = \emptyset \quad \forall\, n \in N$ and $k \in N$.

Hence $G \cap H = \emptyset$ (by the definition of U_n and V_n).

Further $A \subseteq \bigcup_{n \in N} G_n$ and $G_n \subseteq \overline{G_n} \subseteq X - \bar{B} \quad \forall\, n$ imply $A \subseteq G$.

Similarly $B \subseteq G$. Thus for separated sets A and B of X, $\exists$ disjoint sets G and H such that $A \subseteq G$ and $B \subseteq G$.

Hence X is a completely normal space.

§3 T_5 – spaces

Definition 3.1: A T_5- space is a completely normal, T_1- space.

Theorem 3.2: Every T_5- space is a T_4 – space.

Proof: - As every completely normal space is normal we get every T_5 – space is a T_4- space.

Remark:- Converse of Theorem3.2 need not be true.

i.e. every T_4- space need not be a T_5- space.

See remark after Theorem 2.1.

Theorem 3.3: Being a T_5- space is a topological property.

Proof: - We know that being a completely normal space is a topological property and being a T_1- space is a topological property. Hence being a T_5-space is a topological property.

Theorem 3.4: Being a T_5-space is a hereditary property.

Proof: - we know that being a completely normal space is a hereditary property. Also being a T_1 – space is a hereditary property. Hence being a T_5 – space is a hereditary property.

Exercises

Prove or Disprove the following statements.

1) Every completely normal space is normal.
2) Every normal space is completely normal.
3) Any subspace of a completely normal space is completely normal
4) Any subspace of a completely normal space is normal.
5) Being a T_5 – space is a hereditary property.
6) Being a T_5 – space is a topological property.
7) Every T_5 – space is a T_4 – space.
8) Every T_4 – space is a T_5- space.

Unit 18

Completely regular spaces and $T_{3\frac{1}{2}}$ Spaces

Unit 18: *Completely regular spaces and $T_{3\frac{1}{2}}$ spaces*

§1 Definition and examples

Definition 1.1: A topological space $\langle X, \mathfrak{J} \rangle$ is said to be regular if it satisfies the following axiom.

"If F is a closed subset of X and x is a point of X not in F, Then there exist a continuous function $f : X \longrightarrow [\,0\,,1]$ such that $f(\,x\,) = 0$ and $f(\,F\,) = \{1\}$.

Examples 1.2:

1) Fort's space is a completely regular space.
2) Every metric space is a completely regular space.
3) $\langle R, \mathfrak{J}_u \rangle$ is a completely regular space.

§2 Characterizations and properties

Theorem 2.1: A topological space $\langle X, \mathfrak{J} \rangle$ is completely regular if and only if for every $x \in X$ and every open set containing x , there exists a continuous mapping $f : X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f[y] = 1\,,\ \forall\ y \in X - G$.

Proof:- Only if part.

Let X be a completely regular $x\ \in G$ where G is an open set in X. Then X-G is a closed set in X with $x \notin X - G$. As X is completely regular, ∃ a continuous function $f : X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f(\,X - G\,) = \{1\}$

$i.e.$ $f(y) = 1$ for each $y \in X - G$.

If part.

Assume that for every $x \in X$ and every open set containing, there exists a continuous mapping $f : X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f[y] = 1\,,\ \forall\ y \in X - G$.

To prove that X is a completely regular space. Let F be a closed set and $x \notin F$. Then $X - F$ is an open set containing x. Hence by assumption, ∃ a continuous real valued function

$f: X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f(\, X - G\,) = 1 \quad \forall\, y \in X - [X - F]$
$i.e. f(x) = 0$ and $f(y) = 1, \forall\, y \in F.$

Hence $\langle\, X, \mathfrak{J}\rangle$ is a completely regular space.

Theorem 2.2: Let $\langle X, \mathfrak{J}\rangle$ be completely regular space. Let N be neighborhood of $x \in X$. Then $\exists$ a continuous function $f: X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f(y) = 1\,,\ \forall\, y \in X - N$ and conversely.

Proof:- As N is neighborhood of $x \in X$, $\exists$ an open set G in X such that $x \in G \subseteq N$.
Hence $x \notin X - G$, where $X - G$ is a closed set in X.

As X is completely regular, $\exists$ continuous function $f: X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f[X - G] = \{1\}$.
As $G \subseteq N \implies X - N \subseteq X - G$ we get $f(y) = 1$ for each $y \in X - N$.
Conversely, assume that $\exists$ a continuous function $f: X \longrightarrow [0\,,1]$ such that
$f(x) = 0$ and $f(y) = 1\,,\ \forall\, y \in X - N$
To prove that $\langle X, \mathfrak{J}\rangle$ be completely regular space. Let G be an open set in X such that $x \in G$.
As G is neighborhood of $x \in X$, $\exists$ a continuous function $f: X \longrightarrow [0\,,1]$ such that
$f(x) = 0$ and $f(y) = 1\,, \forall\, y \in X - G$.
Hence by Theorem 2.1, $\langle X, \mathfrak{J}\rangle$ is a completely regular space.

Theorem 2.3: Let $\langle X, \mathfrak{J}\rangle$ be a completely regular space.Let F is a closed set in X and $x \notin F$. Then $\exists$ a continuous function $f: X \longrightarrow [0\,,1]$ such that $f(x) = 1$ and $f(F) = \{0\}$.

Proof: Let $x \notin F$ and F be a closed set in X. As X is a completely regular space $\exists$ a continuous function $g: X \longrightarrow [0\,,1]$ such that $g(x) = 0$ and $g(F) = \{1\}$.
Define the function $g: X \longrightarrow [0\,,1]$ by $f(x) = 1 - g(x),\ \forall\, x \in X$.
Then f is a continuous function and $f(0) = 1 - g(0) = 1 - 0 = 1$ and
$f(1) = 1 - g(1) = 1 - 1 = 0.$
Thus $\exists$ a continuous function $f: X \longrightarrow [0\,,1]$ such that $f(0) = 1$ and $g(1) = 0$.

Theorem 2.4: Being a completely regular space is a topological property.

Proof: - Let $\langle X, \mathfrak{J}\rangle$ be a completely regular space. Let $\langle X^*, \mathfrak{J}^*\rangle$ be any topological space and $f: X \longrightarrow X^*$ be a homomorphism. To prove that X^* is a completely regular space. Let F^* be a closed set in X^* and $x^* \notin F^*$ $(x^* \in X^*)$.

As f is onto, $\exists$ $x \in X$ such that $f(x) = x^*$. As f is a continuous function, $f^{-1}[F^*]$ is a closed set in X.

$x^* \notin F^* \implies x^* \notin f^{-1}[F^*]$.

Hence X being a completely regular space, $\exists$ a continuous function $g: X \longrightarrow [0,1]$ such that $g(x) = 0$ and $g[f^{-1}(F^*)] = \{1\}$.

Thus $g[f^{-1}(x^*)] = 0$ and $g[f^{-1}(F^*)] = \{1\}$

$\implies [g \circ f^{-1}](x^*) = 0$ *and* $[g\ o\ f^{-1}](F^*) = \{1\}$

Now $g \circ f^{-1}: X^* \longrightarrow [0,1]$

$f^{-1}: X^* \longrightarrow X$ is a continuous as f is a homeomorphism.

Hence $g \circ f^{-1}$ is a continuous map (see Theorem … in Continuous function)

Thus for $x^* \notin F^*$ $\exists$ a continuous function $go\ f^{-1}: X^* \longrightarrow [0,1]$ such that $[g\ o\ f^{-1}](x^*) = 0$ and $[g\ o\ f^{-1}](F^*) = \{1\}$.

Hence X^* is a completely regular space. As homeomorphic image of a completely regular space is a completely regular space, we get being a completely regular space is a topological property.

Theorem 2.5: Being a completely regular space is a hereditary property.

Proof: - Let $\langle X, \mathfrak{I} \rangle$ be completely regular space and Let $\langle X^*, \mathfrak{I}^* \rangle$ be its subspace. To prove that $\langle X^*, \mathfrak{I}^* \rangle$ is completely regular. Let F^* be any closed set in X^* *and* $x^* \notin F^*$ $(x^* \in X^*)$.

As F^* is a closed set in X^*,$\exists$ a closed set F in X such that,

$F^* = F \cap X^*, x^* \notin F^* \implies x^* \notin F^*$ $(x^* \in X^*)$.

As $\langle X, \mathfrak{I} \rangle$ is a completely regular space and $x^* \notin F$ $\exists$ a continuous function $f: X \longrightarrow [0,1]$ such that $f(x^*) = 0$ and $f(F) = \{1\}$.

Let g denote the restriction of f to X^*. Then g is a real valued continuous function defined on X^* such that $g(x^*) = 0$ and $g(F^*) = \{y\}$.

Hence X^* is a completely regular space. Thus subspace of a completely regular space is a completely regular space. Hence the property of being a completely regular space is a hereditary space.

Theorem 2.6: Every completely regular space is regular.

Proof: Let $\langle X, \mathfrak{J} \rangle$ be a completely regular space. To prove that $\langle X, \mathfrak{J} \rangle$ is regular. Let F be a closed set and $x \notin F$ $(x \in X)$. A s X is a completely regular, $\exists$ a continuous function $f: X \longrightarrow [\,0\,,1]$ such that $f(\,x\,) = 0$ and $f(\,F\,) = \{1\}$. We know that $\langle \mathbb{R}, \mathfrak{J}_u \rangle$ is a Hausdorff space.

Hence [0,1] (being a subspace of $\langle \mathbb{R}, \mathfrak{J}_u \rangle$) is a Hausdorff space.

As $0 \neq 1$ in $[\,0{,}1\,], \exists$ disjoint open sets G and H in [0,1] such that $0 \in G$ and $1 \in H$.

But $f: X \longrightarrow [\,0\,,1]$ $\mathit{is\ continuous}$ $\Longrightarrow$ $f^{-1}(G) \in \mathfrak{J}$ and $f^{-1}(H) \in \mathfrak{J}$.

Further $x \in f^{-1}(G)$ and $F \subseteq f^{-1}(H)$. Thus for $x \notin F$, $\exists$ disjoint open sets $f^{-1}(G)$ and $f^{-1}(H)$ in X such that $x \in f^{-1}(G)$ and $F \subseteq f^{-1}(H)$.

Hence $\langle X, \mathfrak{J} \rangle$ is a regular space.

Theorem 2.7: A normal space is completely regular if and only if it is regular.

Proof: - As every completely regular space is a regular space (see Theorem2.6), the proof of 'only if 'part follows immediately,

To prove if part, assume that X is a normal, regular space. To prove that X is completely regular space. Let F be a closed set and $x \notin F$ $(x \notin X)$. Then $X - F$ is an open set containing x.

As X is a regular space, an open set G in X such that $x \in G \subseteq \bar{G} \subseteq X - F$ (see Theorem 1 in … Regular spaces /$T_3 -$ *space*).

As $\bar{G} \subseteq X - F$ we get $\bar{G} \cap F = \emptyset$.

Thus as $\bar{G}$ and F are disjoint closed sets in a normal space X. Hence $\exists$ continuous function $f: X \longrightarrow [0\,,1]$ such that $f(\bar{G}) = \{0\}$ and $f(F) = \{1\}$ (by Urysohn's Lemma).

As $x \in \bar{G}$ we get $f(x) = 0$ and $f(F) = \{1\}$. Hence X is a completely regular space.

Corollary 2.8: Any compact, T_2 – space is completely regular.

Proof:- We know that compact, T_2 – space is both normal and regular (see Theorems …and … T_2 – space) . Hence by Theorem 2.7, any compact, T_2 – space is completely regular.

Corollary 2.9: Any compact, regular space is completely regular.

Proof:- We know that any compact regular space is normal (see Theorem ….Normal spaces) Hence by Theorem 2.7, it is a completely regular space.

Theorem 2.10: Every locally compact, Hausdorff space is completely regular.

Proof: - Let $\langle X, \mathfrak{J} \rangle$ be a countably compact, Hausdorff space. Let $\langle X^*, \mathfrak{J}^* \rangle$ be one point compactification of $\langle X, \mathfrak{J} \rangle$. $X^* = X \cup \{ \infty \}$ where $\infty \notin X$ and

$\mathfrak{J}^* = \{ G \subseteq X^* \mid X^* - G$ is a closed, compact subset of $X \} \cup \mathfrak{J}$.

Claim 1: $\langle X^*, \mathfrak{J}^* \rangle$is a Hausdorff space.

Let $x \neq y$ in X^* .

Case 1:- $x, y \in X$ and $x \neq y$.

As X is a T_2 – space, ∃ disjoint open sets G and H in $\langle X, \mathfrak{J} \rangle$ such that $x \in G$ and $y \in H$ But then $G, H \in \mathfrak{J}^*$ and hence in this case x and y are separated by disjoint open sets in X^*.

Case 2:- $x = \infty \in X^*$ $i.e.$ $x \notin X$ and $y \notin X$.

As X is a locally compact space and $y \in X$, y is an interior point of some compact subset say K. Let G be an open set in X such that $x \in G \subseteq K$. As K is a compact subset of a T_2 – space, K is a closed in X and hence $X^* - K$ is open in X^*.Thus $x \in G$, $\infty \in X - K$ and G, $X^* - K$ are disjoint open sets in X^*.

Thus from both the cases we get $\langle X^*, \mathfrak{J}^* \rangle$ is a Hausdorff space.

Claim 3:- $\langle X^*, \mathfrak{J}^* \rangle$ is a completely regular. As $\langle X^*, \mathfrak{J}^* \rangle$ is a compact, Hausdorff space, it is completely regular (By Corollary2.7).

Claim 4:- $\langle X, \mathfrak{J} \rangle$ is a completely regular. We know that $\langle X, \mathfrak{J} \rangle$ is a subspace of $\langle X^*, \mathfrak{J}^* \rangle$ and $\langle X^*, \mathfrak{J}^* \rangle$ is a completely regular space. Hence $\langle X, \mathfrak{J} \rangle$ is a completely regular space (see Theorem 2.4).

§3 $T_{3\frac{1}{2}}$ spaces or Tichonov spaces

Definition 3.1: Completely regular, T_1 – space is called a Tichonov space or a $T_{3\frac{1}{2}}$ space.

Theorem 3.2: Every Tichonov space ($T_{3\frac{1}{2}}$ space) is a T_3 – space.

Proof:- As every completely regular space is regular (Theorem 6), every Tichonov space ($T_{3\frac{1}{2}}$ space) is a T_3 – space.

Theorem 3.3: Every space T_4 – space is a Tichonov space ($T_{3\frac{1}{2}}$ $space$).

Proof: Let $\langle X, \mathfrak{I} \rangle$ be a T_4 – space i.e. $\langle X, \mathfrak{I} \rangle$ is a normal T_1 – space. To prove that $\langle X, \mathfrak{I} \rangle$ is a Tichonov space. Let $x \notin F$ where F is a closed set in X ($x \in X$). As X is a T_1 – space, $\{x\}$ is a closed set in X.

$x \notin F \implies \{x\} \cap F = \emptyset$. Hence as X is normal , $\exists$ a continuous function $f: X \longrightarrow [\,0\,,1]$ such that $f(\{x\}) = \{0\}$ and $f(\,F\,) = \{1\}$ (By Urysohn's Lemma).

Thus for $x \notin F$, $\exists$ a continuous function $f: X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$

Hence X is a completely regular space.

Theorem 3.4: If $x \neq y$ in a Tichonov | $T_{3\frac{1}{2}}$ space X, then $\exists$ a continuous function such that $f(x) \neq f(y)$.

Proof:- Let $\langle X\,, \mathfrak{I} \rangle$ be a Tichonov space and $x \neq y$ in X. As X is a T_1 – space $\{y\}$ is a closed set in X. $x \neq y \implies x \notin \{y\}$. X being a completely regular space, $\exists$ a continuous function $f: X \longrightarrow [0\,,1]$ such that $f(x) = 0$ and $f(\{y\}) = \{1\}$. $i.e.$ $f(y) = 1$.

Hence $f(x) \neq f(y)$.

Theorem 3.5: Being Tichonov space ($T_{3\frac{1}{2}}$) is a topological property.

Proof: - We know that being a completely regular space is a topological property and being a T_1 – space is also a topological property. Hence being a Tichonov space ($T_{3\frac{1}{2}}$) is a topological property.

Theorem 3.6: Being a Tichonov ($T_{3\frac{1}{2}}$) space is a hereditary property.

Proof: We know that being a completely regular space is a hereditary property and being a T_1 – space is a hereditary property. Hence being a Tichonov ($T_{3\frac{1}{2}}$) space is a hereditary property.

§4 Solved Problems

Problem 1: Fort's space is a completely regular space ($T_{3\frac{1}{2}}$ space).

Solution: Fort's space is a compact, Hausdorff space (see).

Hence by Corollary2.8, Fort's space is completely regular.

Again as Fort's space is a T_1 – space (being a Hausdorff space) it is a $T_{3\frac{1}{2}}$ space.

Problem 2: Define the topology $\mathfrak{J}$ on $\mathbb{R}$ by $\mathfrak{J} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$. Show that $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a completely regular space.

Solution:

I] $\langle \mathbb{R}, \mathfrak{J} \rangle$ is normal.

The family of closed sets in $\mathbb{R}$ *is* $\varkappa = \{\emptyset, \mathbb{R}\} \cup \{[\, \boldsymbol{a}, \infty)/\, \boldsymbol{a} \in \mathbb{R}\}$.

Hence A and B are disjoint sets in $\mathbb{R}$, then $A = \emptyset$ *or* $B = \emptyset$. Hence if $A = \emptyset$, then $G = \emptyset$ *and* $H = \mathbb{R}$ are disjoint open sets containing A and B respectively. Hence $\langle \mathbb{R}, \mathfrak{J} \rangle$ is normal space.

II] $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a regular space.

Let $F = [\,1, \infty)$. Then F is a closed set in $\mathbb{R}$ and $0 \notin F$. As the only open set containing F is $\mathbb{R}$, we get $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a regular space.

Hence form (I) and (II), $\langle \mathbb{R}, \mathfrak{J} \rangle$ is not a completely regular space.

Problem 3: Every metric space is a completely regular space.

Solution: Let $\langle X, d \rangle$ be a metric space and let $\mathfrak{J}$ denote the topology on X induced by the metric d. Let F be any closed set in X and $x \notin F$ $(x \in X)$. Then $\{x\} \cap F = \emptyset$ *and* $\{x\}$ is a closed set in X (Since $\langle X, \mathfrak{J} \rangle$ is a T_1 – space, (see T_1 – space). As every metric space is normal (see Normal spaces). By Urysohn's Lemma, $\exists$ continuous function $f : X \longrightarrow [0, 1]$ such that $f(\{x\}) = \{0\}$ and $f(F) = \{1\}$. But then $f(x) = 0$ and $f(F) = \{1\}$.

Therefore $\langle X, \mathfrak{J} \rangle$ is a completely regular space.

Exercises

I) Let $\langle X, \mathfrak{J} \rangle$ be a completely regular space. Show that for any pair of disjoint subsets A and B such that A is compact and B is closed in X, there exists a real valued continuous function f on X such that $f(A) = \{0\}$ *and* $f(B) = \{1\}$.

II) Prove or disprove the following statements.

1) Every completely regular space is regular.
2) Every regular space is completely regular.

3) Homeomorphic image of a completely regular space is a completely regular space.
4) Subspace of a completely regular space is a completely regular space.
5) A normal space is completely regular if it is regular.
6) Any subspace of a normal, completely regular is regular.
7) Any subspace of a normal, regular is completely regular.
8) Any compact, regular space is completely regular.
9) Any compact, T_2- $space$ is completely regular.
10) Any countably compact, T_2- $space$ is completely regular.

Unit 19

Product Spaces and Quotient Spaces

§1 Definition and Basic concepts.

§2 Product Invariant Properties.

§3 Quotient topology.

Unit 19: *Product Spaces and Quotient Spaces*

§1 Definition and Basic concepts

Theorem 1.1: Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two T-spaces.

$X \times Y = \{(x, y) \mid x \in X\ ,\ y \in Y\ \}$. Let $\mathfrak{B} = \{G_1 \times G_2 | \ G_1 \in \mathfrak{J}_1$ and $G_2 \in \mathfrak{J}_2\}$

Then $\mathfrak{B}$ is a base for some topology on $X \times Y$.

Proof : Obviously, $\mathfrak{B}$ is a family of subsets of $X \times Y$. As $X \in \mathfrak{J}_1$ and $Y \in \mathfrak{J}_2$, we get

$X \times Y = \bigcup\{B | \ B \in \mathfrak{B}\}$.

Further let $G_1 \times G_2 \in \mathfrak{B}$, $H_1 \times H_2 \in \mathfrak{B}$ and $(x, y) \in (G_1 \times G_2) \cap (H_1 \times H_2)$.

Then $(x, y) \in (G_1 \cap G_2) \times (H_1 \cap H_2)$. As $G_1 \cap G_2 \in \mathfrak{J}_1$ and $H_1 \times H_2 \in \mathfrak{J}_2$ we get

$(G_1 \cap G_2) \times (H_1 \cap H_2) \in \mathfrak{B}$. Thus $(x, y) \in (G_1 \cap G_2) \times (H_1 \cap H_2) = (G_1 \times H_1) \cap (G_2 \times H_2)$.

This shows that both the conditions of the **Theorem** are satisfied. Hence $\mathfrak{B}$ is base for some topology $\mathfrak{J}$ on $X \times Y$.

Definition 1.2: The topology $\mathfrak{J}$ defined on $X \times Y$ for which

$\mathfrak{B} = \{G \times H \mid G \in \mathfrak{J}_1$ and $H \in \mathfrak{J}_2\}$ is a base is called the product topology on $X \times Y$ and the T-space $\langle X \times Y, \mathfrak{J} \rangle$ is called product space, where $\mathfrak{J}$ is product topology on $X \times Y$.

Theorem1.3: Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two T-spaces. If $\mathfrak{B}_1$ is a base for $\mathfrak{J}_1$ and $\mathfrak{B}_2$ is a base for $\mathfrak{J}_2$, then $\mathfrak{B}_1 \times \mathfrak{B}_2$ is base for the product topology on $X \times Y$.

Proof:

(1) $\mathfrak{B}_1 \subseteq \mathfrak{J}_1$, $\mathfrak{B}_2 \subseteq \mathfrak{J}_2 \ \ \Rightarrow \ \ \mathfrak{B}_1 \times \mathfrak{B}_2 \subseteq \mathfrak{B}$.

(2) Let O be any open set containing (x, y) in the product space $X \times Y$. Then as $\mathfrak{B}$ is base for , we get, $\exists\ G \in \mathfrak{J}_1$ and $H \in \mathfrak{J}_2$ such that $(x, y) \in G \times H \subseteq$ O .
   As $x \in G$ and $G \in \mathfrak{J}_1$, $\exists\ B_1 \in \mathfrak{B}_1$ such that $x \in B_1 \subseteq G$.
   Similarly $\exists\ B_2 \in \mathfrak{B}_2$ such that $y \in B_2 \subseteq H$.
   Thus $(x, y) \in B_1 \times B_2 \subseteq G \times H \subseteq$ O.

As $B_1 \times B_2 \in \mathfrak{B}$ we get, for given $(x, y) \in \mathrm{O}$, $\mathrm{O} \in \mathfrak{J}$ $\exists\, B_1 \times B_2 \in \mathfrak{B}$ such that $(x, y) \in B_1 \times B_2 \subseteq \mathrm{O}$.

Hence from (1) and (2), $\mathfrak{B}$ is a base for the product topology $\mathfrak{J}$ on $X \times Y$.

Problem1.4: Let $X = \{a, b, c\}$, $\mathfrak{J}_1 = \{\emptyset, \{a\}, X\}$, $Y = \{p, q, r, s\}$ and

$\mathfrak{J}_2 = \{\emptyset, \{p\}, \{q\}, \{p, q\}, \{r, s\}, \{p, r, s\}, \{q, r, s\}, Y\}$.

Find the base for the product topology of $X \times Y$.

Solution: $\mathfrak{B}_1$ is the base for $\mathfrak{J}_1$, where $\mathfrak{B}_1 = \{\{a\}, X\}$. $\mathfrak{B}_2$ is the base for $\mathfrak{J}_2$, where $\mathfrak{B}_2 = \{\{p\}, \{q\}, \{r, s\}\}$.

$$\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2 = \{\{a\} \times \{p\}, \{a\} \times \{q\}, \{a\} \times \{r, s\}, X \times \{p\}, X \times \{q\}, X \times \{r, s\}\}$$

$$= \left\{ \begin{array}{c} \{(a,p)\},\ \{(a,q)\},\ \{(a,r),(a,s)\},\ \{(a,p),(b,p),(c,p)\}, \\ \{(a,q),(b,q),(c,q)\},\ \{(a,r),(b,r),(c,r),(a,s),(b,s),(c,s)\} \end{array} \right\}$$

This family $\mathfrak{B}$ is the base for the product topology $\mathfrak{J}$ on $X \times Y$.

Theorem1.5: Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two T-spaces. Let $\langle X \times Y, \mathfrak{J} \rangle$ be a product space. Let $\prod_X : X \times Y \to X$ and $\prod_Y : X \times Y \to Y$ be the projection mappings. Then

(I) $\prod_X$ and $\prod_Y$ are continuous, open mappings.

(II) The product topology $\mathfrak{J}$ on $X \times Y$ is the smallest topology for which the projections are continuous.

Proof:

(I)

1) $\prod_X : X \times Y \to X$. To prove that $\prod_X$ is continuous. Let G be any open set in X.
   Let $G \in \mathfrak{J}_1 \Rightarrow G \times Y \in \mathfrak{J}$ (by definition of $\mathfrak{J}$) and hence, $\prod_X^{-1}(G) = G \times Y \in \mathfrak{J}$. But this shows that $\prod_X$ is continuous.

2) To prove that $\prod_X$ is open. Let $O \in \mathfrak{J}$ then by definition of $\mathfrak{J}$,

$$\mathrm{O} = \bigcup \{G_\lambda \times H_\lambda \mid \lambda \in \Lambda,\ G_\lambda \in \mathfrak{J}_1,\ H_\lambda \in \mathfrak{J}_2\}$$

   (by definition of the base).

   Hence,

$$\prod_X(\mathrm{O}) = \prod_X \left[\bigcup \{G_\lambda \times H_\lambda \mid \lambda \in \Lambda,\ G_\lambda \in \mathfrak{J}_1,\ H_\lambda \in \mathfrak{J}_2\}\right]$$

$$= \bigcup \{ \prod_X [G_\lambda \times H_\lambda] \mid \lambda \in \Lambda,\ G_\lambda \in \mathfrak{J}_1,\ H_\lambda \in \mathfrak{J}_2 \}$$

$$= \bigcup \{ G_\lambda \mid \lambda \in \Lambda,\ G_\lambda \in \mathfrak{J}_1 \} \in \mathfrak{J}_1 \ldots (as\ \mathfrak{J}_1\ is\ a\ topology)$$

This shows that $\prod_X$ is an open map.

3) As in 1) and 2) we can prove that $\prod_Y$ is a continuous, open mapping.

(II) Let $\mathfrak{J}^*$ be any other topology on $X \times Y$ such that the projection maps $\prod_X$ and $\prod_Y$ are both continuous. To prove that $\mathfrak{J} \leq \mathfrak{J}^*$.

Let $O \in \mathfrak{J}$. Then by definition of $\mathfrak{J}$,

$$O = \bigcup \{ G_\lambda \times H_\lambda \mid\ G_\lambda \in \mathfrak{J}_1,\ H_\lambda \in \mathfrak{J}_2,\ \lambda \in \Lambda \}$$

$$= \bigcup \{ (G_\lambda \cap H_\lambda) \cap (X \times Y) \mid\ G_\lambda \in \mathfrak{J}_1,\ H_\lambda \in \mathfrak{J}_2,\ \lambda \in \Lambda, \}$$

$$= \bigcup \{ (G_\lambda \times Y) \cap (X \times H_\lambda) \mid\ G_\lambda \in \mathfrak{J}_1,\ H_\lambda \in \mathfrak{J}_2,\ \lambda \in \Lambda, \}$$

$$= \bigcup \{ \textstyle\prod_X^{-1}(G_\lambda) \cap \prod_Y^{-1}(H_\lambda) \mid\ G_\lambda \in \mathfrak{J}_1,\ H_\lambda \in \mathfrak{J}_2,\ \lambda \in \Lambda, \}$$

As for any $G_\lambda \in \mathfrak{J}_1$, $\prod_X^{-1}(G_\lambda) \in \mathfrak{J}^*$ and for any $H_\lambda \in \mathfrak{J}_2$, $\prod_Y^{-1}(H_\lambda) \in \mathfrak{J}^*$, we get,
$\prod_X^{-1}(G_\lambda) \cap\ \prod_Y^{-1}(H_\lambda) \in \mathfrak{J}^*$. $\mathfrak{J}^*$ being topology on $X \times Y$, we get $O \in \mathfrak{J}^*$.
Thus $O \in \mathfrak{J} \implies O \in \mathfrak{J}^*$ and hence $\mathfrak{J} \subseteq \mathfrak{J}^*$.
This shows that the product topology $\mathfrak{J}$ on $X \times Y$ is the smallest topology for which the projections are continuous.

Theorem1.6: For any fixed $y \in Y$, $g: X \times \{y\} \to X$ defined by $g(x,y) = x$, $\forall\, x \in X$ is a homeomorphism.

For any fixed $x \in X$, $h: \{x\} \times Y \to Y$ defined by $h(x,y) = y$, $\forall\, y \in Y$ is a homeomorphism.

Proof: I] To prove that g is a homeomorphism.

(1) g is one-one:

Let, $g(x_1, y) = g(x_2, y)$ for x_1 , $x_2 \in X$.

Then $x_1 = x_2 \implies (x_1\ ,\ y) = (x_2\ ,\ y)$.

But this shows that g is one-one.

(2) g is onto:

Let $x \in X$. Then $(x, y) \in X \times \{y\}$ and $g(x, y) = x$. This shows that g is onto.

(3) g is continuous:

As $\prod_X : X \times Y \to X$ is continuous and g is the restriction of $\prod_X$ to the subspace $X \times \{y\}$, we get, g is a continuous mapping.

(4) g is open:

Let O^* be any open set in $X \times \{y\}$. Then $\exists$ an open set O in $X \times Y$ such that $O^* = O \cap (X \times \{y\})$.

As $O = \bigcup\{G_\lambda \times H_\lambda \mid \ G_\lambda \in \mathfrak{J}_1 ,\ H_\lambda \in \mathfrak{J}_2 , \lambda \in \Lambda\}$

We get,

$$\begin{aligned}
g(O^*) &= g[O \cap (X \times \{y\})] \\
&= g\left[\bigcup\{(G_\lambda \times H_\lambda) \cap (X \times \{y\}) \mid \ G_\lambda \in \mathfrak{J}_1 ,\ H_\lambda \in \mathfrak{J}_2 , \lambda \in \Lambda\}\right] \\
&= g\left[\bigcup\{(G_\lambda \cap X) \times (H_\lambda \cap \{y\}) \mid \ G_\lambda \in \mathfrak{J}_1 ,\ H_\lambda \in \mathfrak{J}_2 , \lambda \in \Lambda\}\right] \\
&= g\left[\bigcup\{G_\lambda \times (H_\lambda \cap \{y\}) \mid \ G_\lambda \in \mathfrak{J}_1 ,\ H_\lambda \in \mathfrak{J}_2 , \lambda \in \Lambda\}\right] \\
&= \bigcup\{g[G_\lambda \times (H_\lambda \cap \{y\})] \mid \ G_\lambda \in \mathfrak{J}_1 ,\ H_\lambda \in \mathfrak{J}_2 , \lambda \in \Lambda\} \\
&= \bigcup\{g[G_\lambda \times \{y\}] \mid \ G_\lambda \in \mathfrak{J}_1,\ y \in H_\lambda , \lambda \in \Lambda\} \\
&= \bigcup\{g[G_\lambda \times \emptyset] \mid \ G_\lambda \in \mathfrak{J}_1,\ y \in H_\lambda , \lambda \in \Lambda\} \\
&= \bigcup\{G_\lambda \mid \ G_\lambda \in \mathfrak{J}_1\} \ \ldots (by\ definition\ of\ g) \\
&\in \mathfrak{J}_1.
\end{aligned}$$

This shows that g is open.

From (1) to (4) we get, g is homeomorphism.

Hence $X \times \{y\}$ is homeomorphic with X for any fixed $y \in Y$.

II] As in I] we can prove that $\{x\} \times Y$ is homeomorphic with Y under the homeomorphism $h: \{x\} \times Y \to Y$ defined by $h(x, y) = y\ ,\ \forall\, y \in Y$.

§2 Product Invariant Properties

Theorem 2.1: Let $\langle X, \mathfrak{J}_1\rangle$ and $\langle Y, \mathfrak{J}_2\rangle$ be two T-spaces. Let $\langle X \times Y, \mathfrak{J}\rangle$ be a product space. The product space $X \times Y$ is connected if and only if both X and Y are connected spaces.

Proof: Only if part.

Let $X \times Y$ be a connected space.

To prove that X and Y both are connected spaces.

We know that $\prod_X : X \times Y \to X$ is continuous, onto, open mapping.

Hence $X \times Y$ is connected $\Longrightarrow X$ is connected.

Similarly, as $\prod_Y : X \times Y \to Y$ is continuous, onto, open mapping, $X \times Y$ is connected implies Y is connected.

Hence, if the product space $X \times Y$ is connected then both X and Y are connected.

If part.

Let X and Y be connected spaces.

To prove that product space $X \times Y$ is connected.

Let (x_1, y_1) and (x_2, y_2) be distinct points of $X \times Y$. As $\{x_1\} \times Y$ is homeomorphic with Y (see Theorem …). We get $\{x_1\} \times Y$ is connected space (see Theorem …). Similarly $X \times \{y_2\}$ is a homeomorphic image of X (see Theorem …). As Y is a connected set we get $X \times \{y_2\}$ is connected set(see Theorem …). Further as $(x_1, y_2) \in (\{x_1\} \times Y) \cap (X \times \{y_2\})$, we get, $(\{x_1\} \times Y) \cap (X \times \{y_2\}) \neq \emptyset$. Hence by Theorem …, we get, $(\{x_1\} \times Y) \cup (X \times \{y_2\})$ is a connected space.

Thus for$(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$, $\exists$ a connected set $(\{x_1\} \times Y) \cup (X \times \{y_2\})$ containing them. Hence $X \times Y$ is a connected space.

Theorem2.2:- Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two topological spaces. Let $\langle X \times Y, \mathfrak{J} \rangle$ be their product space. The product space $X \times Y$ is compact if and only if each of the spaces is compact.

Proof:- Only if part.

Let $X \times Y$ be a compact space.

We know that continuous image of a compact space is compact.

As $\prod_X : X \times Y \to X$ is continuous, onto, open mapping, we get X is compact (see Theorem …).

Similarly, $\prod_Y : X \times Y \to Y$ is continuous, onto, open mapping. Hence Y is a compact space.

If part .

Let X and Y be compact spaces. To prove that $X \times Y$ is a compact space. It is enough to prove that any basic open cover of $X \times Y$ has a finite sub-cover.

Let $\{G_\lambda \times H_\lambda \mid \lambda \in \Lambda\}$ be any basic open cover for $X \times Y$. Then $\{G_\lambda \times H_\lambda \mid \lambda \in \Lambda\}$ is also a basic open cover for $\{x\} \times Y$, for some fixed $x \in X$.

As $\{x\} \times Y$ is a homeomorphic with Y and Y is compact, we get $\{x\} \times Y$ is compact.

Hence $\exists$ finite sub-cover for the given basic open cover for $\{x\} \times Y$.

Let $\{x\} \times Y \subseteq \bigcup_{i=1}^{n} (G_{\lambda_i} \times H_{\lambda_i})$.

Define $G(x) = \bigcap_{i=1}^{n} G_{\lambda_i}$. Then $G(x) \in \mathfrak{J}_1$ $\forall\, x \in X$ and $x \in G(x)$.

Consider the family $\left\{G(x) \times H_{\lambda_i} \mid 1 \leq \mathrm{i} \leq n\right\}$. Then this family forms a finite open cover for $\{x\} \times Y$.

$x \in G(x)$ and $G(x) \in \mathfrak{J}_1 \implies \{G(x)\}_{x \in X}$ forms an open cover for X. As X is compact this open cover of X has a finite sub-cover.

Let $X = \bigcup_{j=1}^{m} G(x_j)$.

Now for each $G(x_j)$, find G_{λ_j} such that $G(x_j) \subseteq G_{\lambda_j}$ $\forall\, j$, $1 \leq j \leq m$, $\lambda_j \in \Lambda$.

Find corresponding H_{λ_j} , $\lambda_j \in \Lambda$ so that $G_{\lambda_j} \times H_{\lambda_j} \in \{G_\lambda \times H_\lambda \mid \lambda \in \Lambda\}$.

Thus the basic open cover $\{G_\lambda \times H_\lambda \mid \lambda \in \Lambda\}$ of $X \times Y$ has a finite sub-cover $\left\{G_{\lambda_j} \times H_{\lambda_j} \mid \lambda_j \in \Lambda, 1 \leq j \leq m\right\}$.

Hence $X \times Y$ is a compact space.

Theorem2.3: The product space $\langle X \times Y, \mathfrak{J} \rangle$ is a first axiom space iff both $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ are first axiom spaces.

Proof: Only if part.

Let $\langle X \times Y, \mathfrak{J} \rangle$ be a first axiom space. We know that the projection map $\prod_X : X \times Y \to X$ is a continuous, onto, open map.

Hence, X is a first axiom spaces. Similarly, as the projection map $\prod_Y : X \times Y \to Y$ is continuous, onto and open map, Y is a first axiom spaces.

If part.

Let $\langle X, \mathfrak{I}_1 \rangle$ and $\langle Y, \mathfrak{I}_2 \rangle$ be first axiom spaces. To prove that $\langle X \times Y, \mathfrak{I} \rangle$ is a first axiom spaces. Let $(x, y) \in X \times Y$. Then $x \in X$ and $y \in Y$. As $\langle X, \mathfrak{I}_1 \rangle$ is a first axiom spaces, $\exists$ a countable local base say $\{B_n(x)\}$ at x in $\langle X, \mathfrak{I}_1 \rangle$. Similarly, as $\langle Y, \mathfrak{I}_2 \rangle$ is a first axiom spaces, $\exists$ a countable local base say $\{D_n(y)\}$ at y in $\langle Y, \mathfrak{I}_2 \rangle$.

Define $\mathfrak{D} = \{ B_i(x) \times D_j(y) \mid i, j \in N \}$.

(1) $B_i(x) \times D_j(y) \in \mathfrak{I} \ \forall \, i, j \in N$.

(2) $(x, y) \in B_i(x) \times D_j(y) \ \forall \, i, j \in N$.

(3) $Let\ (x, y) \in G \times H$ where $G \times H \in \mathfrak{B}$.

Where $\mathfrak{B}$ is a base for the product topology $\mathfrak{I}$. Then $x \in G$ and $G \in \mathfrak{I}_1$imply $\exists \ x \in N$ such that $x \in B_n(x) \subseteq G$. similarly $y \in H$ and $H \in \mathfrak{I}_2$ imply $\exists \ m \in N$ such that $y \in D_m(y) \subseteq G \times H$. Shows that the family $\{ B_i(x) \times D_j(y) \}$ forms a countable base at (x, y) in $\langle X \times Y, \mathfrak{I} \rangle$. Hence $\langle X \times Y, \mathfrak{I} \rangle$ is a First axiom spaces.

Theorem2.4: The product space $\langle X \times Y, \mathfrak{I} \rangle$ is a second axiom space if and only if both $\langle X, \mathfrak{I}_1 \rangle$ and $\langle Y, \mathfrak{I}_2 \rangle$ are second axiom spaces.

Proof:- Only if part .

Let $X \times Y$ be a second axiom space. Consider the projection map $\prod_X : X \times Y \to X$. Then $\prod_X$ is continuous, open and onto map.

Hence, we get X is a second axiom space. Similarly, as the projection map $\prod_X : X \times Y \to X$ is a continuous open and onto, we get Y is a second axiom space.

If part .

Let $\langle X, \mathfrak{I}_1 \rangle$ and $\langle Y, \mathfrak{I}_2 \rangle$ be second axiom spaces. To prove that $\langle X \times Y, \mathfrak{I} \rangle$ is a second axiom space. As $\langle X, \mathfrak{I}_1 \rangle$ is a second axiom space, $\exists$ a countable base say $\{B_n\}$ for $\mathfrak{I}_1$. Similarly, as $\langle Y, \mathfrak{I}_2 \rangle$ is a second axiom space, $\exists$ a countable base say $\{ D_n\}$ for $\mathfrak{I}_2$.

Consider the family $\{ B_i(x) \times D_j(y) \mid i, j \in \mathbb{N} \}$. This will form a countable base for $\mathfrak{I}$. Hence $\langle X \times Y, \mathfrak{I} \rangle$ is a second axiom space.

Theorem 2.5:-The product space $\langle X \times Y, \mathfrak{I} \rangle$ is a completely regular space if and only if both $\langle X, \mathfrak{I}_1 \rangle$ and $\langle Y, \mathfrak{I}_2 \rangle$ are completely regular spaces.

Proof**:- Only if part .**

Let $X \times Y$ be a completely regular space. Consider the projection map $\prod_X : X \times Y \to X$. Then $\prod_X$ is continuous, open and onto map.

Hence , we get X is a completely regular space. Similarly, as the projection map $\prod_X$: $X \times Y \to X$ is a continuous open and onto, we get Y is a completely regular space.

If part.

Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two completely regular spaces.

To prove that the product space $\langle X \times Y, \mathfrak{J} \rangle$ is a completely regular space. Let $(x, y) \in X \times Y$ and O be an open set in $X \times Y$ such that $(x, y) \in$ O. By definition of $\mathfrak{J}$, $\exists\ G \in \mathfrak{J}_1\ and\ H \in \mathfrak{J}_2$ such that $(x, y) \in G \times H \subseteq$ O. Thus $x \in G\ ,\ y \in H$.

X is completely regular $\Longrightarrow \exists$ a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(X - G) = \{1\}$ **.**

Y is completely regular $\Longrightarrow \exists$ a continuous function $g : Y \to [0,1]$ such that $g(y) = 0$ and $g(X - H) = \{1\}$.

Define $h : X \times Y \to [0,1]$ by $h(x, y) = max\ \{f(x)\ ,\ g(y)\}\ \forall\ x\ ,\ y \in X \times Y$.

(1) Then h is a continuous function [since both f and g are continuous]

(2) $h(x, y) = max\ \{f(x)\ ,\ g(y)\} = 0$

(3) Let $(u\ , v) \in (X \times Y) - (G \times H)$

$\Longrightarrow\ (u\ , v) \notin (G \times H)$

$\Longrightarrow\ u \notin G$ or $v \notin H$.

Let $u \notin G$. Then $f(u) = 1$.

$h(u\ , v) = max\ \{f(u)\ ,\ g(v)\} = 1$.

Let $v \notin H$. Then $g(v) = 1$.

$h(u\ , v) = max\ \{f(u)\ ,\ g(v)\} = 1$.

Thus $h[(X \times Y) - (G \times H)] = \{1\}$.

As $G \times H \subseteq$ O , $[X \times Y -$ O$] \subseteq (X \times Y) - (G \times H)$. Hence, $h[X \times Y -$ O$] = \{1\}$.

This shows that $X \times Y$ is a completely regular space.

Theorem2.6: The product space $\langle X \times Y, \mathfrak{J} \rangle$ is a T_0 – space if and only if both $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ are T_0 – spaces.

Proof**: Only if part.**

Let $X \times Y$ be a T_0 – space. Consider the projection map $\prod_X : X \times Y \to X$. Then $\prod_X$ is continuous, open and onto map.

Hence, we get X is a T_0 – space. Similarly, as the projection map $\prod_X : X \times Y \to X$ is a continuous open and onto, we get Y is a T_0 – space.

If part.

Let $\langle X, \Im_1 \rangle$ and $\langle Y, \Im_2 \rangle$ be two T_0 – spaces and $\langle X \times Y, \Im \rangle$ be their product space. To prove that $\langle X \times Y, \Im \rangle$ is a T_0 – space. Let $(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume that $x_1 \neq x_2$. As X is a T_0 – space and $x_1 \neq x_2$ in X, $\exists$ open set G in $\langle X, \Im \rangle$ such that $x_1 \in G$ but $x_2 \notin G$. But then $(x_1, y_1) \in G \times Y$ and $(x_2, y_2) \notin G \times Y$. As $G \times Y$ is open sets in the product space $X \times Y$, the result follows.

Theorem2.7: The product space $\langle X \times Y, \Im \rangle$ is a T_1 – space if and only if both $\langle X, \Im_1 \rangle$ and $\langle Y, \Im_2 \rangle$ are T_1 – spaces.

Proof**: Only if part.**

Let $X \times Y$ be a T_1 – space. Consider the projection map $\prod_X : X \times Y \to X$. Then $\prod_X$ is continuous, open and onto map.

Hence, we get X is a T_1 – space. Similarly, as the projection map $\prod_X : X \times Y \to X$ is a continuous open and onto, we get Y is a T_1 – space.

If part.

Let $\langle X, \Im_1 \rangle$ and $\langle Y, \Im_2 \rangle$ be two T_1 – spaces and $\langle X \times Y, \Im \rangle$ be their product space. To prove that $\langle X \times Y, \Im \rangle$ is a T_1 – space. Let $(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume that $x_1 \neq x_2$. As X is a T_1 – space and $x_1 \neq x_2$ in X, $\exists$ open sets G and H in $\langle X, \Im \rangle$ such that $x_1 \in G$ but $x_2 \notin G$and $x_2 \in H$ but $x_1 \notin H$. But then $(x_1, y_1) \in G \times Y$, $(x_2, y_2) \notin G \times Y$ and $(x_1, y_1) \notin H \times Y$, $(x_2, y_2) \in H \times Y$. As $G \times Y$ and $H \times Y$ are open sets in the product space $X \times Y$, the result follows.

Theorem2.8:- The product space $\langle X \times Y, \Im \rangle$ is a T_2 – space if and only if both $\langle X, \Im_1 \rangle$ and $\langle Y, \Im_2 \rangle$ are T_2 – spaces.

Proof**:- Only if part.**

Let $X \times Y$ be a T_2 – space. Consider the projection map $\prod_X : X \times Y \to X$. Then $\prod_X$ is continuous, open and onto map.

Hence, we get X is a T_2 – space. Similarly, as the projection map $\prod_X : X \times Y \to X$ is a continuous open and onto, we get Y is a T_2 – space.

If part.

Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be two T_2 – spaces.

To prove that the product space $\langle X \times Y, \mathfrak{J} \rangle$ is a T_2 – space. Let (x_1, y_1) and (x_2, y_2) be distinct points of $X \times Y$. Then either $x_1 \neq x_2$ and $y_1 \neq y_2$. Let $x_1 \neq x_2$. As X is T_2 – space, and $x_1 \neq x_2$, in X, $\exists\ G, H \in \mathfrak{J}_1$ such that $x_1 \in G$, $x_2 \in H$ and $G \cap H = \emptyset$. Define $O_1 = G \times Y$ and $O_2 = H \times Y$. Then O_1 , $O_2 \in \mathfrak{J}$, $O_1 \cap O_2 = (G \cap H) \times Y = \emptyset$, $(x_1, y_1) \in O_1$ and $(x_2, y_2) \in O_2$. Hence, the product space $\langle X \times Y, \mathfrak{J} \rangle$ is a T_2 – space.

Theorem2.9: If the product space $\langle X \times Y, \mathfrak{J} \rangle$ is a separable, then both $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ are separable spaces.

Proof:- Only if part.

Let $\langle X \times Y, \mathfrak{J} \rangle$ be a separable space. Then $\exists$ a countable dense set say $A \times B$ in $X \times Y$.

Claim: A is countable dense set in X.

Let $x \in X$. If possible, assume that $x \notin \bar{A}$. Then for any $y \in Y$, $(x, y) \in \overline{A \times B}$.

$x \notin d(A)$, $\exists\ G \in \mathfrak{J}_1$ such that $x \in G$ and $G \cap A - \{x\} = \emptyset$.

But then $G \cap A = \emptyset$ (since $x \notin \bar{A} \implies x \notin A$).

In this case $G \cap Y \in \mathfrak{J}$, $(x, y) \in G \times Y$ and

$(G \times Y) \cap (A \times B) - \{(x, y)\} = (G \cap A) \times (Y \cap B) - \{(x, y)\} = \emptyset$.

This contradicts the fact that $(x, y) \in \overline{A \times B}$.

Hence, each $x \in X$ must be in $\bar{A}$ i.e. $\bar{A} = X$.

Thus, there exists a countable dense set A in X, X is separable space.

Similarly, we can prove that the countable set B is dense in Y.

Hence X and Y are separable spaces when $X \times Y$ is separable space.

If part.

Let $\langle X, \mathfrak{J}_1 \rangle$ and $\langle Y, \mathfrak{J}_2 \rangle$ be separable spaces.

To prove that $\langle X \times Y, \mathfrak{J} \rangle$ is separable.

Let C and D be countable dense sets in X and Y respectively.

Then $C \times D$ is a countable subset of $X \times Y$.

Claim that $\overline{C \times D} = X \times Y$.

Let $(x, y) \in X \times Y$. If $x \notin \overline{C} = C \cup d(c)$, we get $x \notin d(C)$.

Hence $\exists\, G \in \mathfrak{J}_1$ such that $G \cap C - \{x\} = \emptyset$. But then $G \cap C = \emptyset$.

Consider $G \times Y \in \mathfrak{J}$. As $(x, y) \in G \times Y$, we get $(G \cap Y) \cap (C \times D) - \{(x, y)\} \neq \emptyset$.

But $(G \cap Y) \cap (C \times D) - \{(x, y)\} = (G \cap Y) \cap (Y \times D) - \{(x, y)\}$

$$= \emptyset \quad \ldots (G \cap C = \emptyset)$$

This is absurd.

Hence $(x, y) \in G \times Y \implies (x, y) \in d(C \times D)$

$\therefore\ \overline{C \times D} = X \times Y$.

Hence $C \times D$ is a countable dense set in $X \times Y$.

Therefore $X \times Y$ is a separable space.

§3 Quotient topology

We know that product topology is the smallest topology on the domain for which projection maps are continuous .Also we know that indiscrete topology is the smallest topology on the co –domain Y for which any function $f: X \to Y$ is continuous . Now our aim is to find the largest topology on Y for which $f: X \to Y$ is continuous, if exists.

Theorem 3.1: Let $\langle X, \mathfrak{J} \rangle$ be topological space *and* let f be a mapping of X onto a set Y. Define $\mathfrak{J}^* = \{G \subseteq Y \mid f^{-1}(G) \in \mathfrak{J}\}$ .Then

(1) $\mathfrak{J}^*$ is a topology on Y.

(2) $f: \langle X, \mathfrak{J} \rangle \longrightarrow \langle Y, \mathfrak{J}^* \rangle$ is a continuous function.

(3) $\mathfrak{J}^*$ is the largest topology on Y for which $f: X \longrightarrow Y$ is continuous.

(4) $F \subseteq Y$ is closed in $\langle Y, \mathfrak{J}^* \rangle$ if and only if $f^{-1}(F)$ is closed in $\langle X, \mathfrak{J} \rangle$.

Proof: (1) $\mathfrak{J}^*$ is a topology on Y.

(i) $f^{-1}(\emptyset) = \emptyset$, $\emptyset \in \mathfrak{J} \implies \emptyset \in \mathfrak{J}^*$.

$f^{-1}(Y) = X$, $X \in \mathfrak{J} \implies Y \in \mathfrak{J}^*$ (since f is onto)

(ii) Let $A, B \in \mathfrak{J}^*$. Then $f^{-1}(A) \in \mathfrak{J}$ and $f^{-1}(B) \in \mathfrak{J}$.

Therefore, $f^{-1}(A) \cap f^{-1}(B) \in \mathfrak{J}$ i.e. $f^{-1}(A \cap B) \in \mathfrak{J}$.

But this shows that $A \cap B \in \mathfrak{J}^*$.

(iii) $A_\lambda \in \mathfrak{J}^*\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set . Then $f^{-1}(A_\lambda) \in \mathfrak{J}\ \ \forall\, \lambda \in \Lambda$.

$\Im$ being a topology, $\bigcup_{\lambda\in\Lambda} f^{-1}(A_\lambda) \in \Im$ *i.e.* $f^{-1}\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] \in \Im$.

But this shows that $\bigcup_{\lambda\in\Lambda} A_\lambda \in \Im^*$.

From (i), (ii) and (iii) we get, $\Im^*$ is a topology on Y.

(2) $f: \langle X, \Im\rangle \longrightarrow \langle Y, \Im^*\rangle$ is a continuous function.

Let $G \in \Im^*$. Then by definition of $\Im^*$, $f^{-1}(G) \in \Im$. Hence f is continuous.

(3) Let $\Im_1$ denote a topology on Y such that $f: \langle X, \Im\rangle \longrightarrow \langle Y, \Im_1\rangle$ is continuous function.

To prove that $\Im_1 \subseteq \Im^*$.

Let $G \in \Im_1$. Then by continuity of f, $f^{-1}(G) \in \Im$. But then by definition of $\Im^*$, $G \in \Im^*$.

Thus $G \in \Im_1 \Longrightarrow G \in \Im^*$. Hence $\Im_1 \subseteq \Im^*$.

This shows that, $\Im^*$ is the largest topology on Y for which $f: X \longrightarrow Y$ is continuous.

(4) $F \subseteq Y$ is closed in $\langle Y, \Im^*\rangle$

$\Leftrightarrow Y - F \in \Im^*$.

$\Leftrightarrow f^{-1}(Y - F) \in \Im$.

$\Leftrightarrow X - f^{-1}(F) \in \Im$.

$\Leftrightarrow f^{-1}(F)$ is closed in $\langle X, \Im\rangle$.

Definition 3.2: Let $\langle X, \Im\rangle$ be a topological space. Y is any non-empty set. $f: X \to Y$ is an onto mapping. The largest topology $\Im^*$ on Y for *which* f is $\Im$ - $\Im^*$ continuous, is called the quotient topology for Y (relative to f and $\Im$) and the map f is called the quotient map.

Note that $G \in \Im^* \Leftrightarrow f^{-1}(G) \in \Im$.

Theorem 3.3: Let $\langle X, \Im\rangle$ and $\langle Y, \Im'\rangle$ be two topological spaces. $f: X \to Y$ is onto, continuous map. If f is either open or closed, then $\Im'$ is the quotient topology on Y.

Proof: I] Let $f: X \to Y$ be continuous, onto, open.

To prove that $\Im'$ is the quotient topology on Y.

By definition of quotient topology $\Im^*$ on Y, $\Im' \leq \Im^*$.

To prove that $\mathfrak{J}^* \leq \mathfrak{J}'$.

Let $G \in \mathfrak{J}^*$. Then $f^{-1}[G] \in \mathfrak{J}$. As f is open, $f\big[f^{-1}[G]\big] \in \mathfrak{J}'$.

Hence $G \in \mathfrak{J}'$. Therefore, $\mathfrak{J}^* \leq \mathfrak{J}'$.

Combining both inclusions, we get, $\mathfrak{J}' = \mathfrak{J}^*$.

II] Let $f: X \to Y$ be continuous and closed.

To prove $\mathfrak{J}' = \mathfrak{J}^*$.

$\mathfrak{J}' \leq \mathfrak{J}^*$, as $\mathfrak{J}^*$ is largest topology for which f is continuous.

Hence to prove $\mathfrak{J}^* \leq \mathfrak{J}'$. Let $G \in \mathfrak{J}^*$.

Then $f^{-1}[G] \in \mathfrak{J} \Rightarrow X - f^{-1}[G]$ is closed in X.

$\Rightarrow f\big[X - f^{-1}[G]\big]$ is closed in Y.

$\Rightarrow f[f^{-1}(Y) - f^{-1}(G)]$ is closed in Y.

$\Rightarrow f[f^{-1}(Y - G)]$ is closed in Y.

$\Rightarrow (Y - G)$ is closed in Y.

$\Rightarrow G$ is open in Y.

$\Rightarrow G \in \mathfrak{J}'$.

Thus $\mathfrak{J}^* \leq \mathfrak{J}'$.Combining both inclusions we get $\mathfrak{J}' = \mathfrak{J}^*$.

Corollary3.4: Let f be continuous map of a compact space $\langle X, \mathfrak{J} \rangle$ onto a Hausdorff space $\langle Y, \mathfrak{J}^* \rangle$. Then $\mathfrak{J}^*$ is the quotient topology on Y.

Proof: We know that a continuous map of a compact space onto a Hausdorff space is a closed map. Hence f is a closed mapping. By Theorem, $\mathfrak{J}^*$ is the quotient topology on Y.

Corollary 3.5: Let $f: X \to Y$ be a continuous map and let Y have the quotient topology (relative to f). Then $g: Y \to Z$ is continuous if and only if $g \circ f$ is continuous.

Proof: Composition of two continuous functions is always continuous.

Hence 'Only if part' follows.

For 'If part', assume that $g \circ f$ is continuous map.

To prove that g is continuous.

Let G is open in Z.

$g \circ f: X \to Z$ is continuous $\Rightarrow [\, g \circ f]^{-1}(G) \in \mathfrak{J}$.

$\Rightarrow [\,f^{-1} \circ g^{-1}](G) \in \mathfrak{J}$

$\Rightarrow\ f^{-1}[g^{-1}(G)] \in \mathfrak{J}$

$\Rightarrow\ g^{-1}(G) \in \mathfrak{J}^*$ ………. (by definition of $\mathfrak{J}^*$)

Thus any G open in Z we get $g^{-1}(G)$ is open in Y. Hence $g: Y \to Z$ is continuous.

Theorem 3.6: Let $\langle X, \mathfrak{J}\rangle$ be a topological space. Y is any non-empty set. $f: X \to Y$ is an onto mapping If X is compact (connected, separable or Lindelof) then so is Y with the quotient topology.

Proof: Since $f: X \to Y$ is continuous, onto, the result follows.

[Continuous image of a compact (connected, separable or Lindelof) space is a compact (connected, separable or Lindelof) space].

Definition: Let $\langle X, \mathfrak{J}\rangle$ be a topological space and R an equivalence relation on X. Let π be the quotient map of X onto the quotient set $X/_R$ of X over R so that $\pi(x) = [x]$ is the equivalence class to which x belongs. Then the quotient space is the family $X/_R$ with quotient topology (relative to π).

Exercises

(I) Show that $X \times Y$ has each of the properties listed below if and only if both X and Y have the same properties .

(i) T_0

(ii) T_1

(iii) T_2

(iv) regularity.

(v) complete regularity.

(vi) first axiom.

(vii) second axiom.

(viii) separability.

(II) By an example show that the product space of two normal spaces need not be normal .

(III) Let $\langle X, \mathfrak{J} \rangle$ be a topological space. Y is any non-empty set.

$f: X \to Y$ is an onto mapping . Show that Y with the quotient topology is a T_1 – space if and only if $f^{-1}[\{y\}]$ is closed in $X,\ \forall\, y \in Y$.

Unit 20 : The Urysohn Metrization Theorem

The Urysohn Metrization Theorem tells us under which conditions a topological space X is metrizable, i.e. when there exists a metric on the underlying set of X that induces the topology of X. The main idea is to impose such conditions on X that will make it possible to embed X into a metric space Y, by homeomorphically identifying X with a subspace of Y.

Let us start with some definitions. A topologies space X is said to be *regular* if for any point $x \in X$ and any closed set $B \subset X$ not containing x, there exist two disjoint open sets containing x and B respectively. The space X is said to be *normal* if for any two disjoint closed sets B_1 and B_2 there exist two disjoint open sets containing B_1 and B_2 respectively.

Example :

An example of a Hausdorff space which is nor normal is given by the set $\mathbb{R}$, where the usual topology is enhanced by requiring that the set $\{1/n \mid n \in \mathbb{N}\}$ is closed. Examples of spaces which are regular but not normal exist, but are complicated.

Lemma : Every regular space with a countable basis is normal.

Proof : First, using regularity and countable basis, construct a countable covering $\{U_i\}$ of B_1 by open sets whose closures do not intersect B_2. Similarly, construct an open countable covering $\{V_i\}$ of B_2 disjoint from B_1. Then define

$$U_n^{'} := U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \quad \text{and} \quad V_n^{'} := V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i$$

Show that these sets are open and the collection $\{U_n^{'}\}$ covers B_1 and $\{V_n^{'}\}$ covers B_2. Finally show that $U' := \bigcup U_n^{'}$ and $V' := \bigcup V_n^{'}$ are disjoint.

Next, we will prove one of the very deep basic results.

Urysohn Lemma : Let X be a normal space, and let A and B be disjoint subsets of X. There exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ for every $x \in A$, and $f(x) = 1$ for every $x \in B$.

Proof : Let Q be the set of rational numbers on the interval [0, 1]. For each rational number q on this interval we will define an open set $U_q \subset X$ such that whenever $p < q$, we have $\overline{U}_p \subset U_q$.

Hint : Enumerate all the rational numbers on the interval (so that the first two elements are 1 and 0) and then define $U_1 = X / B$ and all other U_q 's can be defined inductively by using normality of X.

Now let us extend the definition of U_q to all rational numbers by defining $U_q = f$ if q is negative, and $U_q = X$ if $q > 1$.

Next, for each $x \in X$ define $Q(x)$ to be the set of those rational numbers such that the corresponding set U_q contains x. Show that $Q(x)$ is bounded below and define $f(x)$ as its infimum.

Now we will show that $f(x)$ is the desired function. First, show that if $x \in \overline{U}_r$, then $f(x) \le r$ and if $x \notin U_r$, then $f(x) \ge r$.

Now prove the continuity $f(x)$ of by showing that for any $x_0 \in X$ and an open interval (c, d) containing $f(x_0)$, there exist a neighbourhood U of x_0 such that $f(U) \subset (c,d)$. [Why would this imply continuity ?] For this choose two rational numbers q_1 and q_2 such that $c < q_1 < f(x_0) < q_2 < d$ and take $U = U_{q2} \setminus \overline{U}_{q1}$.

Next, we will construct the metric space Y for the embedding. Actually, as a topoligical space the space Y is simply the product of $\mathbb{N}$ copies of $\mathbb{R}$ with the product topology. Let $\overline{d}(a,b) = \min\{|a-b|,1\}$ be the so-called standard bounded metric on $\mathbb{R}$ [show that this is indeed a metric]. Then if x and y are two points of Y, define,

$$D(x,y) = \sup\left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$$

Show that this is indeed a metric.

Proposition : The metric D induces the product topology on $Y = \mathbb{R}^{\mathbb{N}}$.

Proof : First, let U be open in the metric topology and let $x \in U$. We will find an open set V in the product topology such that $x \subset V \subset U$. Choose and e-ball centered at x, which lies in U. Then choose N large enough so $1/N < e$. Show that the following set satisfies the requirement :

$$V = (x_1 - e, x_1 + e) \times \ldots \times (x_N - e, x_N + e) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

Conversely, consider a basis element $V = \Pi_{i \in \mathbb{N}} V = \Pi_{i \in \mathbb{N}} V_i$ for the product topology, such that V_i is open in $\mathbb{R}$ and $V_i = \mathbb{R}$ for all but finitely many indices $i_1, \ldots, i_k$. Given $x \in V$, we will find an open ball U in metric topology, which contains x and is contained in V. Choose an interval $(x_i - e_i, x_i + e_i)$ contained in V_i such that $e_i < 1$ and define.

$$e = \min\{e / i \mid i = i_1, \ldots, i_k\}$$

Now show that the ball of radius e centered at x is contained in V..

Next we need the following technical result :

Lemma : Let X be a regular space with a countable basis. There exist a countable collection of continuous functions $f_n : X \to [0,1]$ such that for any $x_0 \in X$ and any neighbourhood U of x_0, there exist an index n such that $f_n(x_0) > 0$ and $f_n = 0$ outside U.

Proof : Given x_0 and U, use regularity to choose two open sets B_n and B_m from the countable basis containing x_0 and contained in U such that $\overline{B}_n \subset B_m$. Then use the Urysohn lemma to construct a function $g_{n,m}$ such that $g_{n,m}(\overline{B}_n) = 1$ and $g_{n,m}(X \setminus B_m) = 0$. Now show that this collection of functions satisfies our requirement.

Finally we will prove the main result :

Urysohn Metrization Theorem : Every regular space X with a countable basis is metrizable.

Proof : Given the collection of functions $\{f_n\}$ from the previous lemma, and $Y = \mathbb{R}^{\mathbb{N}}$ with the product topology, we define a map $F : X \to Y$ as follows :

$$F(x) = (f_1(x), f_2(x), \ldots)$$

Show that is a continuous map. Also show that it is injective.

In order to finish the proof, we need to show that for each open set U in X, the set $F(U)$ is open in $F(X)$. Let z_0 be a point of $F(U)$. Let $x_0 \in U$ be such that $F(x_0) = z_0$ and choose an index N such that $f_N(x_0) > 0$ and $f_N(X \setminus U) = 0$. Now we let

$$W = p_N^{-1}((0,\infty)) \cap f(X)$$

where p_N is the projection $Y \to \mathbb{R}$ onto the Nth multiple. Show that W is an open subset of $F(X)$ such that $z_0 \in W \subset F(U)$.

Give an example of a Hausdorff space with a countable basis which is not metrizable.
