Lectures on
Analytic Number Theory

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Introduction

In additive number theory we make reference to facts about addition in contradistinction to multiplicative number theory, the foundations of which were laid by Euclid at about 300 B.C. Whereas one of the principal concerns of the latter theory is the deconposition of numbers into prime factors, additive number theory deals with the decomposition of numbers into summands. It asks such questions as: in how many ways can a given natural number be ecpressed as the sum of other natural numbers? Of course the decompostion into primary summands is trivial; it is therefore of interest to restrict in some way the nature of the summands (such as odd numbers or even numbers or perfect squares) or the number of summands allowed. These are questions typical of those which will arise in this course. We shall have occasion to study the properties of $\mathcal{V}$-functions and their numerous applications to number theory, in particular the theory of quadratic residues. 1

Formal Power Series

Additive number theory starts with Euler (1742). His tool was power series. His starting point was the simple relation $x^m . x^n = x^{m+n}$ by which multiplication of powers of x is pictured in the addition of exponents. He therefore found it expedient to use power series. Compare the situation in multiplicative number theory; to deal with the product $n.m$, one uses the equation $n^s m^s = (nm)^s$, thus paving the way for utilising Dirichlet series.

While dealing with power series in modern mathematics one asks questions about the domain of convergence. Euler was intelligent enough not to ask this question. In the context of additive number theory power series are purely formal; thus the series $0! + 1!\, x + 2!\, x^2 + \cdots$ is a perfectly good series in our 2 theory. We have to introduce the algebra of formal power series in order to vindicate what Euler did with great tact and insight.

A formal power series is an expression $a_0 + a_1 x + a_2 x^2 + \cdots$. Where the symbol x is an indeterminate symbol i.e., it is never assigned a numerical value. Consequently, all questions of convergence are irrelevant.

Formal power series are manipulated in the same way as ordinary power series. We build an algebra with these by defining addition and multiplication in the following way. If

$$A = \sum_{n=0}^{\infty} a_n x^n, \qquad B = \sum_{n=0}^{\infty} b_n x^n,$$

we define $A + B = C$ where $C = \sum\limits_{n=0}^{\infty} c_n x^n$ and $AB = D$ where $D = \sum\limits_{n=0}^{\infty} d_n x^n$, with the stipulation that we perform these operations in such a way that these equations are true modulo x^N, whatever be N. (This reauirement stems from the fact that we can assign a valuation in the set of power series by defining the order of $A = \sum\limits_{n=0}^{\infty} a_n x^n$ to be k where a_k is the first non-zero coefficient). Therefore c_n and d_n may be computed as for finite polynomials; then

$$\begin{aligned} c_n &= a_n + b_n, \\ d_n &= a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0. \end{aligned}$$

$A = B$ means that the two series are equal term by term, $A = 0$ means that all the coefficiants of A are zero. It is easy to verify that the following relations 3
hold:

$$\begin{aligned} A + B &= B + a & AB &= BA \\ A + (B + C) &= (A + B) + C & A(BC) &= (AB)C \\ A(B + C) &= AB + AC \end{aligned}$$

We summarise these facts by saying that the formal power series form a commutative ring. This will be the case when the coefficients are taken from such a ring, eg. the integres, real numbers, complex numbers.

The ring of power series has the additional property that there are no divisors of zero (in case the ring of coefficients is itself an integrity domain), ie. if $A, B = 0$, either $A = 0$ or $B = 0$. We see this as follows: Suppose $A = 0$, $B = 0$. Let a_k be the first non-zero coefficient in A, and b_j the first non-zero coefficient in B. Let $AB = \sum\limits_{n=0}^{\infty} d_n x^n$; then

$$d_{k+j} = \left(a_\circ b_{k+j} + \cdots + a_{k-1} b_{j+1}\right) + a_k b_j + \left(a_{k+1} b_{j-1} + \cdots + a_{k+j} b_0\right).$$

In this expression the middle term is not zero while all the other terms are zero. Therefore $d_{k+j} \neq 0$ and so $A.B \neq 0$, which is a contradiction.

From this property follows the cancellation law:

If $A \neq \circ$ and $A.B = A.C$, then $B = C$. For, $AB - AC = A(B - C)$. Since $A \neq 0, B - C = \circ$ or $B = C$.

If the ring of coefficients has a unit element so has the ring of power series.

As an example of multiplication of formal power series, let, **4**

$$A = 1 - x \qquad \text{and} \qquad B = 1 + x + x^2 + \cdots$$

$$A = \sum_{n=0}^{\infty} a_n x^n, \quad \text{where } a_0 = 1, a_1 = -1, \text{ and } a_n = 0 \text{ for } n \geq 2,$$

$$B = \sum_{n=0}^{\infty} b_n x^n, \quad \text{where } b_n = 1, n = 0, 1, 2, 3, \ldots$$

$$C = \sum_{n=0}^{\infty} c_n x^n, \quad \text{where } c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0;$$

then

$$c_0 = a_0 b_0 = 1, c_n = b_n - b_{n-1} = 1 - 1 = 0, n = 1, 2, 3, \ldots;$$

so

$$(1 - x)(1 + x + x^2 + \cdots) = 1.$$

We can very well give a meaning to infinite sums and products in certain cases. Thus

$$A_1 + A_2 + \cdots = B,$$
$$C_1 C_2 \cdots = D,$$

both equations understood in the sense module x^N, so that only a finite number of $A's$ or $(C - 1)'s$ can contribute as far as x^N.

Let us apply our methods to prove the identity:

$$1 + x + x^2 + x^3 + \cdots = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \cdots$$

Let

$$\begin{aligned} C &= (1 + x)(1 + x^2)(1 + x^4) \ldots \\ (1 - x)C &= (1 - x)(1 + x)(1 + x^2)(1 + x^4) \ldots \\ &= (1 - x^2)(1 + x^2)(1 + x^4) \ldots \\ &= (1 - x^4)(1 + x^4) \ldots \end{aligned}$$

Continuing in this way, all powers of x on the right eventually disappear, and we have $(1-x)C = 1$. However we have shown that $(1-x)(1+x+x^2+\cdots) = 1$, therefore $(1-x)C = (1-x)(1+x+x^2+\cdots)$, and by the law of cancellation, $C = 1 + x + x^2 + \cdots$ which we were to prove.

This identity easily lends itself to an interpretation which gives an example 5
of the application of Euler's idea. Once again we stress the simple fact that $x^n \cdot x^m = x^{n+m}$. We have

$$1 + x + x^2 + x^3 + \cdots = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\cdots$$

This is an equality between two formal power series (one represented as a product). The coefficients must then be identical. The coefficient of x^n on the right hand side is the number of ways in which n can be written as the sum of powers of 2. But the coefficient of x^n on the left side is 1. We therefore conclude: every natural number can be expressed in one and only one way as the sum of powers of 2.

We have proved that

$$1 + x + x^2 + x^3 + \cdots = (1 + x)(1 + x^2)(1 + x^4)\cdots$$

If we replace x by x^3 and repeat the whole story, modulo x^{3N}, the coefficients of these formal power series will still be equal:

$$1 + x^3 + x^6 + x^9 + \cdots = (1 + x^3)(1 + x^{2.3})(1 + x^{4.3})\cdots$$

Similarly

$$1 + x^5 + x^{2.5} + x^{3.5} + \cdots = (1 + x^5)(1 + x^{2.5})(1 + x^{4.5})\cdots$$

We continue indefinitely, replacing x by odd powers of x. It is permissible to multiply these infinitely many equations together, because any given power of x comes from only a finite number of factors. On the left appears

$$\prod_{k \text{ odd}} (1 + x^k + x^{2k} + x^{3k} + \cdots).$$

On the right side will occur factors of the form $(1 + x^N)$. But N can be written uniquely as $x^\lambda.m$ where m is odd. That means for each N, $1 + x^N$ will occur once and only once on the right side. We would like to rearrange the factors to obtain $(1 + x)(1 + x^2)(1 + x^3)\cdots$

This may be done for the following reason. For any N, that part of the 6
formal power series up to x^N is a polymial derived from a finite number of

factors. Rearranging the factors will not change the polynomial. But since this is true for any N, the entire series will be unchanged by the rearrangement of factors. We have thus proved the identity

$$\prod_{k \text{ odd}} (1 + x^k + x^{2k} + x^{3k} + \cdots) = \prod_{n=1}^{\infty} (1 + x^n) \qquad (1)$$

This is an equality of two formal power series and could be written $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Let us find what a_n and b_n are. On the left we have

$$(1 + x^{1.1} + x^{2.1} + x^{3.1} + \cdots)(1 + x^{1.3} + x^{2.3} + x^{3.3} + \cdots) \\ \times (1 + x^{1.5} + x^{2.5} + x^{3.5} + \cdots) \cdots$$

x^n will be obtainded as many times as n can be expressed as the sum of odd numbers, allowing repetitions. On the right side of (1), we have $(1 + x)(1 + x^2)(1 + x^3) \cdots x^n$ will be obtained as many times as n can be expressed as the sum of integers, no two of which are equal.

a_n and b_n are the number of ways in which n can be expressed respectively in the two manners just stated. But $a_n = b_n$. Therefore we have proved the following theorem of Euler:

Theorem 1. *The number of representations of an integer n as the sum of different parts is the same as the number of representations of n as the sum of odd parts, repetitions permitted.*

We give now a different proof of the identity (1).

$$\prod_{n=1}^{\infty}(1 + x^n) \prod_{n=1}^{\infty}(1 - x^n) = \prod_{n=1}^{\infty}(1 - x^n)(1 + x^n) = \prod_{n=1}^{\infty}(1 - x^{2n}).$$

Again this interchange of the order of the factors is permissible. For, up to 7
any given power of x, the formal series is a polynomial which does not depend on the order of the factors.

$$\prod_{n=1}^{\infty}(1 + x^n) \prod_{n=1}^{\infty}(1 - x^n) = \prod_{n=1}^{\infty}(1 - x^{2n}),$$

$$\prod_{n=1}^{\infty}(1 + x^n) \prod_{n=1}^{\infty}(1 - x^{2n-1}) \prod_{n=1}^{\infty}(1 - x^{2n}) = \prod_{n=1}^{\infty}(1 - x^{2n}).$$

Now $\prod_{n=1}^{\infty}(1-x^{2n}) \neq 0$, and by the law of cancellation, we may cancel it from both sides of the equation obtaining,

$$\prod_{n=1}^{\infty}(1+x^n)\prod_{n=1}^{\infty}(1-x^{2n-1}) = 1.$$

Multiplying both sides by

$$\prod_{n=1}^{\infty}\left(1+x^{2n-1}+x^{2(2n-1)}+x^{3(2n-1)}+\cdots\right)$$

$$\prod_{n=1}^{\infty}(1+x^n)\prod_{n=1}^{\infty}\left(1+x^{2n-1}\right)\prod_{n=1}^{\infty}\left(1+x^{2n-1}+x^{2(2n-1)}+\cdots\right) = \prod_{n=1}^{\infty}\left(1+x^{2n-1}+x^{2(2n-1)}+\cdots\right).$$

For the same reason as before, we may rearrange the order of the factors on the left.

$$\prod_{n=1}^{\infty}(1+x^n)\prod_{n=1}^{\infty}\left(1+x^{2n-1}\right)\left(1+x^{2n-1}+x^{2(2n-1)}+\cdots\right) = \prod_{n=1}^{\infty}\left(1+x^{2n-1}+x^{2(2n-1)}+\cdots\right).$$

However,

$$\prod_{n=1}^{\infty}\left(1+x^{2n-1}\right)\left(1+x^{2n-1}+x^{2(2n-1)}+\cdots\right) = 1,$$

because we have shown that $(1-x)(1+x+x^2+\cdots) = 1$, and this remains true when x is replaced by x^{2n-1}. Therefore the above equation reduces to

$$\prod_{n=1}^{\infty}(1+x^n)\prod_{n=1}^{\infty}5\left(1+x^{2n-1}+x^{2(2n-1)}+\cdots\right) = \prod_{n \text{ odd}}\left(1+x^n+x^{2n}+x^{3n}+\cdots\right)$$

which is the identity (1).

Theorem 1 is easily verified for 10 as follows:10, 1+9, 2+8, 3+7, 4+6, **8**
1+2+7, 1+3+6, 1+4+5, 2+3+5, 1+2+3+4 are the unrestricted partitions. Partitions into odd summands with repetitions are

1+9, 3+7, 5+5, 1+1+1+7, 1+1+3+5, 1+3+3+3, 1+1+1+1+1+5, 1+1+1+1+3+3, 1+1+1+1+1+1+1+3, 1+1+1+1+1+1+1+1+1+1.

We have ten partitions in each category.

It will be useful to extend the theory of formal power series to allow us to find the reciprocal of the series $a_0 + a_1 x + a_2 x^2 + \cdots$ where we assume that $a_0 \neq 0$. (The coefficients are now assumed to form a field). If the series

$$b_0 + b_1 x + b_2 x^2 + \cdots = \frac{1}{a_0 + a_1 x + a_2 x^2 + \cdots},$$

we would have $(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1$. This means that $a_0 b_0 = 1$ and since $a_0 \neq 0$, $b_0 = 1/a_0$. All other coefficients on the left vanish:

$$\begin{gathered} a_0 b_1 + a_1 b_0 = 0, \\ a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{gathered}$$

We may now find b_1 from the first of these equations since all the $a's$ and b_0 are known. Then b_2 can be found from the next equation, since b_1 will then be known. Continuing in this, manner all the $b's$ can be computed by successively solving linear equations since the new unknown of any equation is always accompanied by $a_\nu \neq 0$. The uniquely determined formal series $b_0 + b_1 x + b_2 x^2 + \cdots$ is now called the reciprocal of $a_0 + a_1 x + a_2 x^2 + \cdots$ (We **9**
can not invert if $a_0 = 0$ since in that case we shall have to introduce negative exponents and so shall be going out of our ring of power series). In view of this definition it is meaningful to write $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ since we have shown that $(1-x)(1 + x + x^2 + \cdots) = 1$. Replacing x by x^k, $\frac{1}{1-x^k} = 1 + x^k + x^{2k} + \cdots$ Using this expression, identity (1) may be written

$$\prod_{n=1}^{\infty} (1 + x^n) \prod_{k \text{ odd}} \left(1 + x^k + x^{2k} + \cdots\right) = \prod_{k \text{ odd}} \frac{1}{1 - x^k}.$$

For any N,

$$\prod_{\substack{k \text{ odd} \\ k \leq N}} \frac{1}{1 - x^k} = \frac{1}{\prod\limits_{k \text{ odd}, k \leq N} (1 - x^k)}$$

Since this is true for any N, we may interchange the order of factors in the entire product and get

$$\prod_{n \text{ odd}}^{\infty} \frac{1}{(1 - x^k)} = \frac{1}{\prod\limits_{k \text{ odd}} (1 - x^k)}$$

Therefore, in its revised form identity (l) becomes:

$$\prod_{n=1}^{\infty}(1-x^n) = \frac{1}{\prod\limits_{n \text{ odd}}(1-x^n)}$$

In order to determine in how many ways a number n can be split into k parts, Wuler introduced a parameter z into his formal power series. (The problem was proposed to Euler in St.Petersburgh: in how many ways can 50 be decomposed into the sum of 7 summands?). He considered such expression as $(1+x\mathfrak{z})(1+x^2\mathfrak{z})\cdots$ This is a formal power series in x. The coefficients of x are now polynomials in z, and since these polynomials form a ring they porvide an sdmissible set of coefficients. The product is not a formal power series in z 10
however. The coefficient of z for example, is an infinite sum which we do not allow.

$$\begin{aligned}
(1+x\mathfrak{z})&(1+x^2\mathfrak{z})(1+x^3\mathfrak{z})\cdots \\
&= 1+\mathfrak{z}x+\mathfrak{z}x^2+(\mathfrak{z}+\mathfrak{z}^2)x^3+(\mathfrak{z}+\mathfrak{z}^2)x^4+(\mathfrak{z}+2\mathfrak{z}^2)x^5+\cdots \\
&= 1+\mathfrak{z}(x+x^2+x^3+\cdots)+\mathfrak{z}^2(x^3+x^4+2x^5+\cdots)+\cdots \\
&= 1+\mathfrak{z}A_1(x)+\mathfrak{z}^2A_2(x)+\mathfrak{z}^3A_3(x)+\cdots \qquad (2)
\end{aligned}$$

The expressions $A_1(x), A_2(x), \cdots$ are themselves formal power series in x. They begin with higher and higher powers of x, for the lowest power of x occurring in $A_m(x)$ is $x^{1+2+3+\cdots+m} = x^{m(m+1)/2}$. This term arises by multiplying $(x\mathfrak{z})(x^2\mathfrak{z})(x^3\mathfrak{z})\cdots(x^m\mathfrak{z})$. The advantage in the use of the parameter z is that any power of x multiplying $\mathfrak{z}^m$ is obtained by multiplying m different powers of x. Thus each term in $A_m(x)$ is the product of m powers of x. The $\mathfrak{z}'s$ therefore record the number of parts we have used in building up a number.

Now we consider the finite product $P_N(\mathfrak{z}, x) \equiv \prod\limits_{n=1}^{N}(1+\mathfrak{z}x^n)$.

$P_N(\mathfrak{z}, x)$ is a polynomial in z: $P_N(\mathfrak{z}, x) = 1+\mathfrak{z}A_1^{(N)}(x)+\mathfrak{z}^2A_2^{(N)}(x)+\cdots+\mathfrak{z}^NA_N^{(N)}(x)$, where $A_N^{(N)}(x) = x^{N(N+1)/2}$. Replacing z by Zx, we have

$$\begin{aligned}
\prod_{n=1}^{N}(1+\mathfrak{z}x^{n+1}) &= P_N(\mathfrak{z}x, x) \\
&= 1+\mathfrak{z}xA_1^{(N)}(x)+\mathfrak{z}^2x^2A_2^{(N)}(x)+\cdots
\end{aligned}$$

So

$$(1+\mathfrak{z}x)P_N(\mathfrak{z}x, x) = \left(1+\mathfrak{z}x^{N+1}\right)P_N(\mathfrak{z}, x),$$

$$(1+\mathfrak{z}x)\left(1+\mathfrak{z}xA_1^{(N)}(x)+\cdots+(\mathfrak{z}x)^N A_N^{(N)}(x)\right)$$
$$=\left(1+\mathfrak{z}x^{N+1}\right)\left(1+A_1^{(N)}(x)+\mathfrak{z}^2 A_2^{(N)}(x)+\cdots\right)$$

We may now compare powers of z on both sides since these are polynomials. Taking $\mathfrak{z}^k$, $k \le N$, we have 11

$$x^k A_k^{(N)}(x)+x^k A_{k-1}^{(N)}(x)=A_k^{(N)}(x)+x^{N+1}A_{k-1}^{(N)}(x);$$
$$A_k^{(N)}(x)(1-x^k)=a_{k-1}^{(N)}(x)x^k\left(1-x^{N+1-k}\right),$$
$$A_k^{(N)}(x)=\frac{x^k}{1-x^k}\left(1-x^{N+1-k}\right)A_{k-1}^{(N)}(x),$$
$$A_k^{(N)}(x)\equiv\frac{x^k}{1-x^k}A_{k-1}^{(N)}(x) \pmod{x^N}.$$

From this recurrence relation we immediately have

$$A_1^{(N)}(x)\equiv\frac{x}{1-x} \pmod{x^N},$$
$$A_2^{(N)}(x)\equiv\frac{x\cdot x^2}{(1-x)(1-x^2)} \pmod{x^N}$$
$$\equiv\frac{x^3}{(1-x)(1-x^2)} \pmod{x^N}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_k^{(N)}\equiv\frac{x^{k(k+1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)} \pmod{x^N}$$

Hence

$$\prod_{n=1}^{\infty}(1+\mathfrak{z}x^n)\equiv 1+\frac{\mathfrak{z}x}{1-x}+\frac{\mathfrak{z}^2x^3}{(1-x)(1-x^2)}+\frac{\mathfrak{z}^3x^6}{(1-x)(1-x^2)(1-x^3)}+\cdots \pmod{x^N}$$

Lecture 2

In the last lecture we proved the identity: 12

$$\prod_{n=1}^{\infty}(1+\mathfrak{z}x^k) = \sum_{k=0}^{\infty}\mathfrak{z}^k A_k(x), \tag{1}$$

where

$$A_k(x) = \frac{x^{k(k+1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)} \tag{2}$$

We shall look upon the right side of (1) as a power series in x and *not* as a power-series in z, as otherwise the infinite product on the left side would have no sense in our formalism. Let us inerpret (1) arithmetically. If we want to decompose m into k summands, we have evidently to look for $\mathfrak{z}^k$ and then for x^m, and the coefficient of $\mathfrak{z}^k x^m$ on the right side of (1) gives us exactly what we want. We have

$$\begin{aligned}\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} &= \sum_{n_1=0}^{\infty}x^{n_1}\sum_{n_2=0}^{\infty}x^{2n_2}\cdots\sum_{n_k=0}^{\infty}x^{kn_k}\\ &= \sum_{m=0}^{\infty}p_m^{(k)}x^m,\end{aligned}$$

say, with $\qquad p_0^{(k)} = 1.$

Therefore m occurs only in the form

$$m = n_1 + 2n_2 + \cdots + kn_k, n_j \geq 0,$$

and $p_m^{(k)}$ tells us how often m can be represented by k dfferent summands (with possible repetitions). On the other hand the coefficient of x^m on the left-side of (1) gives us the number of partitions of m into summands not exceeding k. Hence,

Theorem 2. *m can be represented as the sum of k different parts as often as $m - \dfrac{k(k+1)}{2}$ can be expressed as the sum of parts not exceeding k (repetition being allowed).* **13**

(In the first the number of parts is fixed, in the second, the size of parts).

In a similar way, we can extablish the identity

$$\frac{1}{\prod_{n=1}^{\infty}(1-\mathfrak{z}x^n)} = \sum_{k=0}^{\infty} \mathfrak{z}^k B_k(x), \tag{3}$$

with $B_0 = 1$, which again can be interpreted arithmetically as follows.

The left side is

$$\sum_{n_1=0}^{\infty}(\mathfrak{z}x)^{n_1} \sum_{n_2=0}^{\infty}(\mathfrak{z}x^2)^{n_2} \sum_{n_3=0}^{\infty}(\mathfrak{z}x^3)^{n_3} \cdots$$

and

$$B_k(x) = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)} \tag{4}$$

The left-side of (3) gives m with the representation

$$m = n_1 + 2n_2 + 3n_3 + \cdots$$

i.e., as a sum of parts with repetitions allowed. Exactly as above we have:

Theorem 3. *m can be expressed as the sum of k parts (repetitions allowed) as often as $m - k$ as the sum of parts not exceeding k.*

We shall now consider odd summands which will be of interest in connexion with $\mathcal{V}$-function later. As earlier we can establish the identity

$$\prod_{\mathcal{V} \text{ odd}} (1 + \mathfrak{z}x^{\mathcal{V}}) = \sum_{k=0}^{\infty} \mathfrak{z}^k C_k(x) \tag{5}$$

with the provide that $C_\circ(x) = 1$. The trick is the same. One studies temporatily a truncated affair $\prod_{\mathcal{V}=1}^{V}(1 + \mathfrak{z}x^{\mathcal{V}})$, replaces z by $\mathfrak{z}x^2$ and evaluates $C_k(x)$ as in Lecture 1. This would be perfectly legitimate. However one could proceed as **14**

Euler did - this is not quite our story. Multiplying both sides by $1 + \mathfrak{z}x^2$, we have

$$\sum_{k=0}^{\infty} \mathfrak{z}^k C_k(x) = (1 + \mathfrak{z}x^2) \sum_{k=0}^{\infty} \mathfrak{z}^k x^{2k} C_k(x).$$

Now compare powers of z on both sides - and this was what required some extra thought. $C_k(x)$ begins with $x^{1+3+\cdots+(2k-1)} = x^{k^2}$; in fact they begin with later and later powers of x and so can be added up. We have

$$\begin{aligned} C_0 &= 1, \\ C_k(x) &= x^{2k} C_k(x) + x^{2k-1} C_{k-1}(x), k > 0, \\ \text{or} \qquad C_k(x) &= \frac{x^{2k-1}}{1 - x^{2k}} C_{k-1}(x) \end{aligned}$$

from this recurrence relation we obtain

$$\begin{aligned} C_1(x) &= \frac{x}{1 - x^2}, \\ C_2(x) &= \frac{x^3}{1 - x^4} C_1(x) = \frac{x^4}{(1 - x^2)(1 - x^4)}, \\ C_3(x) &= \frac{x^5}{1 - x^6} C_2(x) = \frac{x^9}{(1 - x^2)(1 - x^4)(1 - x^6)}, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ C_k(x) &= \frac{x^{k^2}}{(1 - x^2)(1 - x^4) \cdots (1 - x^{2k})}, \end{aligned}$$

carrying on the same rigmarole.

Now note that all this can be retranslated into something.

Let us give the number theoretic interpretation. The coefficient of $\mathfrak{z}^k x^m$ **15**
gives the number of times m can be expressed as the sum of k different odd summands. On the other hand, the coefficients in the expansion of $\frac{1}{(1-x^2)\cdots(1-x^{2k})}$ give the decomposition into even summands, with repetitions. Hence,

Theorem 4. *m is the sum of k different odd parts as often as $m - k^2$ is the sum of even parts not exceeding $2k$, or what is the same thing, as $\frac{m-k^2}{2}$ is the sum of parts not exceeding k. (since m and k are obviously of the same parity, it follows that $\frac{m-k^2}{2}$ is an integer).*

Finally we can prove that

$$\frac{1}{\prod_{\mathcal{V} \text{ odd}} (1 - \mathfrak{z}x^{\mathcal{V}})} = \sum_{k=0}^{\infty} \mathfrak{z}^k D_k(x) \tag{6}$$

Replacing $\mathfrak{z}$ by $\mathfrak{z}x^2$, we obtain

$$D_k(x) = \frac{x^k}{(1-x^2)\cdots(1-x^{2k})},$$

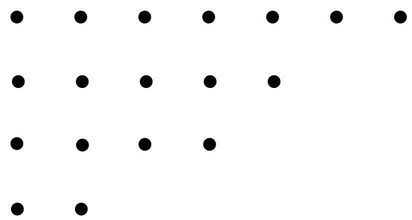
leading to the

Theorem 5. *m is the sum of k odd parts as often as $m-k$ is the sum of even parts not exceeding $2k$, or $\frac{m-k}{2}$ is the sum of even parts not exceeding k. ($\frac{m-k}{2}$ again is integral).*

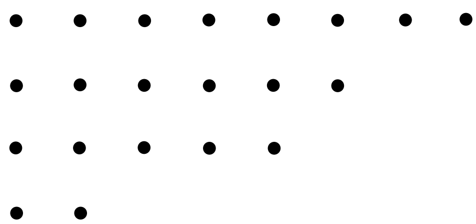
Some other methods

Temporarily we give up power series and make use of graphs to study partitions. A partition of $\mathcal{N}$ may be represented as an array of dots, the number of dots in a row being equal to the magnitude of a summand. Let us arrange the summands according to size.

For instance, let us consider a partition of 18 into 4 different parts **16**



If we read the diagram by rows we get the partiton 18=7+5+4+2. On the other hand reading by columns we have the pertition 18= 4+4+3+3+2+1+1. In general it is clear that if we represent a partition of n into k parts graphically, then reading the graph vertically yields a partition of n with the largest part k, and conversely. This method demonstrates a one-to-one correspondence between partitions of n with k parts and partitions sees that the number of partitions of n with largest part k is equal to the number of partitions of $n-k$ into parts not exceeding k.



Draw a diagonal upward starting from the last but one dot in the column on the extreme left. All the dots to the right of this diagonal constitute a partition of 12 into 4 parts. For each partition of 18 into 4 different parts there corresponds thus a partition of $18 - \frac{4.3}{2} = 12$ into parts. This process works in general for a 17
partition of n with k different parts. If we throw away the dots on and to the left of the diagonal (which is drawn from the last but one point from the bottom in order to exsure that the number of different parts constinues to be exactly k), we are left with a partition of $n - (1 + 2 + 3 + \cdots + (k-1)) = n - \frac{k(k-1)}{2}$. This partition has exactly k parts because each row is longer by at least one dot than the row below it, so an entire row is never discarded. Conversely, starting with a partition of $n - \frac{k(k-1)}{2}$ into k parts, we can build up a unique partition of n into k different parts. Add 1 to the next to the smallest part, 2 to the next longer, 3 to the next and so on. This one-to-one correspondence proves that the number of partitions of n into k different parts equals the number of partitions of $n - \frac{k(k-1)}{2}$ into k parts.

We can prove graphically that the number of partitons of n into k odd summands is the same as the number of partitions of $n - k^2$ into even summands not exceeding k. The last row of the

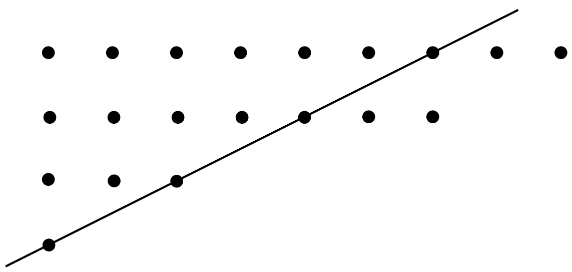


diagram contains at least one dot, the next higher at least three, the one above at least five, and so on. Above and on the diagonal there are $1 + 3 + 5 + \cdots + (2k-1) = k^2$ dots. When these are removed, an even number of dots is left in each row, althogether adding up to $n - k^2$. This proves the result.

Theorem 1 can also be proved graphically, although the proof is not quite 18
as simple. The idea of the proof is examplified by considering the partitons of 35. We have

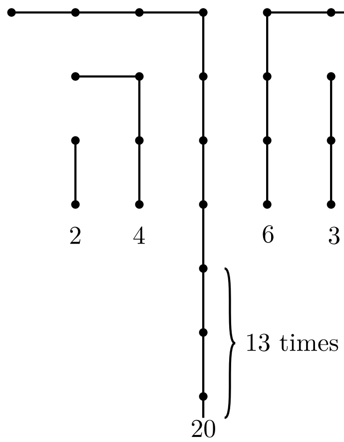
$$\begin{aligned} 35 &= 10 + 8 + 7 + 5 + 4 + 1 \\ &= 5 \times 2 + 1 \times 8 + 7 + 5 + 1 \times 4 + 1 \\ &= 5(2 + 1) + 7 \times 1 + 1(8 + 4 + 1) \end{aligned}$$

$$7+5+5+5+\left(\underbrace{1+\cdots+1}_{\text{13 times}}\right)$$

Thus to each unrestricted partition of 35 we can make correspond a partition into add summands with possible repetitions. Conversely

$$\begin{aligned} 7\times 1+5\times 3+1\times 13 &= 7\times 1+5(1+2)+1(2^3+2^2+2^0)\\ &= 7+5+10+8+4+1. \end{aligned}$$

Now consider the following diagram



Each part is represented by a row of dots with the longest row at ehe top, second longest next to the top, etc. The oddness of the parts allows uo to place the rows symmetrically about a central vertical axis. Now connect the **19**
dots in the following way. Connect the dots on this vertical axis with those on the left half of the top row. Then connect the column to the right of this axis to the other half of the top row. We continue in this way as indicated by the diagram drawing right angles first on one side of the centre and then on the other. We now interpret this diagram as a new partition of 35 each part being represented by one of the lines indicated. In this way we obtain the partition 20+6+4+3+2 of 35 into different parts. It can be proved that this method works in general. That is, to prove that given a partition of n into odd parts, this method transforms it into a unique partition of n into distinct parts; conversely, given a partation into distinct parts, the process can be reversed to find a unique partition into odd parts. This establishes a one-to-one correspondence between the two sorts of partitions. This proves our theorem.

Lecture 3

The series $\sum_{k=0}^{\infty} \mathfrak{z}^k A_k(x)$ that we had last time is itself rather interesting; the $A_k(x)$ **20**
have a queer shape:

$$A_k(x) = \frac{x^{k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)}$$

Such series are called Euler series. Such expressions in which the factors in the denominator are increasing in this way have been used for wide generalisations of hypergeometric series. Euler indeed solved the problem of computing the coefficients numerically. The coefficient of $\mathfrak{z}^k x^m$ is obtained by expanding $\frac{1}{(1-x)\cdots(1-x^k)}$ as a power series. This is rather trivial if we are in the field of complex numbers, since we can then have a decomposition into partial fractiions. Euler did find a nice sort of recursion formula. There is therefore a good deal to be said for a rather elementary treatment.

We shall, however, proceed to more important discussions the problem of unrestricted partitions. Consider the infinite product (this is justifiable modulo x^N)

$$\begin{aligned}
\prod_{m=1}^{\infty} \frac{1}{1-x^m} &= \prod_{m=1}^{\infty} \sum_{n=0}^{\infty} x^{mn} \\
&= \sum_{n_1=0}^{\infty} x^{n_1} \sum_{n_2=0}^{\infty} x^{2n_2} \cdot \sum_{n_j=0}^{\infty} x^{3n_3} \cdots \\
&= 1 + x + 2x^2 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} p_n x^n \qquad (1)
\end{aligned}$$

What does p_n signify? p_n appeared in collecting the term x^n. Following 21
Euler's idea of addition of exponents, we have

$$n = n_1 + 2n_2 + 3n_3 + 4n_4 + \cdots n_j \geq 0, \tag{2}$$

so that p_n is the number of solutions of a *finite* Diophantine equation (since the right side of (2) becomes void after a finite stage) or the number of ways in which n can be expressed in this way, or the number of unrestricted partitions. Euler wrote this as

$$\frac{1}{\prod_{m=1}^{\infty}(1 - x^m)} = \sum_{n=0}^{\infty} p(n)x^n, \tag{3}$$

with the provide that $p(0) = 1$.

We want to find as much as possible about $p(n)$. Let us calculate $p(n)$. Expanding the product,

$$\begin{aligned}\prod_{n=1}^{\infty}(1 - x^n) &= (1 - x)(1 - x^2)(1 - x^3)\cdots \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + + - - \cdots\end{aligned}$$

(Note Euler's skill and patience; he calculated up to x^n and found to this surprise that the coefficients were always $0, \pm 1$, two positive terms followed by two negative terms). We want to find the law of exponents, as every sensible man would. Writing down the first few coeffieicnts and taking differences, we have

$$\begin{array}{ccccccccccccccccc} 0 & & 1 & & 2 & & 5 & & 7 & & 12 & & 15 & & 22 & & 26 \\ & \underline{1} & & 1 & & \underline{3} & & 2 & & \underline{5} & & 3 & & \underline{7} & & 4 & \end{array}$$

the sequence of odd numbers interspersed with the sequence of natural numbers. Euler forecast by induction what the general power would be as follows. 22

$$\begin{array}{ccccccccccccc} 7 & & 2 & & 0 & & 1 & & 5 & & 12 & & 22 \\ & -5 & & -2 & & 1 & & 4 & & 7 & & 10 & \\ & & 3 & & 3 & & 3 & & 3 & & 3 & & \end{array}$$

Write down the coefficients by picking up 0, 1 and every other alternate term, and continue the row towards the left by putting in the remaining coefficients. Now we find that the second differences have the constant value 3. But an arithmetical progression of the second order can be expressed as a polynomial of the second degree. The typical coefficient will therefore be given by an expression of the form

$$a\lambda^2 + b\lambda + c \qquad a(\lambda + 1)^2 + b(\lambda + 1) + c \qquad a(\lambda + 2)^2 + b(\lambda + 2) + c$$

$$a(2\lambda + 1) + b \qquad\qquad a(2\lambda + 3) + b$$

2 a (the constant second difference)

Hence $2a = 3$ or $a = 3/2$. Taking $\lambda = 0$ we find that $c = 0$ and $b = -\frac{1}{2}$, so that the general coefficient has the form $\frac{\lambda(3\lambda-1)}{2}$. Observing that when λ is changed to $-\lambda$, $\frac{\lambda(3\lambda-1)}{2}$ becomes $\frac{\lambda(3\lambda+1)}{2}$, the coefficient of $x^{\lambda(3\lambda-1)/2}$ is $(-)^\lambda$, and hence

$$\prod_{n=1}^{\infty}(1 - x^n) = \prod_{\lambda=-\infty}^{\infty}(-)^\lambda x^{\lambda(3\lambda-1)/2}, \tag{4}$$

which is Euler's theorem.

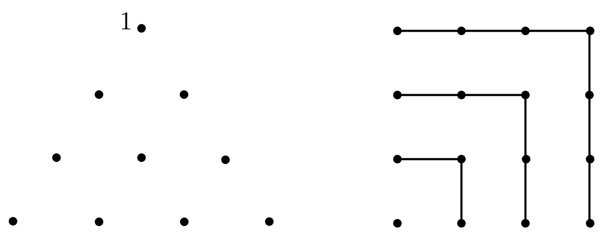
This sequence of numbers $\frac{\lambda(3\lambda-1)}{2}$ played a particular role in the middle ages. They are called *pentagonal numbers* and Euler's theorem is called the pentagonal numbers theorem. We have the so-called triangular numbers:

$$\begin{array}{ccccccccc} 1 & & 3 & & 6 & & 10 & & 15 \\ & 2 & & 3 & & 4 & & 5 & \\ & & 1 & & 1 & & 1 & & \end{array}$$

where the second differences are all 1; the square-numbers **23**

$$\begin{array}{ccccccccc} 1 & & 4 & & 9 & & 16 & & 25 \\ & 3 & & 5 & & 7 & & 9 & \\ & & 2 & & 2 & & 2 & & \end{array}$$

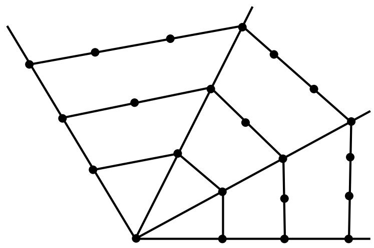
for which the second difference are always 2; and so on.



The triangular numbers can be represented by dots piled up in the form of equilateral triangles; the square numbers by successively expanding squares.

The pentagons however do not fit together like this. We start with one pentagon; notice that the vertices lie perspectively along rays through the origin. So take two sides as basic and magnify them and add successive shelves. The second differences now are always 3:

1		5		12		22
	4		7		10	
		3		3		



In general we can have r-gonal numbers where the last difference are all $r-2$.

We go back to equation (4): **24**

$$\prod_{m=1}^{\infty}(1-x^m) = \sum_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\lambda(3\lambda-1)/2}$$

It is quite interesting to go into the history of this. It appeared in Euler's Introductio in Analysin Infinitorum, Caput XVI, de Partitio numerorum, 1748 (the first book on the differential and integral calculus). It was actually discovered earlier and was mentioned in a paper communicated to the St. Petersburgh Academy in 1741, and in letters to Nicholas Bernoulli (1742) and Goldbach (1743). The proof that bothered him for nine years was first given in a letter dated 9th june 1750 to Goldvach, and was printed in 1750.

The identity (4) is remarkable; it was the first time in history that an identity belonging to the $\mathcal{V}$-functions appeared (later invented and studied systematically by Jocobi). The interesting fact is that we have a power-series in which the exponents are of the second degree in the subscripts. The $\mathcal{V}$-functions have a representation as a series and slso as an infinite porduct.

The proof of identity (4) is quite exciting and elementary. By using distributivity we break up the product

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$$

in the following way:

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots = 1-x-(1-x)x^2-(1-x)(1-x^2)x^3-$$

$$(1-x)(1-x^2)(1-x^3)(1-x^4) - (1-x)\cdots(1-x^4)x^5 - \cdots$$

which may be re-arranged, opening first parenthesis, as

$$1 - x - x^2 \quad -(1-x^2)x^3 \quad -(1-x^2)(1-x^3)x^4 \quad -(1-x^2)(1-x^3)(1-x^4)x^5$$

$$+x^3 \qquad +(1-x^2)x^4 \qquad +(1-x^2)(1-x^3)x^5$$

So 25

$$1 - x - x^2 + x^5 + x^7(1-x^2) + x^9(1-x^2)(1-x^3) + \cdots$$

$$= 1 - x - x^2 + x^5 + \ 2^7 \ +(1-x^3)x^9 \ +(1-x^3)(1-x^4)x^{11}$$

$$-x^9 \qquad -(1-x^3)x^{11}$$

$$1 - x - x^2 + x^5 + x^7 - x^{12}(1-x^3)x^{15} - (1-x^3)(1-x^4)x^{18} - \cdots$$

When this is continued, we get some free terms at the beginning followed by a typical remainder

$$(1-x^k)x^m + (1-x^k)(1-x^{k+1})x^{m+k} + (1-x^k)(1-x^{k+1})(1-x^{k+2})x^{m+2k},$$

which may be rearranged into

$$x^m + (1-x^{k+1})x^{m+k} + (1-x^{k+1})(1-x^{k+2})x^{m+2k} - x^{m+k} - (1-x^{k+1})x^{m+2k} \quad (*)$$

$$= x^m - x^{m+2k+1} - (1-x^{k+1})x^{m+3k+2} - (1-x^{k+1})(1-x^{k+2})$$

$$x^{m+4k+3} - \cdots \quad (**)$$

We have two free terms with opposite signs at the beginning. In (*) the difference between exponents in successive terms is k, while in (**) this increases to $k+1$; this difference is in both cases the exponent of x in the first factor. The remainder after the free terms begine with $-$, so that the sequence of signs is $+ - - + + - - \cdots$ This process perpetuates itself and the question 26 remains which powers actually appear. It is sufficient to mark down a scheme for the exponents which completely characterises the expansion. The scheme is illustrated by what follows.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
			2	3	4	5	6	7	8	9	10	11	12	13	14	15
			5	7	9	11	13	15	17	19	21	23	25	27	29	31
					3	4	5	6	7	8	9	10	11	12	13	14
					12	15	18	21	24	27	30	33	36	39	42	45
							4	5	6	7	8	9	10	11	12	13
							22	26	30	34	38	42	46	50	54	58
									5	6	7	8	9	10	11	12
									35	40	45	50	55	60	65	70
											6	7	8	9	10	11
											51	57	63	69	75	81

We write down the sequence of natural numbers in a row; the sequence less the first two membere is repeated in a parallel row below leaving out the first three placess at the beginning. Adding up we get

$$5 \quad 7 \quad 9 \quad 11 \quad \ldots\ \ldots\ \ldots \quad ,$$

below which is placed the original sequence less the first three members, again translating the whole to the right by two places. We again add up and repeat the procedure. A typical stage in the procedure is exhibited below.

m	$m+k$	$m+2k$	$m+3k$	$m+4k$	$m+5k$
	$k+1$	$k+2$	$k+3$	$k+4$	$k+5$
	$m+2k+1$	$m+3k+2$	$m+4k+3$	$m+5k+4$	$m+6k+5$

The free indices then appear successively as **27**

$$2+3=5 \qquad 3+4+5=12$$
$$3+4=7 \qquad 4+5+6=15,$$

and in general:

$$\lambda+(\lambda+1)+\cdots+(2\lambda-1)=\frac{\lambda(3\lambda-1)}{2},$$
$$(\lambda+1)+(\lambda+2)+\cdots+2\lambda=\frac{\lambda(3\lambda+1)}{2},$$

which are the only exponents appearing. We thus have

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\lambda(3\lambda-1)/2}$$

Lecture 4

In the last lecture we proved the surprising theorem on pentagonal numbers: 28

$$\prod_{m=1}^{\infty}(1-x^m) = \sum_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\lambda(3\lambda-1)/2} \tag{1}$$

We do not need these identities for their own sake, but for their applications to number theory. We have the same sort of power-series on both sides; let us compare the coefficients of x^n. On the left side n appears as the sum of different exponents. But in contradiction to previous situations, the coefficients appear with both positive and negative signs, so that when we collect the terms there may be cancellations. There are gaps in the powers that appear, but among those which appear with non-zero coefficients, we have a pair of positive terms followed by a pair of negative terms and vice versa. In most cases the coefficients are zero; this is because of cancellations, so that roughly terms with positive and negative signs are in equal number. A positive sign appears if we multiply an even number of times. otherwise a negative sign. So an even number of different summands is as frequent generally as an odd number. Hence the following theorem:

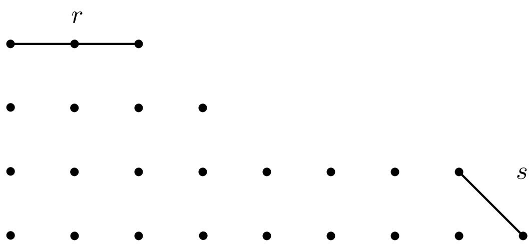
The number of decompositions of n into an even number of different parts is the same as the number of decompositions into an odd number, with the exception that there is a surplus of one sort or the other if n is a pentagonal number of the form $\lambda(3\lambda-1)/2$.

Before proceeding further let us examine a number of concrete instances. Take 6 which is not a pentagonal number. The partitions are 6, $1+5$, $2+4$, 29
$1+2+3$, so that there are two decompositions into an even number of different parts, and two into an odd number. Next take 7, which is a pentagonal number, $7=\frac{\lambda(3\lambda+1)}{2}$ with $\lambda=2$. We can actually foresee that the excess will be in the even partitions. The partitions are 7, $1+6$, $2+5$, $3+4$, $1+2+4$. Take 8 which

again is not pentagonal. We have three in each category: 8, 1 + 7, 2 + 6, 3 + 5, 1 + 2 + 5, 1 + 3 + 4.

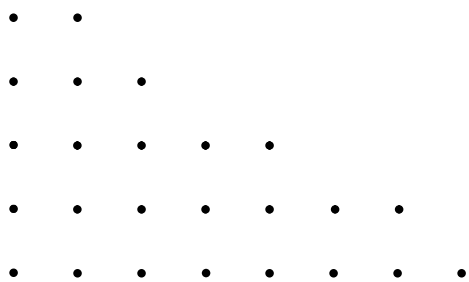
This is a very extraordinary property of pentagonal numbers. One would like to have a direct proof of this. A proof is due to Fabian Franklin (Comptes Rendus, Paris. 1880), a pupil of the famous Sylvester. The proof is combinatorial. We want to establish a one-one correspondence between partitions containing an even number of summands and those containing an odd number - except for pentagonal numbers.

Consider a partition with the summands arranged in increasing order, each summand being denoted by a horizontal row of dots. Mark specifically the first row,



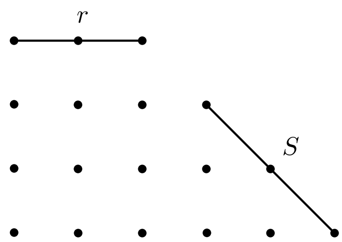
with *r* dots, and the last slope, with *s* dots i.e., points on or below a segment starting from the dot on the extreme right of the last row and inclined at 45° (as in the diagram). We make out two cases.

1. $s < r$. Transfer the last slope to a position immediately above the first row. The diagram is now as shown below:



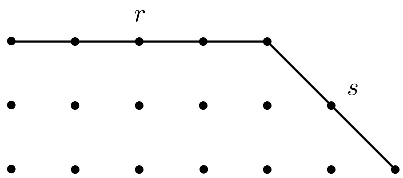
The uppermost row is still shorter than the others. (because in our case $s < r$). By this procedure the number of rows is changed by 1. This establishes the one-one correspondence between partition of the 'odd' type and 'even' type. **30**

2. $s \geq r$. As before consider the first row and the last slope.

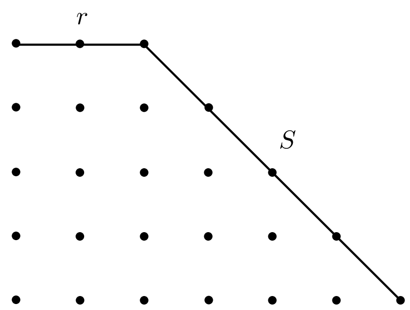


Take the uppermost row away and put it parallel to the last slope. This diminishes the number of rows by 1, so that a partition is switched over from the 'even' class to the 'odd' class or conversely.

Therefore there exists a one-one correspondence between the two classes. So we have proved a theorem, which is a wrong one! because we have not taken account of the exceptional case of pentagonal numbers. The fallacy lies in having overlooked the fact that the last slope may extend right up to the first row; the slope and the row may very well interfere. Let us take one such instance. Let again $s < r$.



If we place the last slope above the first row this works because the number **31**
of points in the first row is also diminished by one, in fact by the disputed point (notice again that no two rows are equal for $s < r-1$). So the interference is of no account. With $s \geq r$ we may again have an interfering case. We again place the top row



behind the last slope, this time with a punishment. We have now shortened the slope by 1. For $s - 1 \geq r$ the method is still good. So the only cases of earnest interference are:

(i) $s < r$ but $x \geq r - 1$. Then $r - 1 \leq s \leq r$ and hence $s = r - 1$

(ii) $s \geq r$ but $s - 1 < r$. Then $s \geq r > s - 1$ and hence $s = r$.

Here we have something which can no longer be overcome. These are the cases of pentagonal numbers. In (ii) the total number of dots is equal to

$$s + (s+1) + (s+2) + \cdots + (2s-1)$$
$$= \frac{s(3s-1)}{2}$$

In (i) this number $= (s+1) + (s-2) + \cdots + 2s$
$$= \frac{s(3s+1)}{2}$$

These decompositions do not have companions. In general every partition into one parity of different summands has a companion of the other parity of different summands; and in the case of pentagonal numbers there is just one in 32
excess in one of the classes.

We now come to the most important application of identity (1). Since

$$\frac{1}{\prod_{m=1}^{\infty}(1-x^m)} = \sum_{n=0}^{\infty} p(n)x^n,$$

we have on combining this with (1),

$$1 = \sum_{n=0}^{\infty} p(n)x^n \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda} x^{\lambda(3\lambda-1)/2} \tag{2}$$

This tells us the follwing story. All the coefficients on the right side of (2) excepting the first must be zero. The typical exponent in the second factor on the right side is $\lambda(3\lambda-1)/2 = \omega_\lambda$, say. (The first few $\omega'_\lambda s$ are 0, 1, 2, 5, 7, 12, 15, ...). Now look for x^n. Since the coefficient in the first factor is $p(n)$ and that in the second always ± 1, we have, since $x^n (n \neq 0)$ does not appear on the left side

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - - + + \cdots = 0$$

or

$$\sum_{0 \leq \omega_\lambda \leq n} p(n-\omega_\lambda)(-)^\lambda = 0 \tag{3}$$

This is a formula of recursion. Omitting the first index of summation (3) gives

$$p(n) = \sum_{0 < \omega_\lambda \leq n} (-)^{\lambda-1} p(n-\omega_\lambda) \tag{4}$$

Let us calculate the first few $p(n)$.

$$\begin{aligned}
p(0) &= 1\\
p(1) &= p(1-1) = p(0) = 1\\
p(2) &= p(2-1) + p(2-2) = 2\\
p(3) &= p(3-1) + p(3-2) = 3\\
p(4) &= p(4-1) + p(4-2) = 5\\
p(5) &= p(5-1) + p(5-2) - p(5-5) = 7
\end{aligned}$$

(Watch! a pentagonal number - and a negative sign comes into action!). These **33**
formulae get longer and longer, but not excessively so. Let us estimate how long these will be. Since $\omega_\lambda \leq n$ we have to look for λ satisfying $\dfrac{\lambda(3\lambda-1)}{2} \leq n$, which gives

$$\begin{aligned}
12\lambda(3\lambda-1) &\leq 24n,\\
36\lambda^2 - 12\lambda &\leq 24n,\\
(5\lambda-1)^2 &= 24n+1,\\
|6\lambda-1| &= \sqrt{24n+1},
\end{aligned}$$

$$|\lambda - \frac{1}{6}| \leq \frac{1}{6}\sqrt{24n+1}.$$

Hence roughly there will be $\frac{1}{3}\sqrt{24n} = \frac{2}{3}\sqrt{6n}$ summands on the left side of (3). So their number increases with the square root of n- the expressions do not get too long after all (for $n = 100$, we have 17 terms).

These formulae have been used for preparing tables of $p(n)$ which have been quite useful. For instance Ramanujan discovered some of the divisibility properties of $p(n)$ by using them. In the famous paper of Hardy and Ramanu- **34**
jan (1917) there is a table of $p(n)$ for $n \leq 200$. These were computed by Macmahon, by using the above formulae and the values were checked with those given by the Hardy-Ramanujan formula. The asymptotic values were found to be very close to what Macmahon computed. Gupta has extended the table for $p(n)$ up to 600.

Before making another application of Euler's pentagonal theorem, we proceed a bit further into the theory of formal power series. We add now one more formal procedure, that of formal differentiation. Let

$$A = a_\circ + a_1 x + a_2 x^2 + \cdots$$

The derivative A' of A is by definition

$$A' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

This is again a power series in our sense. This operation of differentiation which produces one power series from another is a linear operation:

$$(A + B)' = A' + B',$$

where B is a second power series. This is easy to verify; actually we need do this only for polynomials as everything is true modulo x^N. Again,

$$(cA)' = cA'$$

as can be seen directly. Also

$$(A \cdot B)' = A'B + AB'.$$

Let us look into this situation. Start with the simplest case, $A = x^m$, $B = x^n$. Then

$$A' = mx^{m-1}, \qquad B' = nx^{n-1}$$

and $$(AB)' = (x^{m+n})' = (m+n)x^{m+n-1},$$
also
$$\begin{aligned} A'B + AB' &= mx^{m-1+n} + nx^{m+n-1} \\ &= (m+n)x^{m+n-1} \end{aligned}$$

So this is true also for polynomials by linearity, we can do it piecemeal. **35**
And as it is enough if we stop short at x^N, it is true in general,

Let us add one more remark. Let us write down a special case where A and B have reciprocals. Then AB has a reciprocal too (since the units form a group). In this case we have

$$\frac{(AB)'}{AB} = \frac{A'}{A} + \frac{B'}{B},$$

which is the rule for logarithmic differentiation. (It is identical with the procedure in the calculus, as soon as we speak of functions). For A, B and C,

$$(ABC)' = A'(BC) + A(BC)' = A'BC + AB'C + ABC'$$
or
$$\frac{(ABC)'}{ABC} = \frac{A'}{A} + \frac{B'}{B} + \frac{C'}{C},$$

and so on; in general,

$$\frac{\left(\prod_{n=1}^{K} A_k\right)'}{\prod_{k=1}^{K} A_k} = \sum_{k=1}^{K} \frac{A'K}{A_k}$$

We can do this for infinite products also if the products are permissible. Indeed $\prod\limits_{k=1}^{K} A_k$ is legitimate if $A_\ell = 1 + a_{\ell(k)}x^\ell + \cdots$ Consider modulo x^N; break at a finite spot and the factors 1 will come into action.

Lecture 5

Let us consider some applications of formal differentiation of power series. **36**
Once again we start from the pentagonal numbers theorem:

$$\prod_{m=1}^{\infty}(1-x^m) = \sum_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\lambda(3\lambda-1)/2}$$
$$= \sum_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\omega_\lambda}, \tag{1}$$

with $\omega_\lambda = \dfrac{\lambda(3\lambda-1)}{2}$. Taking the logarithmic derivative - and this can be done piecemeal-

$$\sum_{m=1}^{\infty}\frac{-mx^{m-1}}{1-x^m} = \frac{\sum\limits_{\lambda=-\infty}^{\infty}(-)^{\lambda}\omega_\lambda x^{\omega_{\lambda-1}}}{\sum\limits_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\omega_\lambda}}$$

Multiplying both sides by x,

$$\sum_{m=1}^{\infty}\frac{-mx^{m}}{1-x^m} = \frac{\sum\limits_{\lambda=-\infty}^{\infty}(-)^{\lambda}\omega_\lambda x^{\omega_{\lambda}}}{\sum\limits_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\omega_\lambda}} \tag{2}$$

The left side here is an interesting object called a Lambert series, with a structure not quite well defined; but it plays some role in number theory. Let us transform the Lambert series into a power series; it becomes

$$-\sum_{m=1}^{\infty}m\sum_{k=1}^{\infty}x^{km} = -\sum_{k_1}\sum_{m=1}^{\infty}mx^{km},$$

and these are all permissible power series, because though there are infinitely many of them, the inner ones begin with later and later terms.

Rearranging, this gives **37**

$$-\sum_{n=km}\sum_{m=1}^{\infty} mx^n = -\sum_{n=1}^{\infty} x^n \sum_{m/n} m$$
$$= -\sum_{n=1}^{\infty} \sigma(n)x^n,$$

where $\sigma(n)$ denotes the sum of the divisors of n, $\sigma(n) = \sum_{d|n} d$.

(Let us study $\sigma(n)$ for a moment.

$$\sigma(1) = 1, \sigma(2) = 3, \sigma_3 = 4, \sigma(5) = 6; \text{ indeed } \sigma(p) = p + 1$$

for a prime p. And $\sigma(n) = n + 1$ implies that n is prime. $\sigma(n)$ is not too big; there can be at most n divisers of n and so roughly $\sigma(n) = O(n^2)$. In fact it is known that $\sigma(n) = O(n^{1+epsilon})$, $\in > 0$, that is, a little larger than the first power. We shall however not be studying $\sigma(n)$ in detail).

Equation (2) can now be rewritten as

$$\sum_{n=1}^{\infty} \sigma(n)x^n \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda} x^{\omega_\lambda} = \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda-1}\omega_\lambda x^{\omega_\lambda}$$

Let us look for the coefficient of x^m on both sides. Remembering that the first few $\omega'_\lambda s$ are 0, 1, 2, 5, 7, 12, 15 $\cdots$, the coefficient of x^m on the left side is

$$\sigma(m) - \sigma(m-1) - \sigma(m-2) + \sigma(m-6) + \sigma(m-7) - - + + \cdots$$

On the right side the coefficient is 0 most frequently, because the pentagonal numbers are rather rare, and equal to $(-)^{\lambda-1}\omega_\lambda$ exceptionally, when $m = \omega_\lambda$.

$$\sigma(m) - \sigma(m-1) - \sigma(m-2) + + - - \cdots = \begin{cases} 0 & \text{usually,} \\ (-)^{\lambda-1}\omega_\lambda & \text{for } m = \omega_\lambda. \end{cases}$$

We now single out $\sigma(m)$. **38**

We may write

$$\sigma(m) = \sum_{0<\omega_\lambda<m} (-)^{\lambda-1}\sigma(m-\omega_\lambda) + \begin{cases} 0 & \text{usually,} \\ (-)^{\lambda-1}\omega_\lambda & \text{for } m = \omega_\lambda \end{cases}$$

This is an additive recursion formula for $\sigma(n)$. We can make it even more striking. The inhomogeneous piece on the right side is a little annoying. $\sigma(m-m)$ can occur on the right side only for $m = \omega_\lambda$; $\sigma(0)$ does not make sense; however, *for our purpose* let us define

$$\sigma(m-m) = m.$$

Then $\sigma(\omega_\mu - \omega_\mu) = \omega_\mu$, and the previous formula can now be written uninterruptedly as

$$\sigma(m) = \sum_{0<\omega_\lambda\leq m} (-)^{\lambda-1}\sigma(m-\omega_\lambda) \tag{3}$$

We have proved earlier that

$$p(m) = \sum_{0<\omega_\lambda\leq m} (-)^{\lambda-1} p(m-\omega_\lambda) \tag{4}$$

which is a formula completely identical with (3). Here $p(m-m) = p(0) = 1$. It is extraordinary that $\sigma(m)$ and $p(m)$ should have the same recursion formula, differing only in the definition of the term with $n = 0$. This fact was noted by Euler. In fact $p(m)$ is increasing monotonically, while the growth of $\sigma(m)$ is more erratic.

There are more relations between $p(m)$ and $\sigma(m)$. Let us start again with the identity

$$\prod_{m=1}^{\infty}(1-x^m)\sum_{m=0}^{\infty} p(m)x^m = 1 \tag{5}$$

We know that for a pair of power series A, B such that $AB = 1$, on taking 39
logarithmic derivatives, we have $\frac{A'}{A} + \frac{B'}{B} = 0$ or $\frac{A'}{A} = -\frac{B'}{B}$. So from (5),

$$\sum_{m=1}^{\infty}\sigma(m)x^m = \frac{\sum\limits_{n=0}^{\infty} np(n)x^n}{\sum\limits_{n=0}^{\infty} p(n)x^n},$$

or

$$\sum_{m=1}^{\infty}\sigma(m)x^m \sum_{k=0}^{\infty} p(k)x^k = \sum_{n=0}^{\infty} np(n)x^n.$$

Comparing coefficients of x^n,

$$np(n) = \sum_{m+k=n}\sigma(m)p(k),$$

or more explicitly,

$$np(n) = \sum_{m=1}^{\infty} \sigma(m)p(n-m) \tag{6}$$

This is a bilinear relation between $\sigma(n)$ and $p(n)$. This can be proved directly also in the following way. Let us consider all the partitions of n; there are $p(n)$ such:

$$\begin{aligned} n &= h_1 + h_2 + \cdots \\ n &= k_1 + k_2 + \cdots \\ n &= \ell_1 + \ell_2 + \cdots \\ &\cdots\cdots\cdots \end{aligned}$$

Adding up, the left side gives $np(n)$. Let us now evaluate the sum of the right sides. Consider a particular summand h and let us look for those partitions in which h figures. These are $p(n-h)$ partitions in which h occurs at least once, $p(n-2h)$ in which h occurs at least twices; in general, $p(n-rh)$ in which h occurs at least r times. Hence the number of those partitions which contain h *exactly r* times is $p(n-nh) - p(n-\overline{n+1}h)$. Thus the number of times h occurs 40
in all partitions put together is

$$\sum_{nh \leq n} n \left\{ p(n-nh) - (n-\overline{n+1}h) \right\}$$

Hence the contribution from these to the right side will be

$$h \sum_{nh \leq n} n \left\{ p(n-nh) - (n-\overline{n+1}h) \right\} = h \sum_{nh \leq n} p(n-nh)$$

on applying partial summation. Now summing over all summands h, the right side becomces

$$\sum_{h} h \sum_{nh \leq n} p(n-nh) = \sum_{n/m} \frac{m}{n} \sum_{m \leq n} p(n-m),$$

on putting $rh = m$; and this is

$$\sum_{m \leq n} p(n-m) \sum_{n.m} \frac{m}{n} = \sum_{m=1}^{n} p(n-m)\sigma(m).$$

Let us make one final remark.

Again from the Euler formula,

$$\sum_{m=1}^{\infty} \sigma(m) x^m = \frac{\sum\limits_{\lambda=-\infty}^{\infty} (-)^{\lambda-1} \omega_\lambda x^{\omega_\lambda}}{\sum\limits_{\lambda=-\infty}^{\infty} (-)^{\lambda} x^{\omega_\lambda}}$$

$$= \frac{\sum\limits_{\lambda=-\infty}^{\infty} (-)^{\lambda-1} \omega_\lambda x^{\omega_\lambda}}{\prod\limits_{m=1}^{\infty} (1 - x^m)}$$

$$= \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda-1} \omega_\lambda x^{\omega_\lambda} \sum_{m=0}^{\infty} p(m) x^m$$

Comparing the coefficients of x^m on both sides, **41**

$$\sigma(m) = p(m) - 1 \cdot p(m-1) - 2 \cdot p(m-2) + 5 \cdot p(m-5) + 7 \cdot p(m-7) - + \cdots$$

$$= \sum_{0 \leq \omega_\lambda \leq m} (-)^{\lambda-1} \omega_\lambda p(m - \omega_\lambda)$$

This last formula enables us to find out the sum of the divisors provided that we know the partitions. This is not just a curiosity; it provides a useful check on tables of pertitions computed by other means.

We go back to power series leading up to some of Ramanujan's theorems. Jacobi introduced the products

$$\prod_{n=1}^{\infty} (1 - x^{2n})(1 + \mathfrak{z} x^{2n-1})(1 + \mathfrak{z}^{-1} x^{2n-1}).$$

This is a power series in x; though these are infinitely many factors they start with progressively higher powers. The coefficients this time are not polynomials in z but from the field $R(z)$, the field of rational functions of z, which is a perfectly good field. Let us multiply out and we shall have a very nice surprise. The successive coeffieicnts are:

1

x : $\mathfrak{z} + \mathfrak{z}^{-1}$ (note that this is unchanged when $\mathfrak{z} \to \mathfrak{z}^{-1}$)

x^2 : $(1 + 1) = 0$

x^3 : $(\mathfrak{z} + \mathfrak{z}^{-1} - \mathfrak{z} - \mathfrak{z}^{-1}) = 0$

x^4 : $(-1 - 1 + \mathfrak{z}^2 + 1 + 1 + \mathfrak{z}^{-2}) = \mathfrak{z}^2 + \mathfrak{z}^{-2}$ (again unchanged when $\mathfrak{z} \to \mathfrak{z}^{-1}$)

. . . .

We observe that non-zero coeffieicnts are associated only with square exponents. We may threfore provisionally write 42

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+\mathfrak{z}x^{2n-1})(1+\mathfrak{z}^{-1}x^{2n-1}) = 1+\sum_{k=1}^{\infty}(\mathfrak{z}^k+\mathfrak{z}^{-k})x^{k^2}$$
$$= \sum_{k=-\infty}^{\infty}\mathfrak{z}^k x^{k^2} \qquad (7)$$

(with the terms corresponding to $\pm k$ folder together). This is a $\mathcal{V}$- series; only quadratic exponents occur.

We shall now prove the identity (7). But we have got to be careful. Consider the polynomial

$$\Phi_N(x,\mathfrak{z}) = \prod_{n=1}^{N}(1-x^{2n})(1+\mathfrak{z}x^{2n-1})(1+\mathfrak{z}^{-1}x^{2n-1})$$

This consists of terms $\mathfrak{z}^j x^k$ with $-N \leq j \leq N$, $0 \leq k \leq N(N+1)+2N^2 = 3N^2+N$. We can rearrange with respect to powers of z. The coefficients are now polynomials in x. z and z^{-1} occur symmetrically.

$$\Phi_N(x,\mathfrak{z}) = C_\circ(x) + (\mathfrak{z}+\mathfrak{z}^{-1})C_1(x) + (\mathfrak{z}^2+\mathfrak{z}^{-2})C_2(x)+\cdots+(\mathfrak{z}^N+\mathfrak{z}^{-N})C_N(x).$$

Let us calculate the $C's$. It is cumbersome to look for $C_\circ$, for so many cancellations may occur. It is easier to calculate C_N. Since the highest power of z can occur only from the terms with the highest power of x, we have

$$C_N(x) = \prod_{n=1}^{N}(1-x^{2n}) \times x^{1+3+\cdots+(2N-1)}$$
$$= x^{N^2}\prod_{n=1}^{N}(1-x^{2n})$$

Now try to get a recursion among the $C's$. Replacing z by zx^2, we get 43

$$\Phi_N(x,\mathfrak{z}x^2) = \prod_{n=1}^{N}(1-x^{2n})(1+\mathfrak{z}x^{2n+1})(1+\mathfrak{z}^{-1}x^{2n-3}).$$

Compare $\Phi_N(x,\mathfrak{z}x^2)$ and $\Phi_N(x,\mathfrak{z})$; these are related by the equation

$$\Phi_N(x,\mathfrak{z}x^2)(1+\mathfrak{z}x)(1+\mathfrak{z}^{-1}x^{2N-1}) = \Phi_N(x,\mathfrak{z})(1+\mathfrak{z}x^{2n+1})(1+\mathfrak{z}^{-1}x^{-1})$$

The negative power in the last factor on the right is particularly disgusting; to get rid of it we multiply both sides by xz, leading to

$$\Phi_N(x, \mathfrak{z}x^2)(x\mathfrak{z} + x^{2N}) = \Phi_N(x, \mathfrak{z})(1 + \mathfrak{z}x^{2N+1}),$$

or
$$(1 + \mathfrak{z}x^{2N+1})(C_\circ(x) + (\mathfrak{z} + \mathfrak{z}^{-1})C_1(x) + \cdots + (\mathfrak{z}^N + \mathfrak{z}^{-N})C_N(x))$$
$$= (x\mathfrak{z} + x^{2n})(C_\circ(x) + (\mathfrak{z}x^2 + \mathfrak{z}^{-1}x^{-2})C_1(x) +$$
$$+ (\mathfrak{z}^2x^4 + \mathfrak{z}^{-2}x^{-4})C_2(x) + \cdots + (\mathfrak{z}^N x^{2N} + \mathfrak{z}^{-N}x^{-2N})C_N(x))$$

These are perfectly harmless polynomials in x; we may compare coefficients of $\mathfrak{z}^k$. Then

$$C_k(x) + C_{k-1}(x)x^{2N+1} = C_k(x)x^{2k+2N} + x^{2k-1}C_{k-1}(x),$$

or
$$C_k(x)(1 - x^{2N+2k}) = C_{k-1}(x)x^{2k-1}(1 - x^{2N-2k+2})$$

(We proceed from C_k to C_{k-1} since C_N is already known).

$$C_{k-1}(x) = \frac{x^{-2k+1}(1 - x^{2N+2k})}{1 - x^{2N-2k+2}}C_k(x)$$

Since $C_N(x) = x^{N^2}\prod_{n=1}^{N}(1 - x^{2n})$, we have in succession 44

$$C_{N-1}(x) = x^{N^2-2N+1}\frac{1 - x^{4N}}{1 - x^2}\prod_{n=1}^{N}(1 - x^{2n})$$
$$= x^{(N-1)^2}\prod_{n=2}^{N}(1 - x^{2n}) \cdot (1 - x^{4N});$$
$$C_{N-2}(x) = x^{(N-2)^2}\prod_{n=3}^{N}(1 - x^{2n}) \cdot (1 - x^{4N})(1 - x^{4N-2})$$
$$\cdots\cdots\cdots\cdots$$

In general,

$$C_{N-j}(x) = x^{(N-j)^2}\prod_{n=j+1}^{N}(1 - x^{2n})\prod_{m=0}^{j-1}(1 - x^{4N-2m})$$

or, with $j = N - n$,

$$C_n(x) = x^{n^2}\prod_{n=N-n+1}^{N}(1 - x^{2n})\prod_{m=0}^{N-n-1}(1 - x^{4N-2m}) \tag{8}$$

Equation (8) leads to some congruence relations. The lowest terms of $C_n(x)$ have exponent

$$n^2 + 2(N - n + 1) = 2N + (n^2 - 2n + 1) + 1 \geq 2N + 1$$

Hence

$$C_n(x) \equiv x^{k^2} \pmod{x^{2N+1}} \tag{9}$$

From the original formula,

$$\begin{aligned}
\Phi_N(x, \mathfrak{z}) &= \prod_{n=1}^{N} (1 - x^{2n})(1 + \mathfrak{z} x^{2n+1})(1 + \mathfrak{z}^{-1} x^{2n-1}) \\
&\equiv 1 + (\mathfrak{z} + \mathfrak{z}^{-1})x + (\mathfrak{z}^2 + \mathfrak{z}^{-2})x^4 + \cdots \pmod{x^{2N+1}} \\
&\equiv \sum_{k=-\infty}^{\infty} \mathfrak{z}^k x^{k^2} \pmod{x^{2N+1}},
\end{aligned}$$

since the infinite series does not matter, the higher powers being absorbed in 45
the congruence. Hence

$$\Phi_N(x, \mathfrak{z}) \equiv \prod_{n=1}^{\infty} (1 - x^{2n})(1 + \mathfrak{z} x^{2n-1})(1 + \mathfrak{z}^{-1} x^{2n-1}) \pmod{x^{2N+1}}$$

The new terms $x^{2N+2}, \ldots$, are absorbed by mod x^{2N+1}. We have

$$\prod_{n=1}^{\infty} (1 - x^{2n})(1 + \mathfrak{z} x^{2n-1})(1 + \mathfrak{z}^{-1} x^{2n-1}) \equiv \sum_{k=-\infty}^{\infty} \mathfrak{z}^k x^{k^2} \pmod{x^{2N+1}}$$

Thus both expansions agree as far as we wish, and this is what we mean by equality of formal power series. Hence we can replace the congruence by equality, and Jacobi's identity (7) is proved.

As an application of this identity, we shall now give a new proof of the pentagonal numbers theorem. We replace x by y^3, as we could consistently in the whole story; only read modulo y^{6N+3}. Then we have

$$\prod_{n=1}^{\infty} (1 - y^{6n})(1 + \mathfrak{z} y^{6n-3})(1 + \mathfrak{z}^{-1} y^{6n-3}) = \sum_{k=-\infty}^{\infty} \mathfrak{z}^k y^{3k^2}$$

We now do something which needs some justification. Replace z by $-y$. This is something completely strange, and would interfere seriously with our reasoning. For $\Phi_N(y^3, \mathfrak{z})$ we had congruences modulo y^{6N+3}. If we replaced z

by y^3 nobody could forbid that. Since z occurs in negative powers, the powers of y might be lowered too by as much as N. We obtain polynomials in y alone on both sides, but true modulo y^{5N+3}, because we may have lowered powers of 46
y. With this proviso it is justified to replace z by $-y$; so that ultimately we have

$$\prod_{n=1}^{\infty}(1-y^6)(1-y^{6n-2})(1-y^{6n-4}) = \sum_{k=-\infty}^{\infty}(-)^k y^{3k^2+k} \pmod{y^{5N+3}}$$

We can carry over the old proof step by step. Since we now have only even powers of y, this leads to

$$\prod_{m=1}^{\infty}(1-y^{2m}) = \sum_{k=-\infty}^{\infty}(-)^k y^{k(3k+1)}$$

These are actually power series in y^2. Set $y^2 = x$, then

$$\prod_{m=1}^{\infty}(1-x^m) = \sum_{k=-\infty}^{\infty}(-)^K x^{k(3k+1)/2}$$

which is the pentagonal numbers theorem.

Lecture 6

In the last lecture we used the Jacobi formula: 47

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+\mathfrak{z}x^{2n-1})(1+\mathfrak{z}^{-1}x^{2n-1}) = \sum_{k=-\infty}^{\infty}\mathfrak{z}^k x^{k^2} \tag{1}$$

to give a new proof of Euler's pentagonal numbers theorem. We proceed to give another application. We observe again that the right side of (1) is a power series in x; we cannot do anything about the z's and no formal differentiation can be carried out with respect to z. Let us make the substitution $\mathfrak{z} \to -\mathfrak{z}x$. This again interferes greatly with our variable x. Are we entitled to do this? Let us look back into our proof of (1). We started with a curtailed affair

$$\Phi_N(x,\mathfrak{z}) = \prod_{n=1}^{\infty}(1-x^{2n})(1+\mathfrak{z}x^{2n-1})(1+\mathfrak{z}^{-1}x^{2n-1})$$

and this was a polynomial of the proper size and everything went through. When we replace z by $-zx$ and multiply out, the negative powers might accumulate and we might be destroying x^N possibly; nevertheless the congruence relations would be true this time modulo x^{N+1} instead of x^{2N+1} as it was previously; but this is all we went. So the old proof can be reproduced step by step and every thing matches modulo x^{N+1}. (Let us add a side remark. In the proof of (1) we had to replace z by zx^2 - and this was the essential step in the proof. We cannot do the same here as this would lead to congruences $\mod x$ only. Before we had the congruences we had identities and there we could carry out any substitution. Then we adopted a new point of view and introduced congru- 48
ences; and that step bars later the substitution $\mathfrak{z} \to \mathfrak{z}x^2$.

So let us make the substitution $\mathfrak{z} \to -\mathfrak{z}x$ without further compuncton. This gives us

$$\prod_{n=1}^{\infty}(1 - x^{2n})(1 - \mathfrak{z}x^{2n})(1 - \mathfrak{z}^{-1}x^{2n-2}) = \sum_{k=-\infty}^{\infty}(-)^k\mathfrak{z}^k x^{k^2+k}$$

This is not nicely arranged. There appears an extraordinary term without x-corresponding to $n = 1$ in the last factor on the left side; let us keep this apart. Also on the right side the exponent of x is $k(k+1)$, so that every number occurs twice; let us keep these two pieces together. We then have

$$\begin{aligned}(1 - \mathfrak{z})^{-1}\prod_{n=1}^{\infty}&(1 - x^{2n})(1 - \mathfrak{z}x^{2n})(1 - \mathfrak{z}^{-1}x^{2n})\\
&= \sum_{k=0}^{\infty}(-)^k\mathfrak{z}^k x^{k(k+1)} + \sum_{k=0}^{\infty}(-)^{-k-1}\mathfrak{z}^{-k-1}x^{k(k+1)}\\
&\qquad\text{(where in the second half we have replaced } k \text{ by } -k-1\text{)},\\
&= \sum_{k=0}^{\infty}(-)^k x^{k(k+1)}(\mathfrak{z}^k - \mathfrak{z}^{-k-1})\\
&= \sum_{k=0}^{\infty}(-)^k x^{k(k+1)}\mathfrak{z}^k(1 - \mathfrak{z}^{-2k-1})\\
&= \sum_{k=0}^{\infty}(-)^k x^{k(k+1)}\mathfrak{z}^k(1 - \mathfrak{z}^{-1})(1 + \mathfrak{z}^{-1} + \mathfrak{z}^{-2} + \cdots + \mathfrak{z}^{-2k})\end{aligned}$$

We now have an infinite series in x equal to another. Now recollect that our coefficients are from the field $R(z)$ which has no zero divisors. So we may cancel $1 - z^{-1}$ on both sides; this is a non-zero factor in $R(z)$ and has nothing to **49**
do with differentiation. This leads to

$$\prod_{n=1}^{\infty}(1 - x^{2n})(1 - \mathfrak{z}x^{2n})(1 - \mathfrak{z}^{-1}x^{2n}) = \sum_{k=0}^{\infty}(-)^k x^{k(k+1)}(\mathfrak{z}^k + \mathfrak{z}^{k-1} + \cdots + \mathfrak{z}^{-k}).$$

In the field $R(z)$ we can replace z by 1. We can do what we like in the field and that is the essence of the power series method. So putting $z = 1$,

$$\prod_{n=1}^{\infty}(1 - x^{2n})^3 = \sum_{k=0}^{\infty}(-)^k x^{k(k+1)}(2k + 1).$$

This is a power series in x^2; give it a new name, $x^2 = y$. Then

$$\prod_{n=1}^{\infty}(1 - y^n)^3 = \sum_{k=0}^{\infty}(-)^k(2k + 1)y^{k(k+1)/2} \tag{2}$$

This is a very famous identity of Jacobi, originally proved by him by an altogether different method using the theory of functions. Let us juxtapose it with the Euler pentagonal formula:

$$\prod_{n=1}^{\infty}(1-y^n) = \sum_{\lambda=-\infty}^{\infty}(-)^{\lambda}x^{\lambda(3\lambda-1)/2} \tag{2a}$$

Let us proceed to yet another application of the triple product formula; we shall obtain some of Ramanujan's formulas. Taking away the first part of the triple product formula we have

$$\prod_{n=1}^{\infty}(1+\mathfrak{z}x^{2n-1})(1+\mathfrak{z}^{-1}x^{2n-1}) = \sum_{k=-\infty}^{\infty}\mathfrak{z}^k x^{k^2}\frac{1}{\prod\limits_{n=1}^{\infty}(1-x^{2n})} \tag{3}$$

The second part on the right side here is of interest, because it is the generating function of the partition. We had earlier the formula 50

$$\begin{gathered}\prod_{n=1}^{\infty}(1+\mathfrak{z}x^{2n-1}) = \sum_{m=0}^{\infty}\mathfrak{z}^m C_m(n),\\ C_m(x) = \frac{x^{m^2}}{(1-x^2)\cdots(1-x^{2m})}\end{gathered} \tag{4}$$

and these are permissible power series, beginning with later and later powers of x, and so the right side of (4) makes sense, as a formal power series in x.

Substituting (4) in (3), we have

$$\sum_{r=0}^{\infty}\mathfrak{z}^n C_r(x)\sum_{s=0}^{\infty}\mathfrak{z}^{-s}C_s(x) = \sum_{k=-\infty}^{\infty}\mathfrak{z}^k x^{k^2}\frac{1}{\prod\limits_{n=1}^{\infty}(1-x^{2n})} \tag{5}$$

We can compare z^O on both sides for, for very high x^N the left side will contain only finitely many terms and all otheres will disappear below the horizon; we can also add as many terms as we wish. So equating coefficients of z^O, we have

$$\sum_{r=0}^{\infty}C_r(x)C_r(x) = \frac{1}{\prod\limits_{n=1}^{\infty}(1-x^{2n})},$$

or

$$\sum_{r=0}^{\infty}\frac{x^{2r^2}}{(1-x^2)^2\cdots(1-x^{2n})^2} = \frac{1}{\prod\limits_{n=1}^{\infty}(1-x^{2n})}$$

We have even powers of x consistently on both sides; so replace x^2 by y, and write down the first few terms explicitly:

$$1 + \frac{y}{(1-y)^2} + \frac{y^4}{(1-y)^2(1-y^2)^2} + \frac{y^9}{(1-y)^2(1-y^2)^2(1-y^3)^2} + \cdots = \frac{1}{\prod\limits_{n=1}^{\infty}(1-y^n)} \tag{6}$$

This formula is found in the famous paper of Hardy and Ramanujan (1917) 51
and ascribed by them to Euler. It is very useful for rough appraisal of asymptotic formulas. Hardy and Ramanujan make the cryptic remark that it is "*a* formula which lends itself to wide generalisations". This remark was at first not very obvious to me; but it can now be interpreted in the following way. Let us look for $\mathfrak{z}^k$ in (5). Then

$$\sum_{\substack{r,s\\ r-s=k}} C_r(x)C_s(x) = \frac{x^{k^2}}{\prod\limits_{n=1}^{\infty}(1-x^{2n})}$$

or, replacing r by $s+k$, and writting C_s for $C_s(x)$, the left side becomes

$$\sum_{s=0}^{\infty} C_s C_{s+k} = 1\cdot\frac{x^{k^2}}{(1-x^2)\cdots(1-x^{2k})} + \frac{x^{1+(k+1)^2}}{(1-x^2)^2(1-x^4)\cdots(1-x^{2k+2})} + \\ + \frac{x^{4+(k+2)^2}}{(1-x^2)^2(1-x^4)^2(1-x^6)\cdots(1-x^{2k+4})} + \cdots$$

Let us divide by x^{k^2}. The general exponent on the right side is $\ell^2+(k+\ell)^2$, 52
so on division it becomes $2\ell^2+2k\ell$. Every exponent is even, which is a very nice situation. Replace x^2 by y, and we get the 'wide generalisation' of which Hardy and Ramanujan spoke:

$$\frac{1}{(1-y)(1-y^2)\cdots(1-y^k)} + \frac{y^{k+1}}{(1-y)^2(1-y^2)\cdots(1-y^{k+1})} \\ + \frac{y^{2(k+2)}}{(1-y)^2(1-y^2)^2(1-y^3)\cdots(1-y^{k+2})} + \cdots \\ \frac{y^{l(k+l)}}{(1-u)^2\cdots(1-y^l)^2(1-y^{l+1})\cdots(1-y^{k+l})} + \cdots = \frac{1}{\prod\limits_{n=1}^{\infty}(1-y^n)} \tag{7}$$

k is an assigned number and it can be taken arbitrarily.

So such expansions are not unique.

Thus (6) and (7) give two different expansions for

$$\frac{1}{\prod_{n=1}^{\infty}(1-y^n)}.$$

We are now slowly coming to the close of our preoccupation with power series; we shall give one more application due to Ramanujan (1917). In their paper Hardy and Ramanujan gave a surprising asymptotic formula for $p(n)$. It contained an error term which was something unheard of before, $O(n^{-1/4})$, error term *decreasing* as n increases. Since $p(n)$ is an integer it is enough to take a few terms to get a suitable value. The values calculated on the basis of the 53 asymptotic formula were checked up with those given by Macmahon's tables and were found to be astonishingly close. Ramanujan looked at the tables and with his peculiar insight discovered something which nobody else could have noticed. He found that the numbers $p(4)$, $p(9)$, $p(14)$, in general $p(5k+4)$ are all divisible by 5; $p(5)$, $p(12)$, $\cdots p(7k+5)$ are all divisible by 7; $p(11k+6)$ by 11. So he thought thiswas a general property. *A* divisibility property of $p(n)$ is itself surprising, because $p(n)$ is a function defined with reference to addition. The first and second of these results are simpler than the third. Ramanujan in fact suggested more. If we chose a special progression modulo 5^λ, then all the terms are divisible by 5^λ. There are also special progressions modulo $7^{2\lambda-1}$; so for 11. Ramanujan made the general conjecture that if $\delta = 5^a7^b11^c$ and $24n \equiv 1 \pmod{\delta}$, then $p(n) \equiv 0 \pmod{\delta}$. In this form the conjecture is wrong. These things are deeply connected with the theory of modular forms; the cases 5 and 7 relate to modular forms with subgroups of genus 1, the case 11 with genus 2.

Let us take the case of 5. Take $p(5k+4)$. Consider $\Sigma p(n)x^n$; it is nicer to multiply by x and look for x^{5k}. We have to show that the coefficients of x^{5k} in $x\Sigma p(n)x^n$ are congruent to zerp modulo 5. We wish to juggle around with series a bit. Take $\Sigma a_n x^n$; we want to study x^{5k}. Multiply by the series $1 + b_1x^5 + b_2x^{10} + \cdots$ where the b's are integers. We get a new power series

$$\sum a_n x^n \cdot (1 + b_1x^5 + b_2x^{10} + \cdots) = \sum c_n x^n,$$

which is just as good. It is enough if we prove that for this series every fifth 54 coefficient $\equiv 0 \pmod 5$.

For,

$$\sum a_n x^n = \frac{\sum c_n x^n}{1 + b_1x^5 + b_2x^{10} + \cdots}$$

$$= \sum c_n x^n, (1 + d_1 x^5 + d_2 x^{10} + \cdots), \text{ say.}$$

Then if every fifth coefficient of $\Sigma c_n x^n$ is divisible by 5, multiplication by $\Sigma d_n x^{5n}$ will not disturb this. For a prime p look at

$$(1 + x)^p = 1 + \binom{p}{1} x + \binom{p}{2} x^2 + \binom{p}{3} x^3 + \cdots + \binom{p}{p} x^p.$$

All except the first and last coefficients on the right side are divisible by p, for in a typical term $\binom{p}{q} = \frac{p!}{(p-q)!q!}$, the p inm the numerator can be cancelled only by a p in the denominator. So

$$(1 + x)^p \equiv 1 + x^p \pmod{p}.$$

This means that the difference of the two sides contains only coefficients divisible by p. This

$$(1 - x)^5 \equiv 1 + x^5 \pmod{5}$$

We now go to Ramanujan's proof that $p(5k + 4) \equiv 0 \pmod{5}$ We have 55

$$x \sum p(n) x^n = \frac{x}{\prod (1 - x^n)}$$

It is irrelevant here if we multiply both sides by a series containing only $x^5, x^{10}, x^{15}, \cdots$. This will not ruin our plans as we have declared in advance. So

$$x \sum p(n) x^n \prod_{m=1}^{\infty} (1 - x^{5m}) = \frac{x}{\prod (1 - x^n)} \prod_{m=1}^{\infty} (1 - x^{5m})$$

$$\equiv \frac{x}{\prod (1 - x^n)} \prod_{m=1}^{\infty} (1 - x^m)^5 \text{ modulo } 5$$

$$\left(\prod (1 - x^{5m}) - \prod (1 - x^m)^5 \text{has only coefficients divisible by } 5 \right)$$

$$\equiv x \prod_{m=1}^{\infty} (1 - x^m)^4 \text{ modulo } 5$$

$$= x \prod_{m=1}^{\infty} (1 - x^m) \prod_{m=1}^{\infty} (1 - x^m)^3.$$

For both products on the right side we have available wonderful expressions. By (2) and (2a),

$$x \prod (1 - x^m) \prod (1 - x^m)^3 = x \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda(3\lambda-1)/2} \sum_{k=0}^{\infty} (-)^k (2k + 1) x^{k(k+1)/2}$$

The typical term on the right side is

$$\sum_{k=0}^{\infty}(-)^{\lambda+k}x^{1+\lambda(3\lambda-1)/2} + k(k+1)/2$$

The exponent $= 1 + \lambda(3\lambda-1)/2 + k(k+1)/2$, and we want this to be of the form $5m$. Each such combination contributes to x^{5m}. We want 56

$$1 + \frac{\lambda(3\lambda-1)}{2} + \frac{k(k+1)}{2} \equiv 0 \pmod 5$$

Multiply by 8; that will not disturb it. So we want

$$\begin{aligned} 8 + 12\lambda^2 - 4\lambda + 4k^2 + 4k &\equiv 0(5),\\ 3 + 2\lambda^2 - 4\lambda + 4k^2 + 4k &\equiv 0(5),\\ 2(\lambda-1)^2 + (2k+1)^2 &\equiv 0(5). \end{aligned}$$

This is of the form:

$$2.\ \text{a square} + \text{another square} \equiv 0(5)$$

Now

$$\begin{aligned} A^2 &\equiv 0, 1, 4(5),\\ 2B^2 &\equiv 0, 2, 3(5); \end{aligned}$$

and so $A^2 + 2B^2 \equiv 0(5)$ means only the combination $A^2 \equiv 0(5)$ and $2B^2 \equiv 0(5)$; each square must therefore separately be divisible by 5, or

$$2k + 1 \equiv 0(5)$$

So to x^{5m} has contributed only those combinations in which $2k+1$ appeared; and every one of these pieces carried with it a factor of 5. This porves the result.

The case $7k + 5$ is even simpler. We multiply by a series in x^7 leading to $(1 - x^m)^6$ which is to be broken up into two Jacobi factors $(1 - x^m)^3$. These are examples of very beautiful theorems proved in a purely formal way.

We shall deal in the next lecture with one more starting instance, the Rogers-Ramanujan identities which one cannot refrain from talling about.

Lecture 7

We wish to say something about the celebrated Rogers-Ramanujan identities: **57**

$$1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \cdots = \frac{1}{\prod\limits_{\substack{n>0 \\ n\equiv\pm1 \pmod 5}} (1-x^n)}; \tag{1}$$

$$1 + \frac{x^2}{1-x} + \frac{x^{2\cdot3}}{(1-x)(1-x^2)} + \frac{x^{3\cdot4}}{(1-x)(1-x^2)(1-x^3)} + \cdots = \frac{1}{\prod\limits_{\substack{n>0 \\ n\equiv\pm2 \pmod 5}} (1-x^n)} \tag{2}$$

The right hand sides of (1) and (2), written down explicitly, are respectively

$$\frac{1}{(1-x)(1-x^4)(1-x^6)(1-x^9)\ldots}$$
$$\frac{1}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\ldots}$$

One immediately observes that ±1 are quadratic residues modulo 5, and ±2 quadratic non-residues modulo 5. These identities were first communicated by Ramanujan in a letter written to Hardy from India in February 1913 before he embarked for England. No proofs were given at that time. It was a remarkable fact, nevertleless, to have even written down such identities. It is **58**
true that Euler himself did some experimental work with the pentagonal numbers formula. But one does not see the slightest reason why anybody should have tried ±1, ±2 modulo 5. Then in 1917 something happened. In an old

volume of the Proceedings of the London Mathematical Society Ramanujan found that Rogers (1894) had these identities along with extensions of hypergeometric functions and a wealth of other formulae. In 1916 the identities were published in Macmahon's Combinatory Analysis without proof, but with a number-theoretic explanation. This was some progress. In 1917 I.Schur gave proofs, one of them combinatorial, on the lines of F.Franklin's proofof Euler's theorem. Schue also emphasized the mathematical meaninf of the identities.

Let us look at the meaning of these identities. Let us write the right side of (1) as a power-series, say,

$$\frac{1}{(1-x)(1-x^4)(1-x^6)(1-x^9)\ldots} = \sum_{n=0}^{\infty} q'(n)x^n,$$

$q'(n)$ is the number of terms collected from summands 1, 4, 6, ... with repetitions, or, what is the same thing, the number of times in which n can be expressed as the sum of parts $\equiv \pm 1 \pmod 5$, with repetitions. Likewise, if we write

$$\frac{1}{\prod\limits_{n\equiv\pm 2(5)}(1-x^n)} = \sum_{n=0}^{\infty} q''(n)x^n,$$

then $q''(n)$ is the number of representations of n as the sum of parts $\equiv \pm 2 \pmod 5$, with repetitions.

The expressions on the other side appear directly.

Take 59

$$\frac{x^{k^2}}{(1-x)(1-x^2)\cdots(1-x^4)}$$

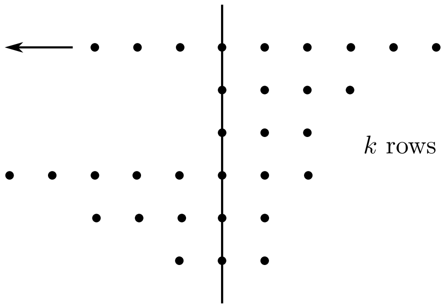
If we write

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = a_0 + a_1x + a_2x^2 + \cdots$$

then the coefficient a_n gives us the number of partitions of n into parts not exceeding k. Let us represent the partitions by dots in a diagram, each vertical column denoting a summand. Then there are at most k rows in the diagram. Since k^2 is the sum of the k first odd numbers,

$$k^2 = 1 + 3 + 5 + \cdots + (2k-1),$$

each partition of n into summands not exveeding k can be enlarged into a partition of $n + k^2$ into summands which differ by at least two, for we can adjoin k^2 dots on the left side, putting one in the lowest row, three in the next, five in the one above and so on finally $2k-1$ in the top most row. Conversely any partition of n into



parts with minimal difference 2 can be mutilated into a partition of $n-k^2$ into 60
summands not exceeding k. Hence there is a one one correspondence between these two types. So the coefficients in the expansion of $\dfrac{x^{k^2}}{(1-x)(1-x^2)\cdots(1-x^k)}$ represent the number of times that a number N can be decomposed into k parts (the partitions are now read horizontally in the diagram) differing by two at least. When this is done for each k and the results added up, we get the following arithmetical interpretation of (1): The number of partitions of n with minimal difference two is equal to the number of partitions into summands congruent to $\pm 1 \pmod 5$ allowing repetitions.

A similar explanation is possible in the case of (2). On the left side we can account for the exponents 2.3, 3.4, …, $k(k+1)$, … in the numerator by means of triangular numbers. In the earlier diagram we adjoin on the left 2, 4, 6, …, $2k$ dots beginning with the lowerst row. The number thus added is $2+4+\cdots\cdots\cdots+2k = k(k+1)$; this disposes of $x^{k(k+1)}$in the numerator. So read horizontally, the diagram gives us a decomposition into parts which differ by 2 at least, but the summand 1 is no longer tolerated. $\dfrac{x^{k(k+1)}}{(1-x)\cdots(1-x^k)}$ gives us therefore the enumeration of x^N by parts differeing by 2 at least, the part 1 being forbidden. We have in this way the following arithmetical interpretation of (2): The number of partitions of n into parts not less than 2 and with minimal difference 2, is equal to the number of partitions of n into parts congruent ± 2 (mod 5), repetitions allowed.

By a similar procedure we can construct partitions where 1 and 2 are for- 61
bidden, partitions differing by at least three, etc. In the case where the difference is 3, we use $1, 4, 7, \ldots$, so that the number of dots adjoined on the left is $1+4+7+\cdots$ to k terms $= k(3k-1)/2$, so a pentagonal number, and this is

no surprise. In fact $\sum \frac{x^{k(3k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)}$ would give us the number of partitions into parts differing by at least 3. And for 4 the story is similar.

The unexpected element in all these cases is the association of partitions of a definite type with divisibility properties. The left-side in the identities is trivial. The deeper part is the right side. It can be shown that there can be no corresponding identities for moduli higher than 5. All these appear as wide generalisations of the old Euler theorem in which the minimal difference between the summands is, of course, 1. Euler's theorem is therefore the nucleus of all such results.

We give here a proof of the Roger-Ramanujan identities which is in line with the treatment we have been following, the motiod of formal power series. It is a transcription of Roger's proof in Hardy's 'Ramanujan', pp.95-98. We use the so-called Gaussian polynomials.

Let us introduce the Gaussian polynomials in a much neater notation than usual. Consider for first the binomial coefficients:

$$\binom{n}{m} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot k}$$

(Observe that both in the numerator and in the denominator there are k factors, which are consecutive integers, and that the factors of equal rank in both numerator and denominator always add up to $n+1$). The $\binom{n}{k}$ are all integers, as is obvious from the recursion formula **62**

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$\binom{n}{n} = 1$, of course, and by definition, $\binom{n}{0} = 1$ We also define $\binom{n}{k} = 0$ for $k > n$ or for $k < 0$. Observe also the eymmetry: $\binom{n}{k} = \binom{n}{n-k}$

The Gaussian polynomials are something of a similar nature. We define the Gaussian polynomial

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_x$$

by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-x^n)(1-x^{n-1})\cdot(1-x^{n-k+1})}{(1-x)(1-x^2)\cdots(1-x^k)}$$

The sum of the indices of x in corresponding factors in the numerator r and denominator is $n+1$, as in $\binom{n}{k}$. That the $\begin{bmatrix} n \\ k \end{bmatrix}$ are polynomials in x is obvious from the recursion formula

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n+1 \\ k-1 \end{bmatrix} x^k$$

where $\begin{bmatrix} n \\ n \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ by definition. The recursion formula is just the same as that for $\binom{n}{k}$ except for the factor in the second term on the right. Also define $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$; also let $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ for $k > n$ or $k < 0$. **63**

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} x^\circ = 1,$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1-x^2}{1-x} = 1 + x;$$

and so on. We also have the symmetry:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$$

The binomial coefficients appear in the expansion

$$(1+y)^2 = \sum_{k=0}^{n} \binom{n}{k} y^k.$$

Likewise, the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}$ appear in expansion:

$$(1+y)(1+xy)(1+x^2y)\cdots(1+x^{n-1}y) = 1 + yG_1(x) + y^2G_2(x) + \cdots + y^nG_n(x)$$

where
$$G_k(x) = x^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}$$

Notice that for $x = 1$, $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$. Changing y to yx we get the recursion formula stated earlier.

We now go back to an identity we has porved sometime back:

$$\prod_{n=1}^{\infty} (1 + \mathfrak{z}x^{2n-1}) = 1 + \mathfrak{z}C_1(x) + \mathfrak{z}^2C_2(x) + \cdots \tag{1}$$

where

$$C_k(x) = \frac{x^{k^2}}{(1-x^2)\cdots(1-x^{2k})}$$

Now write 64

$$x^2 = X, 1-X = X_1 - X^2 = X_2, \ldots, 1-X^k = X_k;$$
$$(1-X)(1-X^2)\cdots(1-X^k) = X, X_2 \ldots X_k = X_k!$$

With this notation,

$$C_k(x) = \frac{x^{k^2}}{X_k!}$$

From Jacobi's triple porduct formula, we have

$$\prod_{n=1}^{\infty}(1+\mathfrak{z}x^{2n-1})(1+\mathfrak{z}^{-1}x^{2n-1}) = \frac{\sum\limits_{\ell=-\infty}^{\infty} \mathfrak{z}^l x^{l^2}}{\prod\limits_{n=1}^{\infty}(1-x^{2n})} \tag{2}$$

By (1), the left side of (2) becomes

$$\sum_{r=0}^{\infty} \mathfrak{z}^r C_r(x) \sum_{s=0}^{\infty} \mathfrak{z}^{-s} C_s(x) = \sum_{n=0}^{\infty} \frac{B_n(\mathfrak{z}, x)}{X_n!},$$

where $X_\circ!$ is put equal to 1. $B_n(\mathfrak{z}, x)$ is the term corresponding to $r+s=n$ when the left side is multiplied out in Cauchy fashion. Thus

$$\begin{aligned} B_n(\mathfrak{z}, x) &= X_n! \sum_{r+s=n} \mathfrak{z}^{r-s} C_r(x) C_s(x) \\ &= X_n! \sum_{r=0}^{n} \mathfrak{z}^{n-2r} \frac{x^{r^2+s^2}}{X_r! X_{n-r}!} \ (r+s=n) \\ &= \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_X x^{(n-r)^2+r^2} \mathfrak{z}^{n-2r} \end{aligned}$$

Notice that the powers of z occur with the same parity as n. Now (2) can be re-written as 65

$$\sum_{n=0}^{\infty} \frac{B_n(\mathfrak{z}, x)}{X_n!} = \frac{\sum\limits_{l=-\infty}^{\infty} \mathfrak{z}^l x^{l^2}}{\prod\limits_{n=1}^{\infty}(1-x^{2n})}$$

Both sides are formal power series in x of the appropriate sort. The $B_n(\mathfrak{z}, x)$ are linear combinations of power series in x with powers of z for coeffieicnts. We can now compare powers of z. We first take only even exponents $\mathfrak{z}^{2m}$; we then have infinitely many equations of formal power series. We multiply the equation arising from $\mathfrak{z}^{2m}$ by $(-)^m x^{m(m-1)}$ and add all these equations together; (amd that is the trick, due to Rogers) we can do this because of linearity. Then

$$\sum_{l=0}^{\infty} \frac{\beta_{2l}(x)}{X_{2l}!} = \frac{\sum\limits_{m=0}^{\infty} (-)^m x^{m(m-1)} x^{(2m)^2}}{\prod\limits_{n=1}^{\infty} (1 - x^{2n})}, \tag{3}$$

where $$\beta_{2l}(x) = \sum_{r=0}^{2l} \begin{bmatrix} 2l \\ r \end{bmatrix}_X x^{(2l-r)^2+r^2} (-)^{l-r} x^{(l-r)(l-r-1)}$$

Writting $l - r = s$,

$$\begin{aligned}\beta_{2l}(x) &= \sum_{s=-l}^{l} \begin{bmatrix} 2l \\ l-s \end{bmatrix} x^{2l^2+2s^2} (-)^s x^{s(s-1)}\\ &= x^{2l^2} \sum_{s=-l}^{l} \begin{bmatrix} 2l \\ l+s \end{bmatrix} (-)^s x^{3s^2} - s\end{aligned}$$

(because of the symmetry between $l - s$ and $l + s$). Separating out the term corresponding to $s = 0$ and folding together the terms corresponding to s and **66**
$-s$,

$$\begin{aligned}\beta_{2l}(x) &= x^{2l^2} \left\{ \begin{bmatrix} 2l \\ l \end{bmatrix} + \sum_{s=1}^{l} (-)^s \begin{bmatrix} 2l \\ l+s \end{bmatrix} x^{s(3s-1)} (1 + x^{2s}) \right\}\\ &= x^{2l^2} \left\{ \sum_{s=1}^{l} (-)^s \begin{bmatrix} 2l \\ l+s \end{bmatrix} x^{s(3s-1)} + \sum_{s=0}^{l} (-)^s \begin{bmatrix} 2l \\ l+s \end{bmatrix} x^{s(3s+1)} \right\}\\ &= x^{2l^2} \left\{ \sum_{s=0}^{l} (-)^{s+1} \begin{bmatrix} 2l \\ l+s+1 \end{bmatrix} x^{(s+1)(3s+2)} + \sum_{s=0}^{l} (-)^s \begin{bmatrix} 2l \\ l+s \end{bmatrix} x^{s(3s+1)} \right\} \qquad (4)\end{aligned}$$

Then

$$\begin{aligned}\beta_{2l}(x) &= x^{2l^2} \sum_{s=0}^{l} (-)^s \begin{bmatrix} 2l \\ l+s \end{bmatrix} x^{s(3s+1)} \left(1 - \frac{1 - X^{l-s}}{1 - X^{l+s+1}} x^{4s+2} \right)\\ &= x^{2l^2} \sum_{s=0}^{l} (-)^s \begin{bmatrix} 2l \\ l+s \end{bmatrix} x^{s(3s+1)} \frac{1 - X^{2s+1}}{1 - X^{l+s+1}}\end{aligned}$$

$$= \frac{x^{2l^2}}{1 - X^{2l+1}} \sum_{s=0}^{l} (-)^s \begin{bmatrix} 2l+1 \\ l+s+1 \end{bmatrix} x^{s(3s+1)} (1 - x^{4s+2}) \tag{5}$$

Let us now compute $\beta_{2l+1}(x)$. For this we compare the coefficients of $\mathfrak{z}^{2m+1}$, multiply the resulting equations by $(-)^m x^{m(m-1)}$ and add up. Then

$$\sum_{l=0}^{\infty} \frac{\beta_{2l+1}(x)}{X_{2l+1}!} = \frac{\sum_{m=0}^{\infty} (-)^m x^{m(m-1)} x^{(2m+1)^2}}{\prod_{n=1}^{\infty} (1 - x^{2n})}, \tag{6}$$

where **67**

$$\beta_{2l+1}(x) = \sum_{r=0}^{2l+1} \begin{bmatrix} 2l+1 \\ r \end{bmatrix} x^{(2l+1-r)^2} + r^2 (-)^{l-r} x^{(l-r)(l-r-1)}$$

Writting $l - r = s$, this gives

$$\begin{aligned}
\beta_{2l+1}(x) &= \sum_{s=-l-1}^{l} \begin{bmatrix} 2l+1 \\ l-s \end{bmatrix} x^{(l+1-s)^2 + (l-s)^2} (-)^s x^{s(s-1)} \\
&= \sum_{s=-l-1}^{l} \begin{bmatrix} 2l+1 \\ l-s \end{bmatrix} (-)^s x^{3s^2 + s + l^2 + (l+1)^2} \\
&= x^{2l^2+2l+1} \Bigg\{ \sum_{s=0}^{l} (-)^s \begin{bmatrix} 2l+1 \\ l+s+1 \end{bmatrix} x^{s(3s+1)} \\
&\qquad + \sum_{s=0}^{l} (-)^{s+1} \begin{bmatrix} 2l+1 \\ l+s+1 \end{bmatrix} x^{(-s-1)(-3s-2)} \Bigg\} \\
&= x^{2l^2+2l+1} \sum_{s=0}^{l} (-)^s \begin{bmatrix} 2l+1 \\ l+s+1 \end{bmatrix} x^{s(3s+1)} (1 - x^{4s+2})
\end{aligned} \tag{7}$$

This expression for $\beta_{2l+1}(x)$ is very neat; it is almost the same as $\beta_{2l}(x)$ but for trivial factors. Let us go back to $\beta_{2l+1}(x)$ in its best shape.

$$\begin{aligned}
\beta_{2l+1}(x) &= x^{2l^2+2l+1} \Bigg\{ \begin{bmatrix} 2l+1 \\ l \end{bmatrix} \\
&\qquad + \sum_{s=1}^{l} \left(\begin{bmatrix} 2l+1 \\ l+s+1 \end{bmatrix} (-)^s x^{s(3s+1)} + \begin{bmatrix} 2l+1 \\ l+s \end{bmatrix} (-)^s x^{s(3s-1)} \right) \Bigg\} \\
&= x^{2l^2+2l+1} \left\{ \begin{bmatrix} 2l+1 \\ l \end{bmatrix} + \sum_{s=1}^{l} \begin{bmatrix} 2l+1 \\ l+s \end{bmatrix} (-)^s x^{s(3s-1)} \left(1 + \frac{1 - X^{l-s+1}}{1 - X^{l+s+1}} x^{2s} \right) \right\}
\end{aligned}$$

Since 68

$$1 + \frac{1 - X^{l-s+1}}{1 - X^{l+s+1}} x^{2s} = \frac{1 - X^{l+s+1} + X^s - X^{l+1}}{1 - X^{l+s+1}} = \frac{(1 - X^{l+1})(1 + X^s)}{1 - X^{l+s+1}},$$

$$\beta_{2l+1}(x) = x^{2l^2+2l+1} \frac{1 - X^{l+1}}{1 - X^{2l+2}}$$
$$\left\{ \begin{bmatrix} 2l+2 \\ l+1 \end{bmatrix} + \sum_{s=1}^{l+1} \begin{bmatrix} 2l+2 \\ l+s+1 \end{bmatrix} (-)^s x^{s(3s-1)} (1 + x^{2s}) \right\}$$

This fits with β_{2l+2}. Now we can read off the recursion formulae. The consequences are too very nice facts. The whole thing hinges upon the courage to tackle these sums. We did not do these things ad hoc.

Let us compare β_{2l+1} with β_{2l}

$$\beta_{2l+1} = x^{2l+1}(1 - X^{2l+1})\beta_{2l};$$

$$\beta_{2l+1} = x^{-2l-1} \frac{1 - X^{l+1}}{1 - X^{2l+2}} \beta_{2l+2};$$

so

$$\beta_{2l+2} = x^{2l+1} \frac{1 - X^{2l+2}}{1 - X^{l+1}} \beta_{2l+1},$$

and $\beta_\circ = 1$. These things collapse beautifully into something which we could not foresee before. Of course the older proof was shorter. This proof fits very well into our scheme.

Lecture 8

Last time we obtained the two fundamental formulae for β_{2l}, β_{2l+1}, from which we deduced the recurrence relations: 69

$$\begin{aligned} \beta_{2m+1} &= x^{2m+1}(1 - x^{2(2m+1)})\beta_{2m}, \\ \beta_{2m+2} &= x^{2m+1}\frac{1 - x^{2(2m+2)}}{1 - x^{2(m+1)}}\beta_{2m+1} \end{aligned} \tag{1}$$

β_{2m} came from B_{2m} by a substitution which was not yet plausible. Let us calculate the first few $\beta' s$ explicitly. By definition

$$\begin{aligned} & B_0 = 1 = \beta_0 \\ & \beta_1 = x(1 - x^2) \qquad \beta_0 = x(1 - x^2) \\ & \beta_2 = x\frac{1 - x^4}{1 - x^2} \qquad \beta_1 = x^2(1 - x^4) \\ & \beta_3 = x^3(1 - x^6) \quad \beta_2 = x^5(1 - x^4)(1 - x^6) \\ & \beta_4 = x^3\frac{1 - x^8}{1 - x^4} \qquad \beta_3 = x^8(1 - x^6)(1 - x^8); \end{aligned}$$

and in general,

$$\begin{aligned} \beta_{2m} &= x^{2m^2}(1 - x^{2m+2})(1 - x^{2m+4})\cdots(1 - x^{4m}) \\ &= X^{m^2}\frac{X_{2m}!}{X_m!}(\text{with } X = x^2); \end{aligned} \tag{2}$$

and similarly,

$$\begin{aligned} \beta_{2m+1} &= x^{2m^2+2m+1}(1 - x^{2m+2})(1 - x^{2m+4})\cdots(1 - x^{4m+2}) \\ &= X^{m^2+m}x \cdot \frac{X_{2m+1}!}{X_m!} \end{aligned} \tag{3}$$

This is a very appealing result. We got the $\beta's$ in the attempt of ours to utilise the Jacobi formula. We actually had 70

$$\frac{\sum\limits_{l=0}^{\infty}(-)^l x^{5l^2-l}}{\prod\limits_{m=1}^{\infty}(1-x^{2m})} = \sum_{m=0}^{\infty}\frac{\beta_{2m}}{X_{2m}!},$$

so that by (2)

$$\frac{\sum\limits_{l=0}^{\infty}(-)^l X^{l(5l-l)/2}}{\prod\limits_{m=1}^{\infty}(1-x^m)} = \sum_{m=0}^{\infty}\frac{X^{m^2}}{X_m!} \tag{4}$$

Similarly we had

$$\frac{\sum\limits_{l=0}^{\infty}(-)^l x^{5l^2+3l+1}}{\prod\limits_{m=1}^{\infty}(1-x^{2m})} = \sum_{m=0}^{\infty}\frac{\beta_{2m+1}}{X_{2m+1}!},$$

so that by (3)

$$\frac{\sum\limits_{l=0}^{\infty}(-)^l X^{l(5l+3)/2}}{\prod\limits_{m=1}^{\infty}(1-x^m)} = \sum_{m=0}^{\infty}\frac{X^{m(m+1)}}{X_m!} \tag{5}$$

Now the right side in the Rogers-Ramanujan formula is

$$\frac{1}{\prod\limits_{m=1}^{\infty}(1-x^{5m-1})(1-x^{5m-4})} = \frac{\prod\limits_{m=1}^{\infty}(1-x^{5m})(1-x^{5m-2})(1-x^{5m-3})}{\prod\limits_{m=1}^{\infty}(1-x^m)}$$

which becomes, on replacing x by x^2, 71

$$\frac{\prod\limits_{m=1}^{\infty}(1-x^{10m})(1-x^{10m-4})(1-x^{10m-6})}{\prod\limits_{m=1}^{\infty}(1-x^{2m})}$$

The numerator is the same as the left side of Jacobi's triple product formula:

$$\prod_{m=1}^{\infty}(1-x^{2m})(1-\mathfrak{z}x^{2m-1})(1-\mathfrak{z}^{-1}x^{2m-1}) = \sum_{l=-\infty}^{\infty}(-)^l\mathfrak{z}^l x^{l^2},$$

with x replaced by x^5 and z by x. Hence

$$\frac{\prod_{l=-\infty}^{\infty}(1-x^{10mm})(1-x^{10m-4})(1-x^{10m-6})}{\prod_{m=1}^{\infty}(1-x^{2m})} = \frac{\sum_{l=-\infty}^{\infty}(-)^l X^{5l^2+l}}{\prod_{m=1}^{\infty}(1-x^{2m})} = \frac{\sum_{l=-\infty}^{\infty}(-)^l X^{(5l^2+l)/2}}{\prod_{m=1}^{\infty}(1-X^m)}$$

now

$$\frac{\sum_{l=-\infty}^{\infty}(-)^l x^{5l^2+l}}{\prod_{m=1}^{\infty}(1-x^{2m})} = \frac{\sum_{k=-\infty}^{\infty}(-)^k \mathfrak{z}^k x^{k^2}}{\prod_{m=1}^{\infty}(1-x^{2m})} = \sum_{n=0}^{\infty}\frac{B_n(\mathfrak{z},x)}{X_n!} = \frac{\sum_{l=-\infty}^{\infty}(-)^l x^{l^2+l} x^{(2l)^2}}{\prod_{m=1}^{\infty}(1-x^{2m})},$$

on replacing $\mathfrak{z}^{2l}$ by $(-)^l x^{l(l+1)}$, and this we can do because of linearity. Hence 72

$$\frac{\sum_{l=-\infty}^{\infty}(-)^l X^{l(5l-1)/2}}{\prod_{m=1}^{\infty}(1-X^m)} = \frac{1}{\prod_{m=1}^{\infty}(1-x^{5m-1})(1-x^{5m-4})}$$

Similarly,

$$\frac{1}{\prod_{m=1}^{\infty}(1-X^{5m-2})(1-X^{5m-3})} = \frac{\prod_{m=1}^{\infty}(1-x^{10m})(1-x^{10m-2})(1-x^{10m-8})}{\prod_{m=1}^{\infty}(1-X^m)}$$

$$= \frac{\sum_{l=-\infty}^{\infty}(-)^l x^{5l^2+3l}}{\prod_{m=1}^{\infty}(1-X^m)}.$$

This time we have to replace $\mathfrak{z}^{2k+1}$ by $(-)^k x^{k(k-1)}$. Then

$$\frac{1}{\prod_{m=1}^{\infty}(1-X^{5m-2})(1-X^{5m-3})} = \frac{\sum_{l=-\infty}^{\infty}(-)^l X^{l(5l+3)/2}}{\prod_{m=1}^{\infty}(1-X^m)}$$

These formulae are of extreme beauty. The present proof has at least to do with things that we had already handled. The pleasant surprise is that these things do come out. The other proofs by Watson, Ramanujan and other use 73 completely unplausible combinations from the very start. Our proof is substantilly that by Rogers given in Hardy's Ramanujan, pp.96-98, though one may not recognize it as such. The proof there contains completely foreign elements, trigonometric functions which are altogether irrelevant here.

We now give up formal power series and enter into an entirely different chapetr - Analysis.

Part II

Analysis

Lecture 9

Theta-functions

A power series hereafter shall for us mean something entirely different from what it did hitherto. x is a complex variable and $\sum\limits_{n=0}^{\infty} a_n x^n$ will have a value, its sum, which is ascertained only only after we introduce convergence. Then **74**

$$f(x) = \sum_{n=0}^{\infty} a_n x^n;$$

x and the series are coordinated and we have a function on the complex domain. We take for granted the theory of analytic functions of a complex variable; we shall be using Caushy's theorem frequently, and in a moment we shall have occasion to use Weierstrass's double series theorem.

Let us go back to the Jacobi identity:

$$\prod_{n=1}^{\infty}(1 - x^{2n})(1 + \mathfrak{z}x^{2n-1})(1 + \mathfrak{z}^{-1}x^{2n-1}) = \sum_{k=-\infty}^{\infty} \mathfrak{z}^k x^{k^2}$$
$$= 1 + \sum_{k=1}^{\infty}(\mathfrak{z}^k + \mathfrak{z}^{-k})x^{k^2}, (\mathfrak{z} \neq 0),$$

which is a power series in x. Two questions arise. First, what are the domains of convergence of both sides? Second, what does equality between the two sides mean? Formerly, equality meant agreement of the coefficients up to any **75**
stage; what it means now we have got to explore. The left side is absolutely convergent - and absolute convergence is enough for us - for $|x| < 1$; (for the infinite product $\prod(1+a_n)$ is absolutely convergent if $\sum |a_n| < \infty$; z is a complex

variable which we treat as a parameter). For the right side we use the Cauchy-Hadamard criterion for the radius of convergence:

$$\rho = \frac{1}{\overline{\lim}\ \sqrt[n]{|a_n|}} = \frac{1}{\overline{\lim}\ \sqrt[k^2]{|\mathfrak{z} + \mathfrak{z}^{-k}|}}$$

Suppose $|\mathfrak{z}| > 1$; then *****************, and

$$|\mathfrak{z}^k + \mathfrak{z}^{-k}| < 2|\mathfrak{z}|^k,$$

and

$$\sqrt[k^2]{|\mathfrak{z}^k + \mathfrak{z}^{-k}|} < \sqrt[k^2]{2}\sqrt[k]{|\mathfrak{z}|} \to 1 \text{ as } k \to \infty$$

$$\therefore \quad \overline{\lim}(\sqrt[k^2]{|\mathfrak{z}^k + \mathfrak{z}^{-k}|}) \leq 1.$$

It is indeed $= 1$, not < 1, because ultimately, if k is large enough, $|\mathfrak{z}|^k > 1$, and so

$$\frac{1}{2}|\mathfrak{z}|^k < |\mathfrak{z}^k + \mathfrak{z}^{-k}|,$$

and we have the reverse inequality. By symmetry in $\mathfrak{z}$ and $1/\mathfrak{z}$, this holds also for $|\mathfrak{z}| < 1$. The case $|\mathfrak{z}| = 1$ does not present any serious difficulty either. So in all cases $\rho = 1$. Thus both sides are convergent for $|x| < 1$, and indeed *uniformly* in any closed circle $|x| \leq 1 - \delta < 1$.

The next question is, why are the two sides equal in the sense of function **76**
theory? This is not trivial. Here equality of values of coefficients up to any definite stage is not sufficient as it was before; the unfinished coefficients before multiplication may go up and cannot be controlled. Here, however, we ae in a strong position. We have to prove that

$$\prod_{n=1}^{N}(1 - x^{2n})(1 + \mathfrak{z}x^{2n-1})(1 + \mathfrak{z}^{-1}x^{2n-1}) \to 1 + \sum_{k=1}^{\infty}(\mathfrak{z}^k + \mathfrak{z}^{-k})x^{k^2}$$

with increasing N, when $|x| < 1$, and indeed uniformly so in $|x| \leq 1 - \delta < 1$. On the left side we have a sequence of polynomials:

$$f_N(x) = \prod_{n=1}^{N}(1 - x^{2n})(1 + \mathfrak{z}x^{2n-1})(1 + \mathfrak{z}^{-1}x^{2n-1}) = \sum_{m=0}^{\infty} a_m^{(N)} x^m, \quad \text{say.}$$

(of course the coefficients are all zero beyond a certain finite stage). Now we know that the left side is a partial product of a convergent infinite product; in fact $f_N(x)$ tends uniformly to a series, $f(x)$, say. Now what do we know about a sequence of analytic functions on the same domain converging uniformly to a limit function? The question is answered by Weierstrass's double series theorem. We can assert that $f(x)$ is analytic in the same domain at least, and further if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then

$$a_m = \lim_{N\to\infty} a_m^{(N)}.$$

The coefficients of the limit function have got something to do with the original coefficients. Now 77

$$a_m^{(N)} = \frac{1}{2\pi i} \int\limits_{|x|=1-\delta} \frac{f_N(x)}{x^{m+1}} dx$$

Let $N \to \infty$; this is permissible by uniform convergence and the $a_m^{(N)}$, s in fact converge to

$$a_m = \frac{1}{2\pi i} \int\limits_{|x|=1-\delta} \frac{f(x)}{x^{m+1}} dx.$$

(Weirestrass' own proof of this theorem was what we have given here, in some disguise; he takes the values at the roots of unity and takes a sort of mean value).

Now what are the coefficients in $1 + \sum(\mathfrak{z}^k + \mathfrak{z}^{-k})x^{k^2}$? Observe that the convergence of $a_m^{(N)}$ to a_m is a peculiar and simple one. $a_m^{(N)}$ indeed converges to a known a_m; as a matter of fact $a_m^{(N)} = a_m$ for N sufficiently large. They reach a limit and stay put. And this is exactly the meaning of our formal identity. So the identity has been proved in the function-theoretic sense:

$$\prod_{n=1}^{\infty}(1 - x^{2n})(1 + \mathfrak{z}x^{2n-1})(1 + \mathfrak{z}^{-1}x^{2n-1}) = 1 + \sum_{k=1}^{\infty}(\mathfrak{z}^k + \mathfrak{z}^{-k})x^{k^2} = \sum_{k=-\infty}^{\infty} \mathfrak{z}^k x^{k^2}.$$

These things were done in full extension by Jacobi. Let us employ the ususl symbols; in place of x write $q, |q| < 1$, and put $z = e^{2\pi iv}$. Notice that the right side is a Laurent expansion in z in $0 < |\mathfrak{z}| < \infty$ (v is unrestricted because we hace used the exponential). We write in the traditional notation 78

$$\prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1}e^{2\pi iv})(1 + q^{2n-1}e^{-2\pi iv})$$

$$= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi inv}$$

$$= v_3(v, q)$$

v_3 (and in fact all the theta functions) are entire functions of v. We have taken $|q| < 1$; it is customary to write $q = e^{\pi i\tau}$, so that $|q| < 1$ implies

$$|e^{\pi i\tau}| = e^{\mathscr{R}\pi i\tau}, \mathscr{R}\pi i\tau < 0$$

i.e., $$\mathscr{R}i\tau < 0 \quad \text{or} \quad \mathscr{I}m\tau > 0$$

τ is a point in the upper half-plane. τ and q are equivalent parameters. We also write

$$\mathscr{V}_3(\mathscr{V}, q) = \mathscr{V}_3(\mathscr{V}/\tau)$$

(An excellent accout of the $\mathscr{V}$-functions can be found in Tannery and Molk: Fonctiones Elliptiques, in 4 volumes; the second volume contains a very well orginized collection of formulas).

One remark is immediate from the definition of $\mathscr{V}_3$, viz.

$$\mathscr{V}_3(\mathscr{V} + 1, q) = \mathscr{V}_3(\mathscr{V}, q)$$

On the other hand, 79

$$\mathscr{V}_3(\mathscr{V} + \tau, q) = \prod_{n=1}^{\infty}(1 - q^{2n})(1 - q^{2n-1}e^{2\pi i\mathscr{V}}e^{2\pi i\tau}) \times (1 + q^{2n-1}e^{-2\pi i\mathscr{V}}e^{-2\pi i\tau})$$
$$= \sum_{n=-\infty}^{\infty} q^{n^2}e^{2\pi in\mathscr{V}}e^{2\pi in\mathscr{V}},$$

and since $q = e^{\pi i\tau}$,

$$\prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n+1}e^{2\pi i\mathscr{V}})(1 + q^{2n-3}e^{-2\pi i\mathscr{V}}) = \sum_{n=-\infty}^{\infty} q^{n^2+2n}e^{2\pi in\mathscr{V}}$$

or

$$\frac{1 + q^{-1}e^{-2\pi i\mathscr{V}}}{1 + qe^{2\pi i\mathscr{V}}}\prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1}e^{2\pi i\mathscr{V}})(1 + q^{2n-1}e^{-2\pi i\mathscr{V}})$$
$$= q^{-1}e^{-2\pi i\mathscr{V}}\sum_{n=-\infty}^{\infty} q^{(n+1)^2}e^{2\pi i(n+1)\mathscr{V}}$$
$$= q^{-1}e^{-2\pi i\mathscr{V}}\mathscr{V}_3(\mathscr{V}, q)$$
$$= (qe^{2\pi i\mathscr{V}})^{-1}\mathscr{V}_3(\mathscr{V}, q)$$

So we have the neat result:

$$\mathscr{V}_3(\mathscr{V}+\tau,q)=q^{-1}e^{-2\pi i\mathscr{V}}\mathscr{V}_3(\mathscr{V},q)$$

1 is a period of $\mathscr{V}_3$ and τ resembles a period. It is quite clear that we cannot **80**
expect 2 peroids in the full sense, because it is imposible for an entire function to have two periods. Indeed if ω_1 and ω_2 are two periods of f, then $f(\mathscr{V}+\omega_1)=f(\mathscr{V})$, $f(\mathscr{V}+\omega_2)=f(\mathscr{V})$, and $f(\mathscr{V}+\omega_1+\omega_2)=f(\mathscr{V})$ and the whole module generated by ω_1 and ω_2 form periods. Consider the fundamental region which is the parallelogram with vertices at $0,\omega_1,\omega_2,\ \omega_1+\omega_2$. If the function is entire it has no poles in the parallelogram and is bounded there (because the parallelogram is bounded and closed), and therefopre in the whole plane. Hence by Liouville's theorem the function reduces to a constant.

While dealing with trigonometric functions one is not always satisfied with the cosine function alone. It is noce to have another function: $\cos(x-\pi/2)=\sin x$. A shift by a half-period makes it concenient for us. Let us consider analogously $\mathscr{V}_3(\mathscr{V}+\frac{1}{2},q)$, $\mathscr{V}_3(\mathscr{V}+\tau/2,q)$, and $\mathscr{V}_3(\mathscr{V}+\frac{1}{2}+\frac{\tau}{2},q)$. Though τ is not strictly a period we can still speak of the funcdmental region, because on shifting by τ we change only by a trivial factor. Replace $\mathscr{V}$ by $\mathscr{V}+\frac{1}{2}$ and everything is fine as 1 is a period.

$$\begin{aligned}\mathscr{V}_3(\mathscr{V}+\frac{1}{2},q)&=\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}e^{2\pi i\mathscr{V}})(1-q^{2n-1}e^{-2\pi i\mathscr{V}})\\&=\sum_{n=-\infty}^{\infty}(-)^n q^{n^2}e^{2\pi in\mathscr{V}}\end{aligned}$$

which is denoted $\mathscr{V}_4(\mathscr{V},q)$

Again **81**

$$\begin{aligned}\mathscr{V}_3(\mathscr{V}+\frac{\tau}{2},q)&=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i\mathscr{V}}e^{\pi i\tau})(1+q^{2n-1}e^{-2\pi i\mathscr{V}}e^{-\pi i\tau})\\&=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2\pi in\mathscr{V}}e^{\pi in\tau}\end{aligned}$$

$$\text{i.e.,}\quad (1+e^{-2\pi i\mathscr{V}})\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n}e^{2\pi i\mathscr{V}})(1+q^{2n}e^{-2\pi i\mathscr{V}})=\sum_{n=-\infty}^{\infty}q^{n^2+n}e^{2\pi in\mathscr{V}}$$

$$= q^{-1/4} e^{-\pi i \mathscr{V}} \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)\pi i \mathscr{V}}$$
$$= q^{-1/4} e^{-\pi i \mathscr{V}} \mathscr{V}_2(\mathscr{V}, q)$$

where $\mathscr{V}_2(\mathscr{V}, q) = \sum\limits_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)\pi i \mathscr{V}}$, by definition. (Here $q^{-1/4}$ does not contain an unknown 4th root of unity as factor, but is an abbreviation for $e^{-\pi i \tau/4}$, so that it is well defined). So

$$\mathscr{V}_2(\mathscr{V}, q) = 2q^{1/4} \cos \pi \mathscr{V} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n} e^{2\pi\tau \mathscr{V}})(1 + q^{2n} e^{-2\pi i \mathscr{V}})$$

Finally **82**

$$\mathscr{V}_3(\mathscr{V} + \frac{1+\tau}{2}, q) = q^{-1/4} e^{-\pi i (\mathscr{V} + \frac{1}{2})} \mathscr{V}_2\left(\mathscr{V} + \frac{1}{2}, q\right)$$
$$= q^{-1/4} \frac{1}{i} e^{-\pi i \mathscr{V}} \mathscr{V}_2\left(\mathscr{V} + \frac{1}{2}, q\right)$$
$$= q^{1/4} e^{-\pi i \mathscr{V}} \sum_{n=-\infty}^{\infty} (-)^n q^{\left(\frac{2n+1}{2}\right)^2} e^{(2n+1)\pi i \mathscr{V}}$$
$$= \frac{2}{i} \cos \pi \left(\mathscr{V} + \frac{1}{2}\right) e^{-\pi i \mathscr{V}}$$
$$\prod_{n=1}^{\infty} (1 - q^{2n}) \left(1 - q^{2n} e^{2\pi i \mathscr{V}}\right) \times \left(1 - q^{2n} e^{-2\pi i \mathscr{V}}\right)$$

Now define
$$\mathscr{V}_1(\mathscr{V}, q) = \mathscr{V}_2\left(\mathscr{V} + \frac{1}{2}, q\right),$$
or

$$\mathscr{V}_1(\mathscr{V}, q) = 2q^{1/4} \sin \pi \mathscr{V} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n} e^{2\pi i \mathscr{V}})(1 - q^{2n} e^{-2\pi i \mathscr{V}})$$
$$= i q^{-1/4} \sum_{m=-\infty}^{\infty} (-)^n q^{\left(\frac{2n+1}{2}\right)^2} e^{(2n+1)\pi i \mathscr{V}}$$

Collecting together we have the four $\mathscr{V}$-functions: **83**

$$\mathscr{V}_1(\mathscr{V}, q) = i q^{-1/4} \sum_{m=-\infty}^{\infty} (-)^n q^{\left(\frac{2n+1}{2}\right)^2} e^{(2n+1)\pi i \mathscr{V}}$$

$$= \sum_{n=0}^{\infty} (-)^n q^{\left(\frac{2n+1}{2}\right)^2} \sin(2n+1)\pi\mathscr{V}$$

$$\mathscr{V}_2(\mathscr{V}, q) = 2\sum_{n=0}^{\infty} q^{\left(\frac{2n+1}{2}\right)^2} \cos(2n+1)\pi\mathscr{V}$$

$$\mathscr{V}_3(\mathscr{V}, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2n\pi\mathscr{V}$$

$$\mathscr{V}_4(\mathscr{V}, q) = 1 + 2\sum_{n=1}^{\infty} (-)^n q^{n^2} \cos 2\pi n\mathscr{V}$$

Observe that the sine function occurs only in $\mathscr{V}_1$. Also if $q, \mathscr{V}$ are rel these reduce to trigonometric expansions.

Lecture 10

Let us recapitulate the formulae we has last time. **84**

$$
\begin{aligned}
\mathscr{V}_1(\mathscr{V}, q) &= \frac{1}{i} \sum_{n=-\infty}^{\infty} (-)^n q^{\left(\frac{2n+1}{2}\right)^2} e^{(2n+1)\pi i \mathscr{V}} \\
&= 2 \sum_{n=\infty}^{\infty} (-)^n q^{\left(\frac{2n+1}{2}\right)^2} \sin(2n+1)\pi\mathscr{V} \\
&= 2q^{1/4} \sin\pi\mathscr{V} \prod_{m=1}^{\infty} \left(1 - q^{2m}\right)\left(1 - q^{2m} e^{2\pi i \mathscr{V}}\right)\left(1 - q^{2m} e^{-2\pi i \mathscr{V}}\right) \qquad (1) \\
\mathscr{V}_2(\mathscr{V}, q) &= \sum_{n=-\infty}^{\infty} q^{\left(\frac{2n+1}{2}\right)^2} e^{(2n+1)\pi i \mathscr{V}} \\
&= 2 \sum_{n=0}^{\infty} q^{\left(\frac{2n+1}{2}\right)^2} \cos(2n+1)\pi\mathscr{V} \\
&= 2q^{1/4} \cos\pi\mathscr{V} \prod_{m=1}^{\infty} \left(1 - q^{2m}\right)\left(1 + q^{2m} e^{2\pi i \mathscr{V}}\right)\left(1 + q^{2m} e^{-2\pi i \mathscr{V}}\right) \qquad (2) \\
\mathscr{V}_3(\mathscr{V}, q) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i \mathscr{V}} \\
&= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2n\pi\mathscr{V} \\
&= \prod_{m=1}^{\infty} \left(1 - q^{2m}\right)\left(1 + q^{2m-1} e^{2\pi i \mathscr{V}}\right)\left(1 + q^{2m-1} e^{-2\pi i \mathscr{V}}\right) \qquad (3)
\end{aligned}
$$

$$\begin{aligned}
\mathscr{V}_4(\mathscr{V}, q) &= \sum_{n=-\infty}^{\infty} (-)^n q^{n^2} e^{2n\pi i \mathscr{V}} \\
&= 1 + 2\sum_{n=1}^{\infty} (-)^n q^{n^2} \cos 2n\pi\mathscr{V} \\
&= \prod_{m=1}^{\infty} (1 - q^{2m})(1 - q^{2m-1} e^{2\pi i \mathscr{V}})(1 - q^{2m-1} e^{-2\pi i \mathscr{V}}) \qquad (4)
\end{aligned}$$

We started with $\mathscr{V}_3$ and shifted the argument $\mathscr{V}$ by 'periods', and we had, **85**
writing $q = e^{\pi i \tau}$,

$$\begin{aligned}
\mathscr{V}_3(\mathscr{V} + 1, q) &= \mathscr{V}_3(\mathscr{V}, q) \\
\mathscr{V}_3(\mathscr{V} + \tau, q) &= q^{-1} e^{-2\pi i \mathscr{V}} \mathscr{V}_3(\mathscr{V}, q).
\end{aligned} \qquad (5)$$

Then we took 'half-periods' and then something new happened, and we gave names to the new functions:

$$\begin{aligned}
\mathscr{V}_3\left(\mathscr{V} + \frac{1}{2}, q\right) &= \mathscr{V}_4(\mathscr{V}, q) \\
\mathscr{V}_3\left(\mathscr{V} + \frac{\tau}{2}, q\right) &= q^{-1/4} e^{-2\pi i \mathscr{V}} \mathscr{V}_2(\mathscr{V}, q) \qquad (6) \\
\mathscr{V}_3\left(\mathscr{V} + \frac{1+\tau}{2}, q\right) &= i q^{-1/4} e^{-\pi i \mathscr{V}} \mathscr{V}_1(\mathscr{V}, q)
\end{aligned}$$

Let us study how these functions alter when the argument $\mathscr{V}$ is changed by 1, τ, $1/2$, $\tau/2$, $(1+\tau)/2$. $\mathscr{V} \to \mathscr{V} + 1$ is trivial; $\mathscr{V} \to \mathscr{V} + 1/2$ is also easy to see by inspection. Let us take $\mathscr{V} + \tau$. (We suppress the argument q for convenience of writing).

$$\begin{aligned}
\mathscr{V}_1(\mathscr{V}) &= \frac{1}{i} q^{1/4} e^{2\pi i \mathscr{V}} \mathscr{V}_3\left(\mathscr{V} + \frac{1+\tau}{2}\right) \\
\therefore \quad \mathscr{V}_1(\mathscr{V} + \tau) &= \frac{1}{i} q^{1/4} e^{\pi i(\mathscr{V} + \tau)} \mathscr{V}_3\left(\mathscr{V} + \tau + \frac{1+\tau}{2}\right) \\
&= \frac{1}{i} q^{1/4} e^{\pi i \mathscr{V}} q q^{-1} e^{-2\pi i(\mathscr{V} + 1 + \tau/2)} \mathscr{V}_3\left(\mathscr{V} + \frac{1+\tau}{2}\right) \\
&= e^{-2\pi i \mathscr{V}} e^{-\pi i(1+\tau)} \mathscr{V}_1(\mathscr{V}, q) \\
&= -A\mathscr{V}_1(\mathscr{V}, q),
\end{aligned}$$

where $A = q^{-1} e^{-2\pi i \mathscr{V}}$; the other conspicuous factor which occurs in similar **86**
contexts is denoted $B = q^{-1/4} e^{-2\pi i \mathscr{V}}$.

The other transformations can be worked out in a similar way by first going over to $\mathscr{V}_3$. We collect the results below in tabular form.

	$\mathscr{V}+1$	$\mathscr{V}+\tau$	$\mathscr{V}+\frac{1}{2}$	$\mathscr{V}+\frac{3}{2}$	$\mathscr{V}+\frac{1+\tau}{2}$
$\mathscr{V}_1$	$-\mathscr{V}_1$	$-A\mathscr{V}_1$	$\mathscr{V}_2$	$iB\mathscr{V}_4$	$B\mathscr{V}_3$
$\mathscr{V}_2$	$-\mathscr{V}_2$	$A\mathscr{V}_2$	$-\mathscr{V}_1$	$B\mathscr{V}_3$	$-iB\mathscr{V}_4$
$\mathscr{V}_3$	$\mathscr{V}_3$	$A\mathscr{V}_3$	$\mathscr{V}_4$	$B\mathscr{V}_2$	$B\mathscr{V}_1$
$\mathscr{V}_4$	$\mathscr{V}_4$	$-A\mathscr{V}_4$	$\mathscr{V}_3$	$iB\mathscr{V}_1$	$B\mathscr{V}_2$

It may be noticed that each column in the table contains all the four functions; so does each now.

The systematique of the notation for the $\mathscr{V}$-functions is rather questionable. Whittaker and Watson write $\mathscr{V}$ instead of $\pi\mathscr{V}$, which has the unpleasant consequence that the 'periods' are then π and $\pi\tau$. Our notation is the same as in Tannery and Molk. An attempt was made by Kronecker to systematise a little the unsystematic notation. Charles Hermite introduced the following notation: **87**

$$\mathscr{V}_{\mu\nu}(\mathscr{V},q)=\sum_{n=-\infty}^{\infty}(-)^{\nu n}q^{\left(\frac{2r+\mu}{2}\right)^2}e^{(2n+\mu)\pi i\mathscr{V}}$$
$$=\sum_{n=-\infty}^{\infty}(-)^{\nu n}e^{\left(\frac{2n+\mu}{2}\right)^2\pi i\tau}e^{(2n+\mu)\pi i\mathscr{V}}$$

where $\mu,\nu=0,1****$. In this notation,

$$\mathscr{V}_{00}(\mathscr{V},q)=\mathscr{V}_3(\mathscr{V},q)$$
$$\mathscr{V}_{01}(\mathscr{V},q)=\mathscr{V}_4(\mathscr{V},q)$$
$$\mathscr{V}_{10}(\mathscr{V},q)=\mathscr{V}_2(\mathscr{V},q)$$
$$\mathscr{V}_{11}(\mathscr{V},q)=i\mathscr{V}_1(\mathscr{V},q).$$

This, however, has not found any followers.

While writing down derivatives, we always retain the convention that a prime refers to differentiation with respect to $\mathscr{V}$:

$$\mathscr{V}'_\alpha(\mathscr{V},q)=\frac{\partial}{\partial\nu}\mathscr{V}_\alpha(\mathscr{V},q)\ (\alpha=1,2,3,4)$$

Taking partial derivatives, we have

$$\frac{\partial}{\partial\tau}\mathscr{V}_{\mu\nu}(\mathscr{V}/\tau) = \sum_{n=-\infty}^{\infty}(-)^{\nu n}\pi i\left(\frac{2n+\mu}{2}\right)e^{\left(\frac{2n+\mu}{2}\right)^2\pi i\tau}e^{(2n+\mu)\pi i\mathscr{V}},$$

and
$$\frac{\partial^2}{\partial\mathscr{V}^2}\mathscr{V}_{\mu\nu}(\mathscr{V}/\tau) = \sum_{n=-\infty}^{\infty}(-)^{\nu n}e^{\left(\frac{2n+\mu}{2}\right)^2\pi i\tau}\pi^2 i^2(2n+\mu)^2 e^{(2n+\mu)\pi i\mathscr{V}},$$

Comparing these we see that they agree to some extent; in fact, **88**

$$4\pi i\frac{\partial}{\partial\tau}\mathscr{V}_{\mu\nu}(\mathscr{V}/\tau) = \frac{\partial^2}{\partial v^2}\mathscr{V}_{\mu\nu}(\mathscr{V}/\tau) \tag{7}$$

This is a partial differential equation of the second order, a parabolic equation with constant coefficients. It is fundamental to write $i\tau = -t$; (7) then becomes the differential equation for heat conduction. $\mathscr{V}$-functions are thus very useful tools in applied mathematics; they were used by Poisson and Fourier in this connection.

Again,

$$\frac{\partial}{\partial q}\mathscr{V}_{\mu\nu}(\mathscr{V}, q) = \sum_{n=-\infty}^{\infty}(-)^{\nu}\left(\frac{2n+\mu}{2}\right)^2 q^{\left(\frac{2n+\mu}{2}\right)^2-1}e^{(2n+\mu)\pi i\mathscr{V}},$$

$$-4\pi^2 q\frac{\partial}{\partial q}\mathscr{V}_{\mu\nu}(\mathscr{V}, q) = \frac{\partial^2}{\partial v^2}\mathscr{V}_{\mu\nu}(\mathscr{V}, q), \tag{8}$$

which is another form of (7). Here the uniformity of notation was helpful; it was not necessary to discuss the different functions separately.

We now pass on to another important topic. The zeros of the theta - functions.

The $\mathscr{V}$-functions are more or less periodic. The exponential factor that is picked up on passing from one parallelogram to another is non-zero and can accumulate. It is evident from the definition that

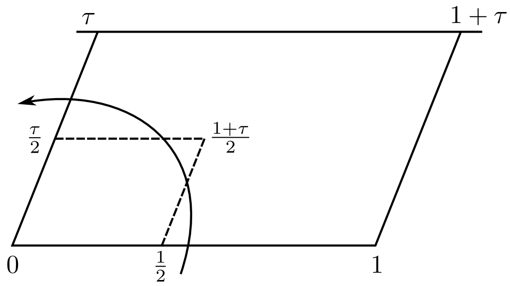
$$\mathscr{V},(0, q) = 0.$$

On the other hand $\mathscr{V}_2$, $\mathscr{V}_3$, $\mathscr{V}_4 \neq 0$. (when the argument $\mathscr{V}$ is 0 we write hereafter simply $\mathscr{V}$). This is so because the infinite products are absolutely convergent. (Let us recall that a product like $1\cdot\frac{1}{2}\cdot\frac{1}{3}\cdots$ is not properly convergent in the product sense). Again from the definitions, **89**

$$\mathscr{V}_2\left(\frac{1}{2}\right) = 0$$

$$\mathscr{V}_3\left(\frac{1+\tau}{2}\right)=0$$
$$\mathscr{V}_4\left(\frac{\tau}{2}\right)=0$$

So far we have one zero per parallelogram for each of the functions; and there can be no other in a parallelogram, as can be seen from the infinite product expansions. The zeros of $\mathscr{V}_1(\mathscr{V}, q)$ are $m_1 + m_2\tau (m_1, m_2$ integers), for $1 - e^{2\pi i m\tau} e^{2\pi i\mathscr{V}} = 0$ implies $m\tau + \mathscr{V}_1 = m_1$ or $\mathscr{V} = m_1 - m\tau$. The zeros of $\mathscr{V}_1(\mathscr{V}, q), \mathscr{V}_2(\mathscr{V}, q), \mathscr{V}_3(\mathscr{V}, q), \mathscr{V}_4(\mathscr{V}, q)$ in the fundamental parallelogram are nicely arranged in order at the points $0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}$ respectively.



All the zeros are therefore given by the formulae: 90

$$\mathscr{V}_1(m_1+m_2\tau)=0$$
$$\mathscr{V}_2\left(m_1+m_2\tau+\frac{1}{2}\right)=0$$
$$\mathscr{V}_3\left(m_1+m_2\tau+\frac{1+\tau}{2}\right)=0$$
$$\mathscr{V}_4\left(m_1+m_2\tau+\frac{\tau}{2}\right)=0$$

It is of interest to study $\mathscr{V}_\alpha(0, q)$ (usually written $\mathscr{V}_\alpha$).

$$\mathscr{V}_1(0)=0$$
$$\mathscr{V}_2(0)=\sum_{n=-\infty}^{\infty} q^{\left(2n+\frac{1}{2}\right)^2}$$
$$=2q^{1/4}\prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m})^2$$

$$= \mathscr{V}_2$$

$$\mathscr{V}_3(0) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$= \prod_{m=1}^{\infty}(1 - q^{2m})(1 + q^{2m-1})$$

$$\mathscr{V}_4(0) = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2}$$

$$= \prod_{m=1}^{\infty}(1 - q^{2m})(1 - q^{2m-1})^2$$

We cannot anything of interest in $\mathscr{V}_1$. Let us look at the others. **91**

$$\mathscr{V}_1'(0, q) = \mathscr{V}_1' = 2\pi \sum_{n=0}^{\infty}(-)^n(2n + 1)q^{\left(\frac{2n+1}{2}\right)^2}$$

$$= 2q^{1/4}\left[\pi \cos \pi\mathscr{V} \prod_{m=1}^{\infty}(\cdots) + \sin \pi\mathscr{V}\left(\prod_{m=1}^{\infty}(\cdots)\right)'\right]_{\mathscr{V}=0}$$

$$= 2\pi q^{1/4} \prod_{m=1}^{\infty}(1 - q^{2m})^3$$

Immediately we see that this yields the interesting identity of Jacobi.

$$\prod_{m=1}^{\infty}(1 - q^{2m})^3 = \sum_{n=0}^{\infty}(-)^n(2n + 1)q^{n^2+n},$$

or, replacing q^2 by x,

$$\prod_{n=1}^{\infty}(1 - x^n)^3 = \sum_{n=0}^{\infty}(-)^n(2n + 1)x^{n(n+1)/2}$$

We had proved this earlier by the method of formal power series. Here we can differentiate with good conscience.

Now

$$\pi\mathscr{V}_2\mathscr{V}_3\mathscr{V}_4 = \mathscr{V}_1'\left(\prod_{m=1}^{\infty}(1 + q^{2m})(1 + q^{2m-1})(1 - q^{2m-1})\right)^2$$

$$= \mathscr{V}_1'\left(\prod_{m=1}^{\infty}(1 + q^{2m})(1 - q^{4m-2})\right)^2,$$

which becomes, on replacing q^2 by x, 92

$$\mathscr{V}_1' \left[\prod_{m=1}^{\infty}(1+x^n)\left(1-x^{2n-1}\right)\right]^2$$

However, $\prod\limits_{m=1}^{\infty}(1+x^n)(1-x^{2n-1}) = 1$. We therefore have the very useful and pleasant formula

$$\mathscr{V}_1{}' = \pi \mathscr{V}_2 \mathscr{V}_3 \mathscr{V}_4$$

Lecture 11

We found that $\mathscr{V}_\alpha(\mathscr{V}, q)$ changes at most its sign when $\mathscr{V}$ is replaced by $\mathscr{V}+1$, **93**
while it picks up a trivial factor A when $\mathscr{V}$ is replaced by $\mathscr{V}+\tau$. If we form quotients, A will cancel out and we may therefore expect to get doubly-periodic functions. Let us form some useful quotients:

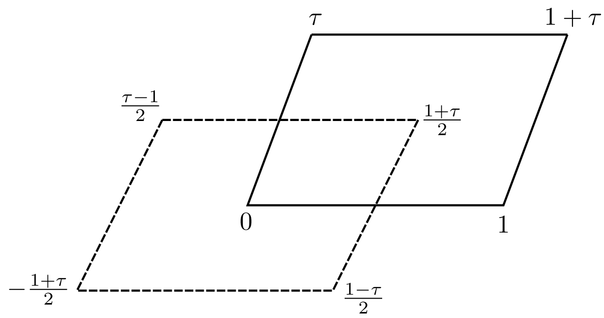
$$f_2(\mathscr{V}) = \frac{\mathscr{V}_2(\mathscr{V}, q)}{\mathscr{V}_1(\mathscr{V}, q)}$$
$$f_3(\mathscr{V}) = \frac{\mathscr{V}_3(\mathscr{V}, q)}{\mathscr{V}_1(\mathscr{V}, q)}$$
$$f_4(\mathscr{V}) = \frac{\mathscr{V}_4(\mathscr{V}, q)}{\mathscr{V}_1(\mathscr{V}, q)}$$

For simplicity of location of poles it is convenient to take $\mathscr{V}_1$ in the denominator since it has a zero at the origin. From the table of the $\mathscr{V}$-functions we find that these functions are not quite doubly periodic:

$$f_2(\mathscr{V}+1) = f_2(\mathscr{V}) \qquad f_3(\mathscr{V}+1) = -f_3(\mathscr{V})$$
$$f_2(\mathscr{V}+\tau) = -f_2(\mathscr{V}) \qquad f_3(\mathscr{V}+\tau) = -f_3(\mathscr{V})$$
$$f_4(\mathscr{V}+1) = -f_4(\mathscr{V})$$
$$f_4(\mathscr{V}+\tau) = f_4(\mathscr{V})$$

So the functions are not doubly periodic; they do not return to themselves. **94**
And we cannot expect that either. For suppose any of the functions f were actually doubly periodic. We know that each has a pole of the first order per parallelogram. Integrating round the parallelogram with vertices at $\pm\frac{1+\tau}{2}, \pm\frac{1-\tau}{2}$ (so that the origin which is the pole is enclosed), we have

$$\int f(\mathscr{V})d\mathscr{V} = 0$$



i.e., the sum of the residues at the poles =0. This means that either the pole is a double is a double pole with zero residue, or there are two simple poles with residues equal in magnitude but opposite in sign. However neither of these is the case. So there is no necessity for any further experimentation.

Let us therefore consider the squares

$$f_2^2(\mathscr{V}), f_3^2(\mathscr{V}), f_4^2(\mathscr{V})$$

these are indeed doubly periodic functions. And they are even functions. So the expansion in the neighbourhood of the pole will not contain the term of power -1. Hence the pole must be a double pole with residue zero. So they are closely related to the Weierstrassian function $\mathscr{P}(\mathscr{V})$, and must indeed be of the form $C\mathscr{P}(\mathscr{V}) + C_1$.

So we have constructed doubly periodic functions. They are essentially 95
$\mathscr{P}(\mathscr{V})$. ω_1 and ω_2 of $\mathscr{P}(\mathscr{V})$ are our 1 and τ. In order to get a better insight we need the exact values of the functions. Let us consider their pole terms. Expanding in the neighbourhood of the origin,

$$\begin{aligned}
\frac{\mathscr{V}_\alpha(\mathscr{V},q)}{\mathscr{V}_1(\mathscr{V},q)} &= \frac{\mathscr{V}_\alpha + \frac{\mathscr{V}_\alpha''}{2!}\mathscr{V}^2 + \cdots}{\frac{\mathscr{V}_1'}{1!}\mathscr{V} + \frac{\mathscr{V}_1'''}{3!}\mathscr{V}^3 + \cdots} \\
&= \frac{\mathscr{V}_\alpha}{v\mathscr{V}_1'}\left(\frac{1 + \frac{1}{2}\frac{\mathscr{V}_\alpha''}{\mathscr{V}_\alpha}v^2 + \cdots}{1 + \frac{\mathscr{V}_1'''}{6\mathscr{V}_1'}v^2 + \cdots}\right) \\
&= \frac{\mathscr{V}_\alpha}{v\mathscr{V}_1'}\left(1 + \frac{1}{2}\frac{\mathscr{V}_\alpha''}{\mathscr{V}_\alpha}v^2 + \cdots\right)\left(1 - \left(\frac{\mathscr{V}_1'''}{6\mathscr{V}_1'}v^2 + \cdots\right)^2 + (\cdots)^3 - \cdots\right) \\
&= \frac{\mathscr{V}_\alpha}{v\mathscr{V}_1'}\left(1 + v^2\left(\frac{\mathscr{V}''}{2\mathscr{V}_\alpha} - \frac{\mathscr{V}_1'''}{6\mathscr{V}_1'}\right) + \cdots\right) \\
\therefore \quad f_\alpha^2 &= \left(\frac{\mathscr{V}_\alpha(\mathscr{V},q)}{\mathscr{V}_1(\mathscr{V},q)}\right)^2
\end{aligned}$$

$$= \frac{\mathscr{V}_\alpha^2}{\mathscr{V}'2_1} \frac{1}{\mathscr{V}^2}\left(1 + \mathscr{V}^2\left(\frac{\mathscr{V}_\alpha''}{\mathscr{V}_1} - \frac{\mathscr{V}_1'''}{3\mathscr{V}_1'}\right) + \cdots\right)$$

Let us now specialise α. We have a special interest in $\mathscr{V}_3$ because it is such a nice function: $\mathscr{V}_3 = \sum_{n=-\infty}^{\infty} q^{n^2}$. We have $\frac{\mathscr{V}_1'^2}{\mathscr{V}_\alpha^2} f_\alpha^2(\mathscr{V}) = \frac{1}{\mathscr{V}_2}$ + non-negative powers of $\mathscr{V}$.

If we take two such and take the difference, the difference will no longer have a pole. Taking $\alpha = 2, 4$, for instance, 96

$$\frac{\mathscr{V}_1'^2}{\mathscr{V}_2^2}\left(\frac{\mathscr{V}_2(\mathscr{V}, q)}{\mathscr{V}_1(\mathscr{V}, q)}\right)^2 - \frac{\mathscr{V}_1'^2}{\mathscr{V}_4^2}\left(\frac{\mathscr{V}_4(\mathscr{V}, q)}{\mathscr{V}_1(\mathscr{V}, q)}\right)^2 = \frac{\mathscr{V}_2''}{\mathscr{V}_2} - \frac{\mathscr{V}_4}{\mathscr{V}_4''} + \text{positive powers of } \mathscr{V} \quad (*)$$

The left side is a doubly periodic function without a pole and so a constant C; the right side is therefore just $\frac{\mathscr{V}_2''}{\mathscr{V}_2} - \frac{\mathscr{V}_4''}{\mathscr{V}_4}$. The vanishing of the other terms on the other terms on the right side, of course, implies lots of identities.

So we have already computed C in one way:

$$C = \frac{\mathscr{V}_2''}{\mathscr{V}_2} - \frac{\mathscr{V}_4''}{\mathscr{V}_4}$$

To evaluate C in other ways we may take in (*) $\mathscr{V} = \frac{1}{2}$, $\mathscr{V} = \frac{\tau}{2}$ or $\mathscr{V} = (1+\tau)/2$. From the table,

$$\begin{aligned}
\mathscr{V}_1\left(\frac{1}{2}, q\right) &= \mathscr{V}_2 & \mathscr{V}_1\left(\frac{1+\tau}{2}, q\right) &= q^{-1/4}\mathscr{V}_3 \\
\mathscr{V}_2\left(\frac{1}{2}, q\right) &= -\mathscr{V}_1 = 0 & \mathscr{V}_2\left(\frac{1+\tau}{2}, q\right) &= -iq^{-1/4}\mathscr{V}_4 \\
\mathscr{V}_4\left(\frac{1}{2}, q\right) &= \mathscr{V}_3 & \mathscr{V}_4\left(\frac{1+\tau}{2}, q\right) &= q^{-1/4}\mathscr{V}_2
\end{aligned}$$

So again from the left side of (*), 97

$$\begin{aligned}
C &= \frac{\mathscr{V}_1'^2}{\mathscr{V}_2^2} \times 0 - \frac{\mathscr{V}_1^2}{\mathscr{V}_4^2}\frac{\mathscr{V}_3^2}{\mathscr{V}_2^2} \\
&= -\frac{\pi^2\mathscr{V}_1^2\mathscr{V}_3^4}{\pi^2\mathscr{V}_2^2\mathscr{V}_3^2\mathscr{V}_4^2} = -\pi^2\mathscr{V}_3^4
\end{aligned}$$

Also

$$C = \frac{\mathscr{V}_1'^2}{\mathscr{V}_2^2}\left(-\frac{\mathscr{V}_4^2}{\mathscr{V}_3^2}\right) - \frac{\mathscr{V}_1'^2}{\mathscr{V}_4^2}\frac{\mathscr{V}_2^2}{\mathscr{V}_3^2}$$
$$= -\frac{\pi^2\mathscr{V}_1'^2\mathscr{V}_4^4}{\pi^2\mathscr{V}_2^2\mathscr{V}_3^2\mathscr{V}_4^2} - \frac{\pi^2\mathscr{V}_1'^2\mathscr{V}_2^4}{\pi^2\mathscr{V}_2^2\mathscr{V}_3^2\mathscr{V}_4^2}$$
$$= -\pi^2\mathscr{V}_4^4 - \pi^2\mathscr{V}_2^4$$

From these we get an identity which is particularly striking:

$$\mathscr{V}_3^4 = \mathscr{V}_2^4 + \mathscr{V}_4^4 \tag{1}$$

We have also

$$\pi^2\mathscr{V}_3^4 = \frac{\mathscr{V}_4''}{\mathscr{V}_4} - \frac{\mathscr{V}_2''}{\mathscr{V}_2} \tag{2}$$

Now let us look at (1) and do a little computing. Explicitly (1) states:

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = \left(q^{1/4}\sum_{n=-\infty}^{\infty} q^{n(n+1)}\right)^4 + \left(\sum_{n=-\infty}^{\infty}(-)^n q^{n^2}\right)^4 \tag{3}$$

This is an identity of some interest.

Let us look for q^N on both sides. The left side gives N in the form $N = n^2 + n_2^2 + n_3^2 + n_4^2$, that is, as the sum of four squares. So dies the second term on the right. If N is even, it is trivial that both sides are in agreement because the first term on the right gives only odd powers of q, and the coefficient of q^n in the second term on the right is **98**

$$\sum_{n_1^2 n_2^2+n_3^2+n_4^2=N} (-)^{n_1+n_2+n_3+n_4}$$

Since N is even either all n_i's are odd, or two of them odd, or none. It is not transperent. What happens when N is odd.

Take the more interesting formula (2):

$$\pi\mathscr{V}_3^4 = \frac{\mathscr{V}_4''}{\mathscr{V}_4} - \frac{\mathscr{V}_2''}{\mathscr{V}_2}$$

By the differential equation,

$$\mathscr{V}_\alpha'' = \left[\frac{\partial^2}{\partial\mathscr{V}^2}\mathscr{V}_\alpha(\mathscr{V}, q)\right]_{\mathscr{V}=0}$$

$$
\begin{aligned}
&= \left[-4\pi^2 q \frac{\partial}{\partial q}\mathscr{V}_\alpha(\mathscr{V}, q)\right]_{\mathscr{V}=0}\\
\therefore \quad \mathscr{V}_3^4 &= 4q\left(\frac{1}{\mathscr{V}_2}\frac{\partial \mathscr{V}_2}{\partial q} - \frac{1}{\mathscr{V}_4}\frac{\partial \mathscr{V}_4}{\partial q}\right)\\
&= 4q\frac{\partial}{\partial q}\log\frac{\mathscr{V}_2}{\mathscr{V}_4}
\end{aligned}
$$

Now 99

$$
\begin{aligned}
\frac{\mathscr{V}_2}{\mathscr{V}_4} &= 2q^{1/4}\frac{\prod\limits_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2}{\prod\limits_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})}\\
&= 2q^{1/4}\frac{\prod\limits_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n})^2}{\prod\limits_{n=1}^{\infty}(1-q^{2n})^2(1-q^{2n-1})^2}\\
&= 2q^{1/4}\frac{\prod\limits_{n=1}^{\infty}(1-q^{4n})^2}{\prod\limits_{n=1}^{\infty}(1-q^{n})^2}\\
&= \frac{2q^1/4}{\prod\limits_{4\nmid n}(1-q^n)^2}
\end{aligned}
$$

Taking the logarithmic derivative,

$$
\begin{aligned}
\mathscr{V}_3^4 &= 4q\left\{\frac{1}{4q} - 2\sum_{4\nmid n}\frac{-nq^{n-1}}{1-q^n}\right\}\\
&= 1 + 8\sum_{4\nmid n}\frac{nq^n}{1-q^n}\\
&= 1 + 8\sum_{4\nmid n} n\sum_{k=1}^{\infty} q^{nk}\\
&= 1 + 8\sum_{\substack{4\nmid m\\ m=1}} q^m \sum_{n|m} n\\
&= 1 + 8\sum_{m=1}^{\infty}\sigma^*(m)q^m
\end{aligned}
$$

with the previous that $\sigma^*(m) = \sum_{d/m}, \underset{4 \nmid \alpha}{d}$, that is the divisor sum with those **100**
divisors omitted which are divisible by 4. This is an interesting identity:

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = 1 + 8\sum_{m=1}^{\infty} \sigma^*(m) q^m \tag{4}$$

On the left q^m can be obtained only as $q^{n_1^2+n_2^2+n_3^2+n_4^2}$, so that the coefficient of q^m on the right is the number of ways in which this representation for m is possible; m is as often the sum of four squares as $8\sigma^*(m)$. Clearly $\sigma^*(m) \neq 0$, since among the admissible divisors, 1 is always present. So $\sigma^*(m) \geq 1$, or every m does admit at least one such representation. We have thus proved *Lagrange's theorem*: Every integer is the sum of at most four squares.

If m is odd, $\sigma^*(m) = \sigma(m)$; if m is even,

$$\begin{aligned}\sigma^*(m) &= \sum_{d|m,\ d \text{ odd}} d + 2\sum_{d|m,\ d \text{ odd}} d \\ &= 3\sum_{d|m,\ d \text{ odd}} d\end{aligned}$$

If we denote by $r_4(m)$ the number of representations of m as the sum of four squares, then

$r_4(m) = 8$ times the sum of odd divisors of m, m odd;
24 times the sum of odd divisors of m, m even.

We have not partitions this time, but representation as the sum of squares. We agree to consider as distinct these representations in which the order of the components has been changed. In partitions we abstracted from the order of **101**
the summands; here we pay attention to order, and also to the sign (i.e., one representation $n_1^2 + n_2^2 + n_3^2 + n_4^2$ is actually counted, order apart, as 16 different representations $(\pm n_1)^2 + (\pm n_2)^2 + (\pm n_3)^2 + (\pm n_4)^2$, if n_1, n_2, n_3, n_4 are all different from 0).

As an example, take $m = 10$. The different representations as the sum of four squares are

$$(\pm 1)^2 + (\pm 1)^2 + (\pm 2)^2 + (\pm 2)^2,$$
$$(\pm 1)^2 + (\pm 3)^2 + (0)^2 + (0)^2,$$

along with their rearrangements, six in each. Thus altogether

$$r_4(10) = 6 \times 16 + 6 \times 8 = 144$$

$$8\sigma^*(10) = 3(1 + 2 + 5 + 10) = 8 \times 18 = 144$$

Lagrange's theorem was first enunciated by Fermat in the seventeenth century. Many mathematicians tried to solve it without success; eventually Jacobi found out the identity

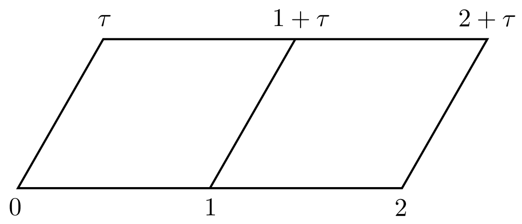
$$r_4(m) = 8\sigma^*(m)$$

Before that, the fact that every integer is the sum of four squares was conjectured by Fermat, Euler did not succeed in proving it. It was proved by Lagrange, and later Euler gave a mere elementary proof. Euler proved that if two numbers are each the sum of four squares, then so is their product, by means of the identity:

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + x_4^2)&(y_1^2 + y_2^2 + y_3^2 + y_4^2)\\ &= (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2 +\\ &\quad + (x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4)^2 + (x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2.\end{aligned}$$

We do not proceed to discuss in detail the representability of a number as the sum of two sequences. 102

If we return not to f_α^2 but to f_α we are not helpless to deal with them. f_4 is not doubly periodic in the fundamental parallelogram, but is doubly periodic in a parallelogram of twice this size with vertices at $0, 2, 2 + \tau, \tau$. It has got a pole at the vertex 0 and another at the vertex 1, with residues adding up to zero.



We may write down another identity:

$$\frac{\mathscr{V}_1'}{\mathscr{V}_4} \cdot \frac{\mathscr{V}_4(\mathscr{V}/\tau)}{\mathscr{V}_1(\mathscr{V}/\tau)} = \frac{1}{2}\left\{\frac{\mathscr{V}_1'\left(\frac{\mathscr{V}}{2}\Big/\frac{\tau}{2}\right)}{\mathscr{V}_1\left(\frac{\mathscr{V}}{2}\Big/\frac{\tau}{2}\right)} - \frac{\mathscr{V}_1'\left(\frac{\mathscr{V}+1}{2}\Big/\frac{\tau}{2}\right)}{\mathscr{V}_1\left(\frac{\mathscr{V}+1}{2}\Big/\frac{\tau}{2}\right)}\right\}$$

This may be deduced by checking that the poles on both sides are the same, Further they are odd functions and so the constant term in the difference must vanish. Put $\mathscr{V} = \dfrac{1}{2}$ on both sides.

Then we get 103

$$\pi \mathscr{V}_3^2 = \frac{1}{2}\left\{\frac{\mathscr{V}_1'\left(\frac{1}{4}\Big/\frac{\tau}{2}\right)}{\mathscr{V}_1\left(\frac{1}{4}\Big/\frac{\tau}{2}\right)} - \frac{\mathscr{V}_1'\left(\frac{3}{4}\Big/\frac{\tau}{2}\right)}{\mathscr{V}_1\left(\frac{3}{4}\Big/\frac{\tau}{2}\right)}\right\}$$

By straightforward calculation, taking logarithmic derivatives, we obtain,

$$\mathscr{V}_3^2 = 4\sum_{m=1}^{\infty} q^m(\sigma_\circ^{(1)}(m) - \sigma_\circ^{(3)}(m)),$$

where the notation employed is:

$$\sigma_k(m) = \sum_{d|m} d^k,$$

$$\sigma_\circ(m) = \sum_{d|m} d^\circ = \text{number of divisors of } m;$$

$$\sigma_\circ^{(j)}(m) = \sum_{d|m,\ d\equiv j \pmod 4} d^\circ$$

comparing coefficients of q^m, and observing that on the left m occurs only in the form $n_1^2 + n_2^2$, we get the beautiful theorem:

> m can be represented as the sum of two squares as often
> as $4(\sigma_\circ^{(1)}(m) - \sigma_\circ^{(3)}(m))$.

Notice that $\sigma_\circ^{(1)}(m) - \sigma_\circ^{(3)}(m)$ is always non negatives; hence $\sigma_\circ^{(1)}(m) \geq \sigma_\circ^{(3)}(m)$ (i.e., the number of divisors of the form $4r + 1$ is never less than the number of divisors of the form $4r + 1$), which is by no means a trivial fact.

In some cases we can actually find out what the difference $\sigma_\circ^{(1)}(m) - \sigma_\circ^{(3)}(m)$ 104
will be. Suppose that m is a prime p. Then the only divisors are 1 and p. The divisor 1 goes into $\sigma_\circ^{(1)}$; and p goes into $\sigma_\circ^{(3)}$ if $p \equiv 3 \pmod 4$. So the difference is zero. However, if $p \equiv 1 \pmod 4$. p goes into $\sigma_\circ^{(1)}$. Hence the number of representations of a prime $p \equiv 1 \pmod 4$ as the sum of two squares is $4 \times 2 = 8$. That the number of representations of a prime $p \equiv 1 \pmod 4$ as the sum of two squares is 8 is a famous theorem of Fermat, proved for the first time by Euler. It is usually proved by using the Gaussian complex numbers.

So far we have been looking upon $\mathscr{V}$ as the variable in the $\mathscr{V}$- functions; now we proceed to consider q as the variable and go to deeper things like the Jacobi transformation.

Lecture 12

We now come to a rather important topic, the transformation of $\mathscr{V}$-functions. 105
So far we have been looking upon $\mathscr{V}_\alpha(\mathscr{V}/\tau)$ as a function of $\mathscr{V}$ only; hereafter we shall be interfering with the 'period' τ also. We want to study how $\mathscr{V}_\alpha(\mathscr{V}/\tau)$ changes when $\mathscr{V}$ is replaced by $\mathscr{V} + 1/\tau$. For this it is enough if we replace $\mathscr{V}$ by $\mathscr{V}\tau = \omega$ and see how the function behaves when ω is changed to $\omega+1$. This would amount to turning the whole plane around in the positive sense about the origin through arg τ. We take $\mathscr{V}_1$, because it is easier to handle, since the zeros become the periods too. Consider

$$f(\mathscr{V}) = \mathscr{V}_1(\mathscr{V}\tau/\tau)$$

Then

$$\begin{aligned} f(\mathscr{V}+1) &= \mathscr{V}_1((\mathscr{V}+1)\tau/\tau) \\ &= \mathscr{V}_1(\mathscr{V}\tau + \tau/\tau) \\ &= -e^{-\pi i\tau} e^{-2\pi i\mathscr{V}\tau} \mathscr{V}_1(\mathscr{V}\tau/\tau) \\ &= e^{-\pi i\tau} e^{-2\pi i\mathscr{V}\tau} f(\mathscr{V}) \end{aligned}$$

τ Similarly consider $f(\mathscr{V} - 1/\tau)$ (We choose to take $-\frac{1}{\tau}$ rather than $\frac{1}{\tau}$ since we want the imaginary part of the parameter to be positive:

$$\operatorname{Im}\frac{1}{\tau} = \operatorname{Im}\frac{\bar{\tau}}{\tau\bar{\tau}} < 0 \text{ and so } \operatorname{Im} -\frac{1}{\tau} > 0)$$

$$\begin{aligned} f(\nu - \frac{1}{\tau}) &= \mathscr{V}_1((\nu - \frac{1}{\tau})\tau/\tau) \\ &= \mathscr{V}_1(\mathscr{V}\tau - 1/\tau) \\ &= -\mathscr{V}_1(\mathscr{V}\tau/\tau) \end{aligned}$$

$$= -f(\mathscr{V})$$

So f is a sort of $\mathscr{V}$-function which picks up simple factors for the 'periods' 1 and $-\frac{1}{\tau}$. $f(\mathscr{V})$ has clearly zeros at 0 and $\tau' = -\frac{1}{\tau}$, or generally at $\mathscr{V} = m_1 + m_2\tau'$; m_1, m_2 integers, which is a point-lattice similar to the old one turned around. **106**

Similarly let us define

$$
\begin{aligned}
g(\mathscr{V}) &= \mathscr{V}_1(\mathscr{V}/\tau') = \mathscr{V}_1\left(\mathscr{V}/-\frac{1}{\tau}\right)\\
g(\mathscr{V}+1) &= \mathscr{V}_1(\mathscr{V}+1/\tau')\\
&= -\mathscr{V}_1(\mathscr{V}/\tau')\\
&= -g(\mathscr{V})\\
g\left(\mathscr{V}-\frac{1}{\tau}\right) &= g(\mathscr{V}+\tau')\\
&= \mathscr{V}_1(\mathscr{V}+\tau'/\tau')\\
&= -e^{-\pi i\tau'}e^{-2\pi i\mathscr{V}}\mathscr{V}_1(\mathscr{V}/\tau')\\
&= -e^{\pi i/\tau}e^{-2\pi i\mathscr{V}}g(\mathscr{V})
\end{aligned}
$$

Let us form the quotient: **107**

$$
\begin{aligned}
\Phi(\mathscr{V}) &= \frac{f(\mathscr{V})}{g(\mathscr{V})}\\
\phi(\mathscr{V}+1) &= \frac{f(\mathscr{V}+1)}{g(\mathscr{V}+1)}\\
&= -e^{-\pi i\tau}e^{-2\pi i\mathscr{V}\tau}\phi(\mathscr{V})\\
\Phi\left(\mathscr{V}-\frac{1}{\tau}\right) &= \frac{f(\mathscr{V}+\tau')}{g(\mathscr{V}+\tau')}\\
&= \frac{f(\mathscr{V})}{e^{\pi i/\tau}e^{-2\pi i\mathscr{V}}g(\mathscr{V})}\\
&= e^{-\pi i/\tau}e^{2\pi i\mathscr{V}}\Phi(\mathscr{V})
\end{aligned}
$$

Φ takes on simple factors in both cases of this peculiar sort that we can eliminate them both at one stroke. We write

$$
\begin{aligned}
e^{-\pi i\tau(2\mathscr{V}+1)}\Phi(\mathscr{V}) &= \Phi(\mathscr{V}+1),\\
e^{\pi i(2\mathscr{V}-1/\tau)}\Phi(\mathscr{V}) &= \Phi(\mathscr{V}-1/\tau)
\end{aligned}
$$

Let us try the following trick. Let us supplement $\Phi(\mathscr{V})$ by an outside function $h(\mathscr{V})$ so that the combined function $\Phi(\mathscr{V})$ is totally doubly periodic. Write

$$\Psi(\mathscr{V}) = \Phi(\mathscr{V})e^{h(\mathscr{V})}$$

We want to choose $h(\mathscr{V})$ in such a manner that **108**

$$\Psi(\mathscr{V}+1) = \Psi(\mathscr{V}+\tau') = \Psi(\mathscr{V})$$

This implies two equations:

$$e^{-\pi i\tau(2\mathscr{V}+1)}e^{h(\mathscr{V}+1)-h(\mathscr{V})} = 1$$
$$e^{\pi i(2\mathscr{V}-1\tau)}e^{h(\mathscr{V}-1/\tau)-h(\mathscr{V})} = 1;$$

or

$$h(\mathscr{V}+1) - h(\mathscr{V}) = \pi i\tau(2\mathscr{V}+1) + 2\pi im$$
$$h(\mathscr{V}+\tau' - h(\mathscr{V}) = -\pi i(2\mathscr{V}+\tau') + 2\pi im'.$$

We can solve both at one stroke. Since on the right side we have a linear function of $\mathscr{V}$ in both cases, a quadratic polynomial will do what we want.

$$(\mathscr{V}+\delta)^2 - \mathscr{V}^2 = 2\mathscr{V}\delta + \delta^2 = \delta(2\mathscr{V}+\delta),$$

and taking $h(\mathscr{V}) = \pi i\tau\mathscr{V}^2$,

$$h(\mathscr{V}+1) - h(\mathscr{V}) = \pi i\tau(2\mathscr{V}+1)$$
$$h(\mathscr{V}+\tau') - h(\mathscr{V}) = \pi i\tau\tau'(\mathscr{V}+\tau') = -\pi i(2\mathscr{V}+\tau'),$$

so that both the equations are satisfied. Putting it in, we have

$$\Psi(\mathscr{V}) = e^{\pi i\tau\mathscr{V}^2}\frac{\mathscr{V}_1(\mathscr{V}\tau/\tau)}{\mathscr{V}_1\left(\mathscr{V}/-\frac{1}{\tau}\right)}$$

This has the property that **109**

$$\psi(\mathscr{V}+1) = \psi(\mathscr{V}+\tau) = \psi(\mathscr{V})$$

So we have double periodicity. This function is also an entire function because the numerator and denominator have the same simple zeros. So this is a pole-free function and hence a constant C, C a constant with respect to the variable $\mathscr{V}$, but may be a function of the parameter $\tau, C = C(\tau)$. We thus have

I.

$$e^{\pi i\tau\mathscr{V}^2}\mathscr{V}_1(\mathscr{V}\tau/\tau) = C(\tau)\mathscr{V}_1\left(\mathscr{V}/-\frac{1}{\tau}\right)$$

What we need now are the corresponding formulas for the other functions. Replacing $\mathscr{V}$ by $\mathscr{V}+\frac{1}{2}$,

$$e^{\pi i\tau(\mathscr{V}+\frac{1}{2})^2}\mathscr{V}_1\left(\left(\mathscr{V}+\frac{1}{2}\right)\tau/\tau\right) = C(\tau)\mathscr{V}_1\left(\mathscr{V}+\frac{1}{2}\Big/-\frac{1}{\tau}\right),$$

or
$$e^{\pi i\tau(\mathscr{V}^2+\mathscr{V}+1/4)}ie^{-\pi i\tau/4}e^{-\pi i\mathscr{V}\tau}\mathscr{V}_4(\mathscr{V}\tau/\tau) = C(\tau)\mathscr{V}_2\left(\mathscr{V}/-\frac{1}{\tau}\right)$$

We notices here that two different $\mathscr{V}$-functions are related. This gives

II.

$$ie^{\pi i\tau\mathscr{V}^2}\mathscr{V}_4(\mathscr{V}\tau/\tau) = C(\tau)\mathscr{V}_2\left(\mathscr{V}/-\frac{1}{\tau}\right).$$

Replacing in I $\mathscr{V}$ by $\mathscr{V}+\tau'/2 = \mathscr{V}-1/(2\tau)$, we get

III.

$$ie^{\pi i\tau\mathscr{V}^2}\mathscr{V}_2(\mathscr{V}\tau/\tau) = C(\tau)\mathscr{V}_4\left(\mathscr{V}/-\frac{1}{\tau}\right)$$

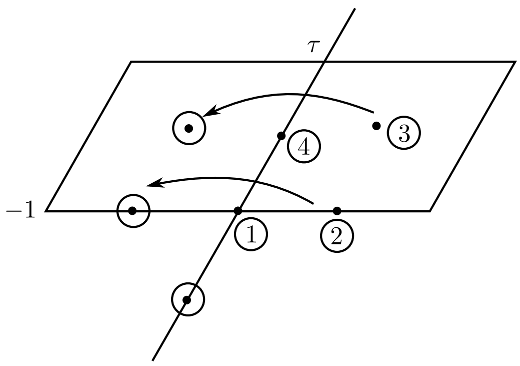
Finally putting $\mathscr{V}+\frac{1}{2}$ for $\mathscr{V}$ In, III,

IV.

$$ie^{\pi i\tau\mathscr{V}^2}\mathscr{V}_3(\mathscr{V}\tau/\tau) = C(\tau)\mathscr{V}_3\left(\mathscr{V}/-\frac{1}{\tau}\right)$$

The way the functions change over in I-IV is quite plausible. For consider **110**
the location of the zeros.

When we take the parallelogram and turn it around what was originally a zero for $\mathscr{V}_4$ becomes one for $\mathscr{V}_2$ and vice versa; and what used to be in the middle, the zero of $\mathscr{V}_3$, Remains in the middle. So the formulae are plausible in structure.



The most important thing now is, what is $C(\tau)$? To evaluate $C(\tau)$ let us differentiate I and put $\mathscr{V}=0$. We have

V.

$$\tau\mathscr{V}_1^{\,1}(0/\tau) = C(\tau)\mathscr{V}_1^{\,1}\left(0/-\frac{1}{\tau}\right)$$

From II, III, and IV, putting $\mathscr{V} = 0$,

$$i\mathscr{V}_4(0/\tau) = C(\tau)\mathscr{V}_2\left(0/-\frac{1}{\tau}\right)$$
$$i\mathscr{V}_2(0/\tau) = C(\tau)\mathscr{V}_4\left(0/-\frac{1}{\tau}\right)$$
$$i\mathscr{V}_3(0/\tau) = C(\tau)\mathscr{V}_3\left(0/-\frac{1}{\tau}\right)$$

Multiplying these together and recalling that $\pi\mathscr{V}_1' = \mathscr{V}_2\mathscr{V}_3\mathscr{V}_4$, we obtain VI.

$$-i\mathscr{V}_1^{\,1}(0/\tau) = (C(\tau))^3\mathscr{V}_1'\left(0/-\frac{1}{\tau}\right).$$

Dividing by VI, by V, **111**

$$\frac{1}{i\tau} = C^2(\tau),$$

or

$$C(\tau) = \pm\sqrt{\frac{1}{i\tau}}$$

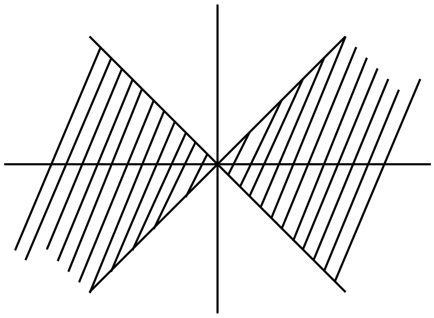
In II, III, IV, it is $\frac{C(\tau)}{i}$ that appears; so let us write this is

$$\frac{C(\tau)}{i} = \pm\frac{1}{i}\sqrt{\frac{1}{i\tau}} = \pm\sqrt{\frac{i}{\tau}}$$

Now $k(i/\tau) > 0 \cdot \frac{C(\tau)}{i}$ is completely determined, analytically, in particular by IV:

$$\frac{C(\tau)}{i} = \frac{\mathscr{V}_3(0/\tau)}{\mathscr{V}_3(0/-\frac{1}{\tau})} = \frac{\sum e^{\pi i\tau n^2}}{\sum e^{\pi i\tau' n^2}}$$

Both the numerator and denominator are analytic functions if $\operatorname{Im}\tau > 0$. So $\frac{C(\tau)}{i}$ is analytic and therefore continuous. i/τ must lie in the right half-plane, and thus $\sqrt{\frac{i}{\tau}}$ in either of the sectors with central angle $\pi/2$, but because of continuity it cannot lie on the border lines. So it is in the interior of entirely one sector. To decide which one it is enough if we make one choice.



Take $\tau = it$, $t > 0$; then 112

$$\frac{C(it)}{i} = \frac{\sum e^{-\pi t n^2}}{\sum e^{-(\pi/t)n^2}}$$

Both numerator and denominator are positive. So $\frac{C(\tau)}{i}$ lies in the right half. So $|arg \sqrt{\frac{i}{\tau}}| < \frac{\pi}{4}$ and $\sqrt{\frac{i}{\tau}}$ denotes the principal branch. The last equality gives:

$$\sum_{n=-\infty}^{\infty} e^{-\pi i n^2/\tau} = \sqrt{\frac{\tau}{i}} \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}$$

This is a very remarkable formula. It gives a functional relation: the transformation $\tau \to -1/\tau$ almost leaves the function unchanged; it changes only by a simple algebraic function. This is one of the achievements of Jacobi.

In the earlier equations we can now put $C(\tau) = \sqrt{(i/\tau)}$. In particular envisage $\mathscr{V}_1'$:

$$\mathscr{V}_1'(0/\tau) = \left(\sqrt{\frac{i}{\tau}}\right)^3 \mathscr{V}_1'\left(0/-\frac{1}{\tau}\right)$$

or
$$\mathscr{V}_1'\left(0/-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \cdot \frac{\tau}{i} \mathscr{V}_1'(o/\tau)$$

But

$$\mathscr{V}_1'(0/\tau) = 2\pi e^{\pi i \tau/4} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})^3$$

$$\therefore \qquad e^{-\pi i \tau/4} \prod_{m=1}^{\infty} (1 - e^{-2\pi i m/\tau})^3 = \sqrt{\frac{\tau}{i}} \cdot \frac{\tau}{i} e^{\pi i \tau/4} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})^3$$

Extracting cube roots on both sides, **113**

$$e^{-\pi i\tau/12}\prod_{m=1}^{\infty}(1-e^{-2\pi im/\tau}) = \epsilon\sqrt{\frac{\tau}{i}}e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1-e^{2\pi im\tau})$$

where $\epsilon^3 = 1$. Dedekind first introduced the function

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1-e^{2\pi im\tau})$$

Then

$$\eta\left(-\frac{1}{\tau}\right) = \epsilon\sqrt{\frac{\tau}{i}}\eta(\tau)$$

This is challenging; we have to decide which ϵ to take: $\epsilon^3 = 1$. The quotient $\eta(-\frac{1}{\tau})\Big/\sqrt{\frac{\tau}{i}}\eta(\tau)$ is an analytic (hence contains) function in the upper half-plane and so must be situated in on of the three open sectors. Now make a special choice; put $\tau = i$. Then $\eta(i) = \epsilon(+1)\eta(i)$, or $\epsilon = 1$.

$$\therefore \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$$

What we have done by considering the lattice of periods can be done in more sophisticated ways. One can have a whole general theory of the transformations from 1, τ, to 1, $\dfrac{a\tau+b}{c\tau+d}$. The quotients appear first and can be carried over. We start with $\mathscr{V}_1$ and come back to it; there may be difficulty, however in deciding the sign. **114**

Lecture 13

We arrived at the following result last time: 115

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau).$$

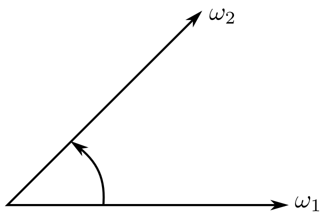
We began by investigating a transformation of $\mathscr{V}_1(\mathscr{V}, \tau)$. Instead of looking upon 1 and τ as generators of the period lattice, we looked upon τ and -1 as generators (turning the plane around through arg τ): $1, \tau \to \tau, -1$. We have of course still the same parallelogram of periods. Since we should like to keep the first period 1, we reduced everything by $\tau : \tau, -1 \to 1, -\frac{1}{\tau}$; so we had to investigate $\mathscr{V}_1(\mathscr{V}\tau/\tau)$. $\mathscr{V}_1(\mathscr{V}\tau/\tau)$ and $\mathscr{V}_1(\mathscr{V}/-\frac{1}{\tau})$ have the same parallelogram of periods.

We could do this a little more generally. Let us introduce linear combinations:

$$\omega_1 = c\tau + d, \quad \omega_2 = a\tau + b,$$

and go from ω_1 to ω_2 in the positive sense. In order that we must have these also as generating vectors for the same lattice, we should have a, b, c, d integers with

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1.$$



Moreover we want the first period to be always 1. (This is the difference between our case and the Weierstrassian introduction of periods, where we have complete homogeneity). So replacing by linearity, the periods are 1 and $\tau' = \frac{a\tau+b}{c\tau+d}$.

Be sure that we want to go from 1 to τ' through an angle less than π in the positive sense. For this we want τ' to have a positive imaginary parts $Im\overline{\tau'} > 0$, or **116**

$$\frac{\tau' - \overline{\tau'}}{i} > 0$$

i.e.,
$$\frac{1}{i}\left(\frac{a\tau + b}{c\overline{\tau} + d} - \frac{a\overline{\tau} + b}{c\overline{\tau} + d}\right) > 0$$

i.e.,
$$\frac{1}{i}\frac{ad\tau + bc\overline{\tau} - ad\overline{\tau} - bc\tau}{|c\tau + d|^2} > 0$$

i.e.,
$$\frac{1}{i}\frac{(ad - bc)(\tau - \overline{\tau})}{|c\tau + d|^2} > 0$$

or since $\tau - \overline{\tau}$ is purely imaginary,

$$ad - bc = \pm 1.$$

We could do the same thing in all our different steps. The most important step, however, cannot be carried through, because we get lost at an important point; and rightly so, it becomes cumber some because a number-theoretic problem is involved there. Let us see what we have done. Compare

$$\mathscr{V}_1((c\tau + d)\mathscr{V}/\tau) \text{ and } \mathscr{V}_1\left(\mathscr{V}\Big/\frac{a\tau + b}{c\tau + d}\right)$$

We want periods 1, τ'; indeed all things obtainable from $\omega_1 = c\tau + d$ and $\omega_2 = a\tau + b$; or $m_1\omega_1 + m_2\omega_2$ must in their totality comprise all periods. For the first $c\tau + d$ is indeed a period, and for the second $a\tau + b$.

Now define **117**

$$f(\mathscr{V}) = \mathscr{V}_1((c\tau + d)\mathscr{V}/\tau)$$

$f(\mathscr{V} + 1)$ is essentially $f(\mathscr{V})$:

$$\begin{aligned} f(\mathscr{V} + 1) &= \mathscr{V}_1((c\tau + d)\mathscr{V} + c\tau + d/\tau)\\ &= (-)^{c+d}e^{-c^2\pi i\tau}e^{-*****(c\tau+d)\mathscr{V}}\mathscr{V}_1((c\tau + d)\mathscr{V}/\tau), \text{ from the table,}\\ &= (\cdots\cdots)f(\mathscr{V})\\ f(\mathscr{V} + \tau') &= \mathscr{V}_1((c\tau + d)v + a\tau + b/\tau)\\ &= (-)^{a+b}e^{-a^2\pi i\tau}e^{-2\pi i a(c\tau+d)v}\mathscr{V}_1((c\tau + d)\mathscr{V}/\tau)\\ &= (\cdots\cdots)f(\mathscr{V}) \end{aligned}$$

$f(\mathscr{V})$ has, leaving trivial factors aside, periods 1, τ' *****. So too for the second function $\mathscr{V}_1\left(\mathscr{V}\Big/\frac{a\tau+b}{c\tau+d}\right)$.

We can form quotients and proceed as we did earlier.

Let us consider for a moment the $\mathscr{V}'s$ with double subscripts. This is a digression, but teaches us a good deal about how to work with $\mathscr{V}$-functions. Recall that

$$\begin{aligned}
\mathscr{V}_{\mu\nu}(\nu/\tau) &= \sum_n (-)^{\nu n} e^{\pi i\tau\left(n+\frac{1}{2}\right)^2} e^{2\pi i\nu\left(n+\frac{1}{2}\right)}\\
\mathscr{V}_1(\nu/\tau) &= \mathscr{V}_n(\nu/\tau)\\
\mathscr{V}_2(\nu/\tau) &= \mathscr{V}_{10}(\nu/\tau)\\
\mathscr{V}_3(\nu/\tau) &= \mathscr{V}_{00}(\nu\tau)\\
\mathscr{V}_4(\nu/\tau) &= \mathscr{V}_{01}(\nu/\tau)
\end{aligned}$$

We take one liberty from now on. Take μ, ν to be arbitrary integers, no **118**
longer 0, 1. That will not do very much harm either. In fact,

$$\mathscr{V}_{\mu,\nu+2}(\nu/\tau) = \mathscr{V}_{\mu,\nu}(\nu/\tau)$$

It is unfortunately not quite so easy for the other one:

$$\mathscr{V}_{\mu+2,\nu}(\nu/\tau) = (-)^{\nu}\mathscr{V}_{\mu,\nu}(\nu/\tau)$$

For

$$\begin{aligned}
\mathscr{V}_{\mu+2,\nu}(\nu/\tau) &= \sum_n (-)^{\nu} e^{\pi i\tau(n+\mu/2)^2} e^{2\pi i\nu(n+1+\mu/2)}\\
&= \sum_n (-)^{\nu}(-)^{\nu(n+1)} e^{\pi i\tau(n+1+\mu/2)^2} e^{2\pi i\nu(n+1+\mu/2)}\\
&= (-)^{\nu}\mathscr{V}_{\mu\nu}(\nu/\tau),
\end{aligned}$$

on shifting the summation index from n to $n+1$. The original table will be considerably reduced now; only in place of $\nu+1$, $\nu+\frac{1}{2}$, $\nu+\frac{\tau}{2}$, $\nu+\frac{1+\tau}{2}$ it will be now necessary to have the combination $\nu+\frac{k}{2}+\frac{l}{2}\tau$. The expression for $\mathscr{V}_{\mu\nu}\left(\nu+\dfrac{k}{2}+\dfrac{l}{2}\tau/\tau\right)$ will include everything that we have done so far in one single formula.

$$\begin{aligned}
\mathscr{V}_{\mu\nu}\left(\nu+\frac{k}{2}+\frac{l}{2}\tau/\tau\right) &= \sum_n (-)^{\nu n} e^{\pi i\tau(n+\frac{\mu}{2})^2} e^{2\pi i\nu(n+\frac{\mu}{2})} e^{\pi i(k+l\tau)(n+\frac{\mu}{2})}\\
&= i^{k\mu}\sum_n (-)^{(\nu+k)^n} e^{\pi i\tau(n+\mu/2+l/2)^2} e^{-\pi i\tau l^2/4} e^{2\pi i\nu(n+\mu/2+l/2)} e^{-\pi i l\nu}\\
&= i^{k\mu} e^{-\pi i\tau l^2/4} e^{-\pi i l\nu}\mathscr{V}_{\mu+l,\nu+k}(\nu/\tau) \qquad (*)
\end{aligned}$$

This one formula has the whole table in it. 119

We now turn to our purpose, viz. To consider the quotient

$$\frac{\mathscr{V}_1((c\tau+d)\nu/\tau)}{\mathscr{V}_1\left(\nu\middle/\frac{a\tau+b}{c\tau+d}\right)}$$

We wish to discuss the behaviour a little more explicitly of $f(\nu)$.

$$\begin{aligned} f(\nu) &= \mathscr{V}_{11}((c\tau+d)\nu/\tau)\\ f(\nu+1) &= \mathscr{V}_{11}((c\tau+d)\nu+c\tau+d/\tau)\\ f(\nu+\tau') &= \mathscr{V}_{11}((c\tau+d)\nu+a\tau+b/\tau)\end{aligned}$$

putting $k=2c$, $l=2d$, $\mu=\nu=1$ in (*), 120

$$\begin{aligned} f(\nu+1) &= (-)^d e^{-\pi i\tau c^2} e^{-2\pi ic(c\tau+d)\nu}\mathscr{V}_{1+2c,1+2d}((c\tau+d)\nu/\tau)\\ &= (-)^{c+d} e^{-\pi i\tau c^2} e^{-\pi ic(c\tau+d)\nu} f(\nu)\end{aligned}$$

Similarly, putting $k=2a$, $l=2b$, $\mu=\nu=1$,

$$f(\nu+\tau') = (-)^{a+b} e^{-\pi i\tau a^2} e^{-2\pi ia(c\tau+d)\nu} f(\nu).$$

Also defining $g(\nu)$:

$$\mathscr{V}_1\left(\nu\middle/\frac{a\tau+b}{c\tau+d}\right) = g(\nu) = \mathscr{V}_{11}(\nu/\tau'),$$

we have

$$g(\nu+1) = \mathscr{V}_{11}(\nu+1/\tau') = -\mathscr{V}_{11}(\nu/\tau').$$

And putting $k=0$, $l=2$, $\mu=3$, $\nu=1$ in (*),

$$\begin{aligned} g(\nu+\tau') &= e^{-\pi i\tau'} e^{-2\pi i\nu}\mathscr{V}_{31}(\nu/\tau')\\ &= -e^{-\pi i\tau'} e^{-2\pi i\nu} g(\nu).\end{aligned}$$

We form now in complete analogy with the old procedure

$$\begin{aligned} \Phi(\nu) &= \frac{f(\nu)}{g(\nu)}\\ \Phi(\nu+1) &= (-)^{c+d+1} e^{-\pi i\tau c^2} e^{-2\pi ic(c\tau+d)\nu\Phi(\nu)},\\ \Phi(\nu+\tau') &= (-)^{a+b+1} e^{-\pi i\tau c^2} e^{-2\pi ia(c\tau+d)\nu} e^{\pi i\tau'+2\pi i\nu}\Phi(\nu)\end{aligned}$$

Φ takes up exponential factors which contain ν linearly. As before ewe can submerge this under a general form. Define 121

$$\Psi(\nu) = \Phi(\nu)e^{h(\nu)},$$

where $h(\nu)$ is to be so determined that

$$\Psi(\nu + 1) = \Psi(\nu + \tau') = \Psi(\nu)$$

we therefore want

$$e^{h(\nu+1)-h(\nu)}(-)^{c+d+1}e^{-c^2\pi i\tau - 2\pi ic(c\tau+d)\nu} = 1,$$
$$e^{h(\nu+\tau')-h(\nu)}(-)^{a+b+1}e^{-a^2\pi i\tau + \pi i\tau' + 2\pi i\nu}e^{-2\pi ia(c\tau+d)\nu} = 1.$$

It will be convenient to observe that $c + d + cd + 1 = (c + 1)(d + 1)$ is even, for at least one of c, d should be odd as otherwise c, d would not be co-prime and we would not have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

So $(-)^{c+d+1} = (-)^{cd} = e^{\pi icd}$. h is given by the equations:

$$\begin{aligned} h(\nu + 1) - h(\nu) &= 2\pi ic(c\tau + d)\nu + \pi ic(c\tau + d), \\ h(\nu + \tau') - h(\nu) &= 2\pi ia(c\tau + d) + \pi ia(a\tau + b) - \pi i\tau' \\ &= 2\pi c(a\tau + b)\nu + \pi ic\tau'(a\tau + b). \end{aligned}$$

We have to introduce a suitable function $h(\nu)$. Since the difference equation can be solved by means of a second degree polynomial, put

$$h(\nu) = A\nu^2 + B$$

for each separately and see whether it works for both. 122

$$\begin{aligned} h(\nu + \delta) - h(\nu) &= 2A\nu\delta + A\delta^2 + B\delta \\ &= \delta(2A\nu + A\delta + B) \end{aligned}$$

Putting $\delta = 1, \tau'$, we find that $A = \pi ic(c\tau + d)$ works in both cases. Also for $\delta = 1$,

$$A + B = \pi ic(c\tau + d),$$
$$A\left(\frac{a\tau + b}{c\tau + d}\right)^2 + B\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\pi ic(a\tau + b)^2}{c\tau + d}$$

So $B = 0$ fits both. Hence

$$h(\nu) = A\nu^2, A = \pi i(c\tau + d)c$$

$$\therefore \qquad \Psi(\nu) = e^{\pi ic(c\tau+d)\nu^2} \frac{f(\nu)}{g(\nu)}$$

And this is a doubly periodic entire function (because the numerator and denominator have the same simple zeros) and therefore a constant. We thus have the transformation formula

$$\mathscr{V}_{11}\left(\nu\Big/\frac{a\tau + b}{c\tau + d}\right) = Ce^{\pi ic(c\tau+d)\nu^2}\mathscr{V}_{11}((c\tau + d)\nu/\tau)$$

where C may depend on the parameters τ, a, b, c, d:

$$C = C(\tau; a, b, c, d)$$

More generally we can have a parallel formula for any μ, ν. As before we get an equation for C^2. And there the thing stops. Formerly we were in a very good position with the special matrix **123**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For general a, b, c, d we get into trouble.

Lecture 14

We were considering the behaviour of $\mathscr{V}_{11}(\nu/\tau)$ under the general modular transformation: 124

$$\mathscr{V}_{11}\left(\nu\Big/\frac{a\tau+b}{c\tau+d}\right) = C(\tau)e^{\pi i C(c\tau+d)\nu^2}\mathscr{V}_{11}((c\tau+d)\nu/\tau), \tag{1}$$

a, b, c, d integers with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = +1.$

We want to determine $C(\tau)$ as far as possible. We shall do this up to a $\pm$ sign. ν is unimportant at the moment; even if we put $\nu = 0$, $C(\tau)$ survives. Put $\nu = \frac{1}{2}, \frac{\tau'}{2}, \frac{1+\tau'}{2}$ in succession, ans use out auxiliary formula which contracted the whole table into one thing:

$$\mathscr{V}_{\mu\nu}\left(\nu+\frac{k}{2}+\frac{l\tau}{2}\Big/\tau\right) = i^{k\mu}e^{-\pi i\tau l^2/4}e^{-\pi i\nu}\mathscr{V}_{\mu+l,\nu+k}(\nu/\tau) \tag{*}$$

Putting $\nu = \frac{1}{2}$ in (1), and writing $\tau' = \dfrac{a\tau+b}{c\tau+d}$,

$$\mathscr{V}_{11}\left(\frac{1}{2}\Big/\tau'\right) = C(\tau)e^{\pi ic(c\tau+d)/4}\mathscr{V}_{11}\left(\frac{c\tau+d}{2}\Big/\tau\right) \tag{2}$$

This is the right moment to call for formula (*). From (*) with $\nu = 0$, $\mu = \nu = 1$, $k = 1$, $l = 0$, we get

$$\mathscr{V}_{11}\left(\frac{1}{2}\Big/\tau'\right) = i\mathscr{V}_{12}(0/\tau')$$

Also from (*) with $\nu = 0$, $\mu = \nu = 1$, $k = d$, $l = C$, we get

$$\mathscr{V}_{11}\left(\frac{c\tau+d}{2}\Big/\tau\right) = i^d C^{-\pi ic^2/4}\mathscr{V}_{1+c,1+d}(o/\tau).$$

Substituting these two formulas in the left and right sides of (2) respectively, we get 125

$$i\mathscr{V}_{12}(0/\tau') = C(\tau)e^{\pi ic(c\tau+d)/4}i^d e^{-\pi i\tau c^2/4}\mathscr{V}_{1+c,1+d}(0/\tau)$$

Now, recalling that

$$\begin{aligned}\mathscr{V}_{\mu,\nu+2}(\nu/\tau) &= \mathscr{V}_{\mu\nu}(\nu/\tau)\\ \mathscr{V}_{\mu+2,\nu}(\nu/\tau) &= (-)^{\nu}\mathscr{V}_{\mu\nu}(\nu/\tau),\end{aligned} \tag{**}$$

the last formula becomes

$$i\mathscr{V}_{10}(0/\tau') = C(\tau)e^{\pi icd/4}i^d\mathscr{V}_{1+c,1+d}(0,\tau) \tag{3}$$

Putting $\nu = \tau'/2$ in (1), we have

$$\mathscr{V}_{11}\left(\frac{\tau'}{2}\Big/\tau'\right) = C(\tau)e^{\pi ic/4\tau'(a\tau+b)}\mathscr{V}_{11}\left(\frac{a\tau+b}{2}\Big/\tau\right).$$

Making use of (*) in succession on the left and right sides (with proper choice of indices) as we did before, this gives

$$e^{-\pi i\tau'/4}\mathscr{V}_{12}(0/\tau') = C(\tau)e^{\pi ic\tau'(a\tau+b)/4}i^b e^{-\pi ia^2\tau/4}\mathscr{V}_{1+a,1+b}(0/\tau),$$

and this, after slight simplification of the exponents on the right sides, gives in view of (**),

$$-\mathscr{V}_{01}(0/\tau') = C(\tau)i^b e^{\pi iab/4}\mathscr{V}_{1+a,1+b}(0/\tau) \tag{4}$$

Putting $\nu = (1+\tau')/2$ in (1), 126

$$\mathscr{V}_{11}\left(\frac{1+\tau'}{2}\Big/\tau'\right) = C(\tau)e^{\pi ic/4(1+\tau')(la+c)\tau+b+d)}\mathscr{V}_{11}\left(\frac{(a+c)\tau+l+d}{2}\Big/\tau\right)$$

Again using (*) and (**) as we did earlier, this gives

$$ie^{-\pi i\tau'/4}\mathscr{V}_{22}(0/\tau') = C(\tau)e^{\pi ic/4(1+\tau')((a+c)\tau+l+d)}i^{l+d}e^{-\pi i(0+c)^2/4}\mathscr{V}^{(0/\tau)}_{1+a+c,1+l+d}$$

This of course can be embellished a little:

$$\begin{aligned}i\mathscr{V}_{00}(0/\tau') &= C(\tau)e^{\pi i/4(1+\tau')(c(a+c)\tau+cb+id+1)}i^{b+d}e^{-\pi i/4}e^{-\pi i\tau(a+c)^2/4}\mathscr{V}^{(0/\tau)}_{1+a+c,1+b+d}\\ &= C(\tau)e^{\pi i/4(a+c)((a+c)\tau+b+d)}i^{l+d}e^{-\pi i/4}e^{-\pi i\tau(a+c)^2/4}\mathscr{V}_{1+a+c,1+b+d}\\ \therefore\ i\mathscr{V}_{00}(0/\tau') &= C(\tau)e^{\pi i/4(a+c)(b+d)}i^{b+d}e^{-\pi i/4}\mathscr{V}^{(0/\tau)}_{1+a+c,1+b+d}\end{aligned} \tag{5}$$

Now utilise the formula:

$$\mathscr{V}_1'(0/\tau) = \pi\mathscr{V}_2(0/\tau)\mathscr{V}_3(0/\tau)\mathscr{V}_4(0/\tau)$$

Multiplying (3), (4) and (5),

$$\begin{aligned}\mathscr{V}_{11}'(0/\tau') = \ & (C(\tau))^3(-)^{b+d}e^{\pi i/4(ab+cd+(a+b)(b+d)-1)} \\ & \times i\pi\mathscr{V}_{1+c,1+d}(0/\tau)\mathscr{V}_{1+a,1++b}(0/\tau)\mathscr{V}_{1+a+c,1+b+d}(0/\tau)\end{aligned}$$

Observe that the sum of the first subscripts on the right side $= 3+2a+2c \equiv 1 \pmod 2$. So either all three numbers $1+a$, $1+c$, $1+a+c$ are odd, or one of them is odd and two even. Then first case is impossible since we should then have both a and c even and so $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 1$. So two of them are even and one odd. The even suffixes can be reduced to zero and the odd one to 1 by repeated application of (**). Similarly for the second suffixes. So the $\mathscr{V}$-factors on the right will be $\mathscr{V}_{00}$, $\mathscr{V}_{01}$, $\mathscr{V}_{10}$. What we hate is the combination 1, 1 and this does not occur. (If it did occur we should have $\mathscr{V}_{11}$ which vanishes at the origin). Although we can not identify the $\mathscr{V}$-factors on the right, we are sure that we get exactly the combinations that are desirable: 01, 10, 00. The dangerous combination is just out. **127**

Let us reduce the subscripts by stages to 0 or 1 as the case may be. When we reduce the second subscript nothing happens, whereas when we reduce index by steps of 2, each time a factor ±1 is introduced, by virtue of (**). By the time the subscript $1+c$ is reduced to 0 or 1, a factor $(-)^{\left[\frac{1+c}{2}\right]}(1+d)$ will have accumulated in the case of $\mathscr{V}_{1+c,1+d}$. Similarly in the case of $\mathscr{V}_{1+a,1+b}$ and $\mathscr{V}_{1+a+c,1+b+d}$. Altogether therefore we have a factor **128**

$$(-)^{\left[\frac{1+c}{2}\right](1+d)+\left[\frac{1+a}{2}\right](1+b)+\left[\frac{1+a+c}{2}\right](1+b+d)},$$

and when this compensating factor is introduced we can write $\mathscr{V}_{00}$, $\mathscr{V}_{11}'$ and $\mathscr{V}_{10}$. Hence our formula becomes

$$\mathscr{V}_{11}'(0/\tau) = (C(\tau))^3(-)^{\alpha}e^{\pi i/4(ab+cd+(a+c)(b+d)-1)}i\pi\mathscr{V}_{00}(0/\tau)\mathscr{V}_{01}(0/\tau)\mathscr{V}_{10}(0/\tau) \quad (6)$$

where

$$\alpha = b+d+\left[\frac{1+c}{2}\right](1+d)+\left[\frac{1+a}{2}\right](1+b)+\left[\frac{1+a+c}{2}\right](1+b+d)$$

From (1), differentiating and putting $\nu = 0$, we have

$$\mathscr{V}_{11}'(0/\tau') = C(\tau)(C\tau+d)\mathscr{V}_{11}'(0/\tau) \quad (7)$$

Dividing (6) by (7)

$$\begin{aligned}(C(\tau))^2 &= (c\tau+d)(-)^\alpha e^{-\pi i/4(ab+cd+(a+c)(b+d)-1)}\\ &= \frac{c\tau+d}{i}(-)^\alpha e^{-\pi i/4(ab+cd+(a+c)(b+d)-3)}\end{aligned}$$

(we may assume $c > 0$, since $c = 0$ implies $ad = 1$ or $a, d = \pm 1$, which give just translations).

$$\therefore \quad C(\tau) = \pm\sqrt{\frac{c\tau+d}{i}}\, i^\alpha e^{-\pi i/8(ab+cd+(a+c)(b+d)1-3)}$$

For the square root we take the principal branch. Since $\mathrm{Im}(c\tau+d) > 0$, $\mathscr{R}\dfrac{c\tau+d}{i} > 0$, so that $\dfrac{c\tau+d}{i}$ is a point in the right half-plane. The sign is still uncertain.

The factor $e^{-\pi i/8(\cdots)}$ looks like a 16th root of unity, but is really not so. Since **129**
$ad+bc$ has the same parity as $ad-bc=1$, the exponent is even, and therefore what we really have is only an 8th root of unity.

What could we do now? We really do not know of any fruitful way. We cannot copy what we did formerly. There we had a very special case: $\tau' = -1/\tau$, or the modular substitution involved was $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$. The $\pm$ sign depends only on a, b, c, d, not on τ, so that it is enough if we make a special choice of τ in the equation. Formerly we could take $\tau = \tau' = i$ and it worked so beautifully because τ is a study the fixed points of the transformation $\tau' = \dfrac{a\tau+b}{c\tau+d}$. The fixed points ξ are given by

$$\xi = \frac{a\xi+b}{c\xi+d},$$

or

$$c\xi^2 + (d-a)\xi - b = 0$$

i.e.,

$$\begin{aligned}\xi &= \frac{(a-d)\pm\sqrt{(a-d^2)+4bc}}{2c}\\ &= \frac{(a-d)\pm\sqrt{(a+d)^2-4}}{2}\end{aligned}$$

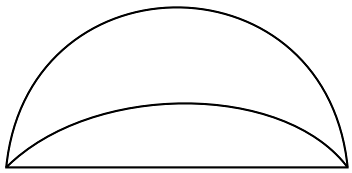
since $ad-bc=1$.

Hence we have several possibilities. If the square root is imaginary we have twp points one in each of the upper and lower half-planes, and for this $|a+d| <$ **130**
2, so that the square root becomes $\sqrt{-4}$ or $\sqrt{-3}$ according as $|a+d| = 0$or 1. This is the *elliptic case*. If $|a+d| = 2$, we have one rational fixed point; this is the *parabolic case*. And in the huge infinity of cases, $|a+d| > 2$, we have two

real fixed points -the *hyperbolic* case. Here the fixed are not accessible to us because they are quadratic algebraic numbers on the real axis.

In the elliptic case with $|a + d| < 2$ we could finish the thing without much trouble. In the parabolic case we are already in a fix. Much more difficult is the hyperbolic case.

If ξ_1 and ξ_2 are the fixed points, τ and τ' will lie on the same circle through ξ_1 and ξ_2, and repetitions of the transformation would give a sequence of points on the same circle which may converge to either ξ_1 or ξ_2. So the ambiguity in the $\pm$ sign will remain.



It will be much more difficult when we pass from $\mathscr{V}$ to η, because then we shall have to determine a cube root.

Lecture 15

We were discussing the possibility of getting a root of unity determined for the transformation of $\mathscr{V}_{11}'\left(\nu\Big/\dfrac{a\tau+b}{c\tau+d}\right)$. There do exist methods for determining this explicitly. Only we tried to carry out the analogue with the special case as far as possible, not with complete success. other methods exist. The first of these is due to Hermite, done nearly 100 years ago. He used what are called Gaussian sums. There are difficulties there too and we want to avoid them. Another method is that of Dedekind using Dedekind sums. 131

In the special case of the transformation from τ to $\tau' = -\frac{1}{\tau}$ we were faced with an elliptic substitution $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. These are of two sorts:

1. $a + d = 0$
2. $a + d = \pm 1$

In both cases we can completely forget about the root of unity if we remember the following fact. Our formula had the following shape:

$$\mathscr{V}_{11}'\left(0\Big/\frac{a\tau+b}{c\tau+d}\right) = \sqrt{\frac{c\tau+d}{i}}\cdot\frac{c\tau+d}{i}\rho(a,b,c,d)\mathscr{V}_{11}'(0/\tau) \qquad (*)$$

where ρ is a root of unity which is completely free of τ. we can then get things straightened out. We have only to consider the fixed points of the transformation given by

$$\xi = \frac{a-d\pm\sqrt{(a+d)^2-4}}{2c}$$

Put ξ on both sides of the formula; since $\xi = \dfrac{a\xi+b}{c\xi+d}$, both sides look alike and $\mathscr{V}_{11}'$ does not vanish for appropriate τ in the upper half-plane (we may take $c > 0$), so that ρ is given directly by the formula. 132

Case 1. $a + d = 0$

$$\xi = \frac{-2d \pm \sqrt{-d}}{2c} = \frac{-d+i}{c}$$

(reject the negative sign since we want a point in the upper half-plane).

$$\frac{c\xi + d}{i} = \frac{i}{1} = i$$

$$\therefore \qquad \mathscr{V}'_{11}(0/\xi) = \rho(a,b,c,d)\mathscr{V}'_{11}(0/\xi)$$

So $\rho(a,b,c,d) = 1$ *and remains 1 in the general formula when we go away from* ξ*.*

Case 2. $a + d = \pm 1$

$$\xi = \frac{\pm 1 - 2d + \sqrt{-3}}{2c}$$

$$\therefore \qquad c\xi + d = \frac{\pm 1 + i\sqrt{3}}{2} = e^{\pi i/3} \text{ or } e^{2\pi i/3}$$

$$\frac{c\xi + d}{i} = e^{\pi i/3 - \pi i/2} \text{ or } e^{2\pi i/3 - \pi i/2}$$

$$= e^{\pm \pi i/6}$$

$$\sqrt{\frac{c\xi + d}{i}} = e^{\mp \pi i/12}$$

Putting ξ on both sides of (*), **133**

$$1 = e^{\mp \pi i/4}\rho(a,b,c,d)$$

$$\therefore \qquad \rho(a,b,c,d) = e^{\pm \pi i/4}$$

when $a + d = \pm 1$, (we may take $c > 0$; the case $c = 0$ is uninteresting and if $c < 0$ we can make it $c > 0$).

There are unfortunately no more cases like these.

Parabolic case. The analysis here is a little longer but it is worth while working it out. Now $a + d = \pm 2$, and there is only one fixed point

$$\xi = \frac{a-d}{2c} - \frac{-2d \pm 2}{2c} = \frac{-d \pm 1}{c} = -\frac{\delta}{\gamma}$$

where $(\gamma, \delta) = 1$ and we may choose $\gamma > 0$. The fixed point is now a rational point on the real axis. We try to approach it. This is a little difficult because we do not know what the function will do thee. But by an auxiliary transformation we can throw this point into the point at infinity. Consider the auxiliary transformation

$$T = \frac{A\tau + B}{\gamma\tau + \delta}, \begin{vmatrix} A & B \\ \gamma & \delta \end{vmatrix} = 1$$

The denominator becomes zero for $\tau = \xi$. Let

$$T' = \frac{A\tau' + B}{\gamma\tau' + \delta}$$

(notice that γ and δ have got something to do with the properties of two other numbers c, d). Now (*) gives **134**

$$\mathscr{V}'_{11}(0/T) = \left(\sqrt{\frac{\gamma\tau + \delta}{i}}\right)^3 \rho(A, B, \gamma, \delta)\mathscr{V}'_{11}(0/\tau),$$

$$\mathscr{V}'_{11}(0/T') = \left(\sqrt{\frac{\gamma\tau' + \delta}{i}}\right)^3 \rho(A, B, \gamma, \delta)\mathscr{V}'_{11}(0/\tau').$$

Dividing, we get

$$\frac{\mathscr{V}'_{11}(0/T')}{\mathscr{V}'_{11}(0/T)} = \left(\frac{\sqrt{\frac{\gamma\tau'+\delta}{i}}}{\sqrt{\frac{\gamma\tau+\delta}{i}}}\right)^3 \frac{\mathscr{V}'_{11}(0/\tau')}{\mathscr{V}'_{11}(0/\tau)} \tag{1}$$

The left side gives the behaviour at infinity. We cannot of course put $\tau = \xi$. Put $\tau = \xi + it$, $t > 0$, and later make $t \to 0$. τ is a point in the upper half-plane.

$$\begin{aligned} \tau - \xi &= it, \\ \tau' - \xi &= \frac{a\tau + b}{c\tau + d} - \frac{a\xi + b}{c\xi + d} \\ &= \frac{\tau - \xi}{(c\tau + d)(c\xi + d)} \\ &= \frac{it}{1 \pm ict} \end{aligned}$$

This is also in the upper half-plane. $\tau' \to \xi$ as $t \to 0$

Let us calculate T and T'. For this consider

$$T' - T = \frac{A\tau' + B}{\gamma\tau' + \delta} - \frac{A\tau + B}{\gamma\tau + \delta}$$

$$= \frac{\tau' - \tau}{(\gamma\tau' + \delta)(\gamma\tau + \delta)}$$
$$= \mp\frac{c}{\gamma^2}$$

This is quite nice; the difference is a real number. 135

$$\begin{aligned}
T &= \frac{A\tau + B}{\gamma\tau + \delta} = \frac{A(\xi + it) + B}{\gamma(\xi + it) + \delta}\\
&= \frac{Ait + A\xi + B}{\gamma it}, \text{ since } \gamma\xi + \delta = 0,\\
&= \frac{A}{\gamma} + \frac{-A\frac{\delta}{\gamma} + B}{\gamma it}\\
&= \frac{A}{\gamma} + \frac{B\gamma - A\delta}{\gamma\tau t}\\
&= \frac{A}{\gamma} + \frac{1}{\gamma^2 t}, \text{ since } B\gamma - A\delta = -1
\end{aligned}$$

$\to i\infty$ (along the ordinate $x = \frac{A}{\gamma}$) as $t \to 0$

$$T' = \frac{A}{\gamma} \pm \frac{c}{\gamma^2} + \frac{i}{r^2 t}$$
$$\to i\infty \text{ as } t \to 0$$

Now recall the infinite product formula 136

$$\mathscr{V}'_{11}(0/T) = 2\pi i e^{\pi iT/4} \prod_{n=1}^{\infty}(1 - e^{2\pi inT})^3$$

Let $T = \frac{A}{\gamma} + \frac{i}{\gamma^2 t}$. Then

$$e^{2\pi inT} = e^{2\pi inA/\gamma} e^{-2\pi nt/\gamma^2} \to 0$$
$$\therefore \quad \mathscr{V}'_{11}(0/T) = 2\pi i e^{\pi iT/4}(\text{a factor tending to } 1)$$

We do not know what happens to $e^{\pi iT/4}$. But we need only the quotient. So

$$\frac{\mathscr{V}'_{11}(0/\tau)}{\mathscr{V}'_{11}(0/\tau)} \sim e^{\pi i(T' - T)/4}$$
$$= e^{\mp\pi ic/4\gamma^2} \tag{2}$$

Consider similarly the quotient $\mathscr{V}'_{11}(0/\tau')/\mathscr{V}'_{11}(0/\tau)$. We have, since $\gamma\xi+\delta =$ 0,

$$\frac{\gamma\tau+\delta}{i} = \frac{\gamma(\xi+it)+\delta}{i} = \gamma^t$$

or $\sqrt{\dfrac{\gamma\tau+\delta}{i}} = \sqrt{\nu t}$, where we take the positive square root

$$\frac{\gamma\tau'+\delta}{i} = \frac{\gamma\left(\xi+\frac{it}{1\pm ict}\right)+\delta}{i} = \frac{\gamma t}{1\pm ict}$$

$$\therefore \quad \sqrt{\frac{\gamma\tau'+\delta}{i}} = \sqrt{rt}\sqrt{\frac{1}{1\pm ict}} \quad \text{(both branches principal)}$$
$$\sim \sqrt{\gamma t} \text{ as } t \to 0.$$

Hence the quotient $\sqrt{\dfrac{\gamma\tau'+\delta}{i}}\Big/\sqrt{\frac{\gamma\tau+\delta}{i}}$ behaves like 1. **137**

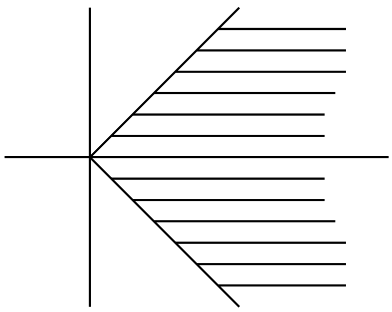
And so we have what we were after:

$$\frac{\mathscr{V}'_{11}(0/\tau')}{\mathscr{V}'_{11}(0/\tau)} \to e^{\mp\pi ic/4\gamma^2} \text{ as } t \to 0 \tag{3}$$

$$\frac{c\tau+d}{i} = \frac{c(\xi+it)+d}{i} = \frac{\frac{a-d}{2}+cit+d}{i}$$
$$= \frac{a+\frac{d}{2}+cit}{i} = \pm\frac{1+cit}{i}$$
$$= \mp i + ct$$
$$\to \mp i \text{ as } t \to 0.$$

What will the square root of this do?

$\sqrt{\dfrac{c\tau+d}{i}} = \sqrt{ct\mp i}$, and this does lie in the proper half plane because $ct > 0$. For small t it will be very near the imaginary axis near $\mp i$. So the square root lies in the sector, in the lower half plane if we choose $\sqrt{-i} = e^{-\pi i/4}$, and in the upper half-plane if we choose $\sqrt{+i} = e^{\pi i/4}$ Hence $\sqrt{\dfrac{c\tau+d}{i}} \to e^{\mp\pi i/4}$ as $t \to 0$.



Using this fact as well as (2) and (3) in (*) we get **138**

$$e^{\mp\pi ic/4\gamma^2} = e^{\mp 3\pi i/4}\rho(a,b,c,d)$$

$$\therefore \qquad \rho(a,b,c,d) = e^{\mp\frac{\pi i}{4}(c/\gamma^2 - 3)}$$

We observe that the common denominator $(\gamma,\delta) = 1$ plays a role, however a and b do not enter.

Hyperbolic case. The thing could also be partly considered in the hyperbolic case. It will take us into deeper things like real quadratic fields and we do not propose to do it.

Let us return to what we had achieved in the specific case. We had a formula for $\eta(\tau)$:

$$\eta(\tau') = \sqrt{\frac{c\tau + d}{i}}\epsilon(a,b,c,d)\eta(\tau),$$

where $\epsilon(a,b,c,d)$ is a 24th root of unity. This shape we have in all circumstances. The difficulty is only to compute ϵ. We shall not determine it in general, and we can do away with it even for the purpose of partitions by using a method developed recently by Selberg.

However in each specific case we can compute ϵ.

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$$

Now

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi in\tau}),$$

$$\eta(\tau + b) = e^{\pi ib/12}\eta(\tau)$$

Out of these two facts we can get every other one, because the two substitutions **139**

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

form generators of the full modular group. This can be shown as follows. Take $c > 0$.

$$a\tau + b = q_0(c\tau + d) - a, \tau - b_1, c > |a_1|,$$

$$\text{or} \qquad \frac{a\tau + d}{c\tau + d} = q_0 - \frac{a_1\tau + b_1}{c\tau + d}, \quad \begin{vmatrix} c & d \\ a_1 & b_1 \end{vmatrix} = 1$$

(if $a < 0$ this step is unnecessary). Similarly

$$\frac{c\tau + d}{a_1\tau + b_1} = q_1 - \frac{a_2\tau + b_2}{a_1\tau + b_1}$$

We thus get a continued fraction expansion. The partial quotients get similar and simpler every time and end with $\frac{\tau + b}{0 + 1} = \tau + q_k$. so we can go back and take linear combinations; all that we have to do is either to add an integer to τ or take $-1/\tau$.

As an example, let us consider

$$\eta\left(\frac{3\tau + 4}{2\tau + 3}\right)$$

Let us break $\frac{3\tau + 4}{2\tau + 3}$ into simpler substitutions,

$$\begin{aligned}
\tau_3 &= \frac{3\tau + 4}{2\tau + 3} = 1 - \frac{1}{\tau_2}, \\
\tau_2 &= -2 + \tau_1; \\
\tau_1 &= -\frac{1}{\tau + 1} \\
\eta(\tau_1) &= \eta\left(-\frac{1}{\tau + 1}\right) = \sqrt{\frac{\tau + 1}{i}}\eta(\tau + 1) \\
&= \sqrt{\frac{\tau + 1}{i}} e^{\pi i/12}\eta(\tau). \\
\eta(\tau_2) &= \eta(\tau_1 - 2) = e^{-\pi i/6}\eta(\tau_1) \\
&= \sqrt{\frac{\tau + 1}{i}} \cdot e^{-\pi i/12}\eta(\tau) \\
\eta\left(-\frac{1}{\tau_2}\right) &= \sqrt{\frac{\tau_2}{i}}\eta(\tau_2) \\
&= \sqrt{\frac{\tau + 1}{i}}\sqrt{\frac{\tau_2}{i}} e^{-\pi i/12}\eta(\tau)
\end{aligned}$$

The two square roots taken separately are each a principal branch, but taken **140** together they may exceed one. We can write this as

$$\begin{aligned}
\eta\left(-\frac{1}{\tau_2}\right) &= \sqrt{\frac{\tau+1}{i}}\sqrt{\frac{-2-\frac{1}{\tau+1}}{i}}e^{-\pi i/12}\eta(\tau)\\
&= \sqrt{\frac{\tau+1}{i}}\sqrt{\frac{-2\tau-3}{i(\tau+1)}}e^{-\pi i/12}\eta(\tau)\\
&= \sqrt{\frac{-2\tau-3}{-1}}e^{-\pi i/12}\eta(\tau)\\
&= \pm\sqrt{3+2\tau}e^{-\pi i/12}\eta(\tau)
\end{aligned}$$

Here we are faced with a question: which square root are we to take? 141

We write $\sqrt{3+2\tau} = e^{\pi i/4}\sqrt{\frac{2\tau+3}{i}}$

Let us look into each root singly. For $\tau = it$ where do they go?

$$\sqrt{\frac{\tau+1}{i}} = \sqrt{\frac{it+1}{i}}$$

$$\to \infty \text{ with argument } 0 \text{ as } t \to \infty.$$

$$\sqrt{\frac{-2\tau-3}{i(\tau+1)}} = \sqrt{\frac{-2it-3}{i(\tau+1)}} \to \sqrt{2i} \text{ as } t \to \infty,$$

or its argument $= \pi/4$

The product $\sqrt{\frac{\tau+1}{i}}\sqrt{\frac{-2\tau-3}{i(\tau+1)}}$ has here argument $\pi/4$, so that it continues to be the principal branch. Of course in a less favourable case, if we had two other arguments, together they would have run into something which was no longer a principal branch. Finally,

$$\eta\left(\frac{3\tau+4}{2\tau+3}\right) = e^{\pi i/4}\sqrt{\frac{2\tau+3}{i}}\eta(\tau)$$

and here there is no ambiguity. Actually in every specific case that occurs one can compute step and make sure what happens.

There does exist a complete formula which determines $\epsilon(a, b, c, d)$ explicitly by means of Dedekind sums $S(h, k)$.

Part III

Analytic theory of partitions

Lecture 16

Our aim will be now to get an explicit formula for $p(n)$ and things connected with it. Later we shall return to the function $\eta(\tau)$ and the discussion of the sign of the square root. That will again lead us into some aspects of the theory of $\mathscr{V}$-functions connected with quadratic residues. 142

Let us come to our topic. Euler had, as we know, the identity:

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{\prod\limits_{m=1}^{\infty}(1-x^m)}.$$

This is the starting point of the function-theoretic treatment of $p(n)$.

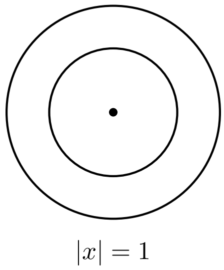
$$p(n) = \frac{1}{2\pi i}\int_c \frac{f(x)}{x^{n+1}}dx,$$

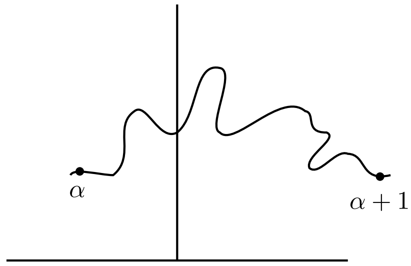
where $f(x) = \prod\limits_{m=1}^{\infty}(1-x^m)^{-1}$ and C is a suitable closed path contained in the unit circle, in which the function is analytic, and enclosing the origin. Since $\sum p(n)x^n$ is a power series beginning with 1, this means a little more. n may be negative also; and when n is negative $f(x)x^{-n-1}$ is regular at $x = 0$. Therefore we include negative exponents also in our discussion; we put $p(-n) = 0$, $n > 0$, when is convenient. Hereafter we shall take n to be an integer $\gtreqless 0$; we shall choose n and keep it fixed throughout our discussion.

It is a little more comfortable to change the variable and put $x = e^{2\pi i\tau}$, 143
$\operatorname{Im}\tau > 0$, which is familiar to us.

$dx = e^{2\pi i\tau}\cdot 2\pi i d\tau$ and the whole thing boils down to

$$p(n) = \int_{\alpha}^{\alpha+1} f(e^{2\pi i\tau})e^{-2\pi i n\tau}d\tau$$





It is enough to take the integral along a path from an arbitrary point α to the point $\alpha + 1$, because the integrand is periodic, with period 1. (This path replaces the original path C that we had in the x-plane before we changed the variable). The method of Hardy and Ramanujan was to take a curve rather close to the unit circle which is a natural boundary for the function (this will come out in the course of the argument). They cut up the path of integration into pieces called Farey arcs, and the trick was to replace the function by a simpler approximating function on each specific Farey arc. We shall use not exactly this method, but consider a special path from α to $\alpha+1$, which we shall discuss.

We shall keep our formula in abeyance for a moment and give a short discussion of Farey series ('series' here is not to be understood in the sense of
infinite series, but as just an aggregate of numbers). Cauchy did make all the **144**
observation attributed by Hardy and Wright to Farey; Farey made his remarks in the Philosophical Magazine, 1816. He put only questions; Cauchy had all the answers earlier.

We deal with the interval $(0, 1)$. Choose all reduced fractions whose denominators do not exceed $1, 2, 3, \cdots$ in succession. Let us write down the first few, with the fractions arranged in increasing order of magnitude.

				$\frac{0}{1}$				$\frac{1}{1}$					order 1
			$\frac{0}{1}$			$\frac{1}{2}$			$\frac{1}{1}$				order 2
		$\frac{0}{1}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{1}{1}$			order 3
	$\frac{0}{1}$		$\frac{1}{4}$	$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$	$\frac{3}{4}$		$\frac{1}{1}$		order 4
$\frac{0}{1}$		$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$		$\frac{1}{1}$	order 5

The interesting fact is that we can write down a new in the following way. We repeat the old row and introduce some new fractions. If $\dfrac{h}{k} < \dfrac{h'}{k'}$ are adjacent fractions in a row, the new one introduced between these in the next row is

$\dfrac{h+h'}{k+k'}$, provided that the denominator is of the proper size. Following Hardy and Littlewood we call $\dfrac{h+h'}{k+k'}$ the 'mediant' between $\dfrac{h}{k}$ and $\dfrac{h'}{k'}$. We have $\dfrac{h}{k} < \dfrac{h+h'}{k+k'} < \dfrac{h'}{k'}$, so that the order is automatically the natural order. We call that row which has denominator $k \leq N$, the *Farey series of order* N. We get this by forming mediants from the preceding row. Farey made the following observation. Take two adjacent fractions in a row; then the determinant formed by their numerators and denominators is equal to -1. For instance, in the fifth row $\dfrac{1}{3}$ and $\dfrac{2}{5}$ are adjacent and $\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1$***********. If we now prove that new fractions are always obtained by using mediants, then we can be sure, by 145
induction, that this determinant is always -1. For, let

$$\begin{vmatrix} h & h' \\ k & k' \end{vmatrix} = -1; \text{ then}$$

$$\begin{vmatrix} h & h+h' \\ k & k+k' \end{vmatrix} = -1 = \begin{vmatrix} h+h' & h' \\ k+k' & k' \end{vmatrix}$$

If indeed only mediants occur, Farey's observation is justified. And this is so. Observe that these fractions must all appear in their lowest terms; otherwise, the common factor will show up and the determinant would not be -1. Suppose that we want to find out where a particular fraction appears. Say, we have in mind a specific fraction $\dfrac{H}{K}$. It should occur for the first time in the Farey series of order $N = K$ and it should not be present on any series of order $< K$. Now look at $N = K - 1$ where $\dfrac{H}{K}$ is not present. If we put it in, it will belong somewhere according to its size, i.e., we can find fractions $\dfrac{h_1}{k_1}, \dfrac{h_2}{k_2}$, with $k, k_2 < N$ such that $\dfrac{h_1}{k_1} < \dfrac{H}{K} < \dfrac{h_2}{k_2}$. Assume that the determinant property and the mediant property are true for order $N < K$. (They are clearly true up to order 5, as we verify by inspection, so that we can start induction). Now prove them for $N = K$. Try to determine H and K by interpolation between $\dfrac{h_1}{k_1}$ and $\dfrac{h_2}{k_2}$. Put

$$Hk_1 - Kh_1 = \lambda,$$

$$-Hk_2 + Kh_2 = \mu,$$

so that λ and μ are integers > 0. Solving for H and K by Cramer's rule, **146**

$$H = \frac{\begin{vmatrix} \lambda & -h_1 \\ \mu & h_2 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 \\ h_1 & k_1 \end{vmatrix}}, \quad K = \frac{\begin{vmatrix} \lambda & -k_1 \\ \mu & k_2 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 \\ h_1 & k_1 \end{vmatrix}}$$

By induction hypothesis, the denominator = 1, and so

$$H = \lambda h_2 + \mu h_1$$
$$K = \lambda k_2 + \mu k_1$$
$$\text{or} \qquad \frac{H}{K} = \frac{\lambda h_2 + \mu h_1}{\lambda k_2 + \mu k_1}.$$

What do we know about K? K did not appear in a series of order $K - 1$; k_1 and k_2 are clearly less than K. What we have found out so far is that any fraction lying between $\frac{h_1}{k_1}$ and $\frac{h_2}{k_2}$ can be put in the form $\frac{\lambda h_2 + \mu h_1}{\lambda k_2 + \mu k_1}$. Of these only one interests us - that one with lowest denominator. This comes after the ones used so far. Look for the one with lowest denominator; this corresponds to the smallest possible λ, μ, i.e., $\lambda = \mu = 1$. Hence first among the many later appearing ones is $\frac{H}{K} = \frac{h_1 + h_2}{k_1 + k_2}$, i.e., if in the next Farey series a new fraction is called for, that is produced by a mediant. So what was true for $K - 1$ is true for K; and the thing runs on.

One remark is interesting, which was used in the Hardy - Littlewood- Ramanujan discussion. In the Farey series of order N, let $\frac{h_1}{k_1}$ and $\frac{h_2}{k_2}$ be adjacent fractions. $\frac{h_1}{k_1} < \frac{h_2}{k_2} \cdot \frac{h_1 + h_2}{k_1 + k_2}$ does not being these. It is of higher order. This says that $k_1 + k_2 > N$. For two adjacent fractions in the Farey series of order N, **147**
the sum of the denominators exceeds N. Both k_1 and $k_2 \leq N$, so

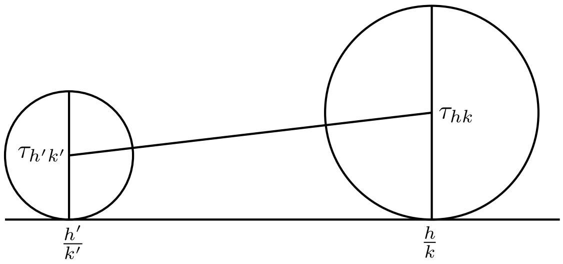
$$2N \geq k_1 + k_2 > N.$$

k_1 and k_2 are equal only in the first row, otherwise it would ruin the determinant rule. So

$$2N > k_1 + k_2 > N, N > 1.$$

This was very often used in the Hardy - Ramanujan discussion. (The Farey series is an interesting way to start number theory with. We can derive from it Euclid's lemma of decomposition of an integer into primes. This is a concrete way of doing elementary number theory).

We now come to the special path of integration. For this we use *Ford Circles* (L.R. Ford, American Mathematical Monthly, 45 (1938), 568-601). We describe a series of circles in the upper half-plane. To each proper fraction $\dfrac{h}{k}$ we associate a circle C_{hk} with centre $\tau_{hk} = \dfrac{h}{k} + \dfrac{i}{2k^2}$ and radius $\dfrac{1}{2k^2}$, so the circles all touch the real axis.



Take another Ford circle $C_{h'k'}$, with centre at $\tau_{h'k'}$. Calculate the distance between the centres.

$$|\tau_{hk} - \tau_{h'k'}|^2 = \left(\frac{h}{k} - \frac{h'}{k'}\right)^2 + \left(\frac{1}{2k^2} - \frac{1}{2k'^2}\right)^2.$$

The sum of the radii $= \frac{1}{2k^2} + \frac{1}{2k'^2}$ **148**

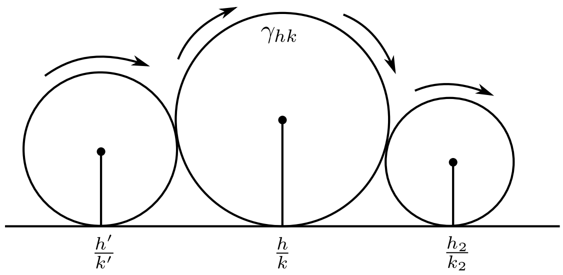
$$|\tau_{hk} - \tau_{h'k'}|^2 - \left(\frac{1}{2k^2} + \frac{1}{2k'^2}\right) = \left(\frac{h}{k} - \frac{h'}{k'}\right)^2 - \frac{1}{k^2h'^2}$$
$$= \frac{(hk' - h'k)^2 - 1}{k^2k'^2} \geq 0,$$

since h, k are coprime and so $\begin{vmatrix} h & k \\ h' & k' \end{vmatrix}$ is an integer $\neq 0$. So two Ford circles never intersect. And they touch if and only if

$$\begin{vmatrix} h & k \\ h' & k' \end{vmatrix} = \pm 1,$$

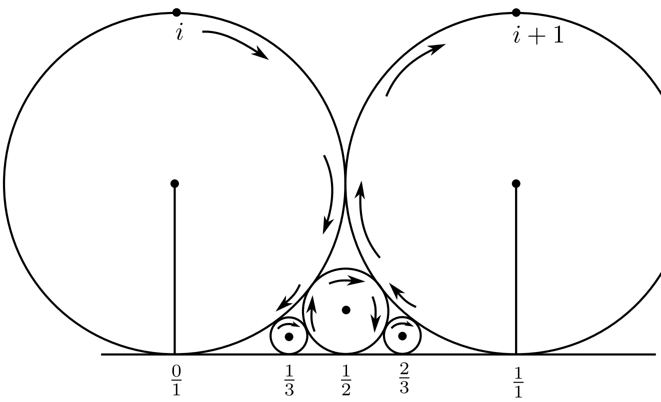
i.e., if in a Farey series $\dfrac{h}{k}, \dfrac{h'}{k'}$ have appeared as adjacent fractions.

Now we come to the description of the path of integration from α to $\alpha + 1$. For this consider the Ford circle C_{hk}.



In a certain Farey series of order N we have adjacent fractions $\dfrac{h_1}{k_1} < \dfrac{h}{k} < \dfrac{h_2}{k_2}$. (We know exactly which are adjacent ones in a specific series). Draw also 149
the Ford circles $C_{h_1k_1}$ and $C_{h_2k_2}$. These touch C_{hk}. Take the arc γ_{hk} of C_{hk} from one point of contact to the other in the clockwise sense (the arc chosen is the one not touching the real axis). This we do for all Farey fractions of a given order. We call the path belonging to Farey series of order N P_N. Let us describe this in a few cases.

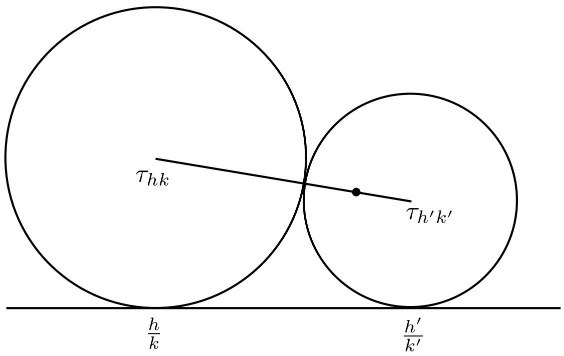
We fix $\alpha = i$ and pass to $\alpha + 1 = i + 1$. Take $N = 1$; we have two circles of radii 2 each with centres at $\dfrac{i}{2}$ and $1 + \dfrac{i}{2}$



ρ_1 will be the path consisting of arcs from i to $\frac{1}{2} + \frac{i}{2}$ and $\frac{1}{2} + \frac{i}{2}$ to $i + 1$. Later because of the periodicity of $f(e^{2\pi i\tau})$ we shall replace the second piece by the arc from $-\frac{1}{2}+\frac{i}{2}$ to i. Next consider Farey series of order 2; $\dfrac{0}{1}$ and $\dfrac{1}{1}$ are no longer adjacent. The path now comprises the arc of C_{01} from i to the point of contact with C_{12}, the arc of C_{12} from this point to the point of contact with C_{11} **150** and the arc of C_{11} from this point to $i+1$. Similarly at the next stage we pass from i on C_{01} to $i+1$ on C_{11} through the appropriate arcs on the circles C_{13}, C_{12}, C_{23} in order. So the old arcs are always retained but get extended and new arcs spring into being and the path gets longer and longer. At no stage does the path intersect itself, but these are points of contact. The path is complicated and was not invented in one sitting. The Farey dissection of Hardy and Ramanujan can be pictured as composed of segments parallel to the imaginary axis. Here it is more complicated.

We need a few things for our consideration. We want the point of contact of C_{hk} and $C_{h'k'}$. This is easily seen to be the point

$$\tau_{hk}\frac{\frac{1}{2k'^2}}{\frac{1}{2k^2}+\frac{1}{2k'^2}}+\tau_{h'k'}\frac{\frac{1}{2k^2}}{\frac{1}{2k^2}+\frac{1}{2k'^2}}=\left(\frac{h}{k}+\frac{i}{2k^2}\right)\frac{k^2}{k^2+k'^2}+\left(\frac{h'}{k'}+\frac{i}{2k'^2}\right)\frac{k'^2}{k^2+k'^2}$$
$$=\frac{h}{k}+\left(\frac{h'}{k'}-\frac{h}{k}\right)\frac{k'^2}{k^2+k'^2}+\frac{i}{k^2+k'^2}$$



and this, since **151**

$$\frac{h}{k}<\frac{h'}{k'}\text{ and }\begin{vmatrix}h' & h\\ k' & k\end{vmatrix}=1,\ \text{is}=\frac{h}{k}+\frac{k'}{k(k^2+k'^2)}+\frac{i}{k^2+k'^2}$$
$$=\frac{h}{k}+\xi'_{hk}$$

where $\zeta'_{hk} = \dfrac{k'}{k(k^2+k'^2)} + \dfrac{i}{k^2+k'^2}$. We notice that the imaginary part $1/(k^2+k'^2)$ gets smaller and smaller as $k+h'$ lies between N and $2N$. Each arc runs from $\frac{h}{k}+\zeta'_{hk}$ to $\dfrac{h}{k}+\zeta''_{hk}$. Such an arc is the arc ν_{hk}. No arc touches the real axis.

We continue our study of the integral. Choose a number N, the order of the Farey series, and cut the path of integration P_N into pieces γ_{hk}.

$$\begin{aligned} p(n) &= \int\limits_{P_N} f(e^{2\pi i\tau})e^{-2\pi in\tau}d\tau \\ &= \sum_{\substack{(h,k)=1\\0\leq h<k\leq N}} \int\limits_{\gamma_{hk}} f(e^{2\pi i\tau})e^{-2\pi in\tau}d\tau \end{aligned}$$

Now utilise the points of contact: put

$$\tau = \frac{h}{k}+\zeta;$$

$$p(n) = \sum_{\substack{(h,k)=1\\0\leq h<k\leq N}} \int\limits_{\zeta'_{hk}}^{\zeta''_{hk}} f(e^{2\pi i(\frac{h}{k}+\zeta)})e^{-2\pi in(\frac{h}{k}+S)}dS$$

(γ_{hk} goes from $\frac{h}{k}+\zeta'_{hk}$ to $\frac{h}{k}+\zeta''_{hk}$; these are all arcs of radii $1/2k^2$). We make a further substitution: put $\zeta = \dfrac{iz}{k^2}$, so that we turn round and have everything in the right half-plane, instead of the upper half-plane. (All these are only preparatory changes; there is no actual mathematical progress as yet). Then 152

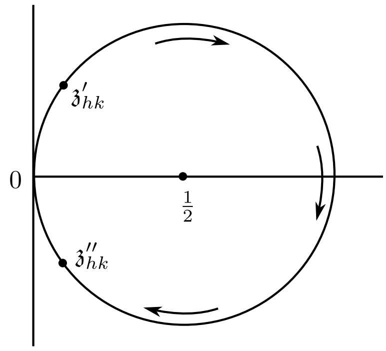
$$p(n) = i\sum_{\substack{(h,k)=1\\0\leq h<k\leq N}} \frac{e^{-2\pi inh/k}}{k^2}\int\limits_{\mathfrak{z}'_{hk}}^{\mathfrak{z}''_{hk}} f(e^{2\pi i(\frac{h}{k}+\frac{i\mathfrak{z}}{k^2})})e^{2\pi n\mathfrak{z}/k^2}d\mathfrak{z}$$

Now find out $\mathfrak{z}'_{hk}$ and $\mathfrak{z}''_{hk}$.

$$\begin{aligned} \mathfrak{z}'_{hk} &= \frac{k^2+ikk'}{k^2+k'^2} \\ \mathfrak{z}''_{hk} &= \frac{k^2-ikk''}{k^2+k''^2} \end{aligned}$$

So what we have achieved so for is to cut the integral into pieces. 153

The whole thing lies on the right half-plane. The original point of contact is 0 and everything lies on the circle $|\mathfrak{z} - \frac{1}{2}| = \frac{1}{2}$. This is a normalisation. We now study the complicated function on each arc separately. We shall find that it is *practically* the function $\eta(\tau)$ about which we know a good deal:



$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\sqrt{\frac{c\tau + d}{i}}\eta(\tau),$$

ϵ being a complicated 24th root of unity.

Lecture 17

We continue our discussion of $p(n)$. Last time we obtained 154

$$p(n) = \sum_{\substack{(h,k)=1 \\ 0 \le h < k \le N}} \frac{i}{k^2} e^{-2\pi i n h/k} \int_{\mathfrak{z}'_{hk}}^{\mathfrak{z}''_{hk}} f\left(e^{2\pi i(\frac{h}{k} + \frac{\mathfrak{z}}{k^2})}\right) e^{2\pi i n \mathfrak{z}/k^2} d\mathfrak{z}$$

n is a fixed integer here, $n \gtreqless 0$ and $p(n) = 0$ for $n < 0$; and this will be of some use later, trivial as it sounds. N is the order of the Farey series. We have to deal with a complicated integrand and we can foresee that there will be difficulties as we approach the real axis. However, f is closely related to η:

$$f(e^{2\pi i\tau}) = e^{\pi i\tau/12}(\eta(\tau))^{-1},$$

since

$$f(x) = \frac{1}{\prod\limits_{m=1}^{\infty}(1 - x^m)},$$

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau}).$$

For us $\tau = \dfrac{h}{k} + \dfrac{i\mathfrak{z}}{k^2}$.

We can now use the modular transformation. We want to make $\operatorname{Im}\tau$ large so that we obtain a big negative exponent. So we put $\tau' = \dfrac{a\tau + b}{c\tau + d}$, a, b, c, d being chosen in such a way for small τ, τ' becomes large. This is accomplished by 155
taking $k\tau - h$ in the denominator; $k\tau - h = 0$ when $z = 0$ and close to zero when z is close to the real axis. We can therefore put $\tau' = \dfrac{a\tau + b}{k\tau - h}$ where a, b should be integers such that $\begin{vmatrix} a & b \\ k & -h \end{vmatrix} = 1$. This gives $-ah - bk = 1$ or $ah \equiv -1$

(mod k). Take a solution of this congruence, say h' i.e., choose h' in such a way that $h'h \equiv -1 \pmod{k}$, which is feasible since $(h, k) = 1$. As soon as h' is found, we can find b. Thus the matrix of the transformation would be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}$$

So we have found a suitable τ' for our purpose.

$$\begin{aligned} \tau' &= \frac{h'\left(\dfrac{h}{k} + \dfrac{i\mathfrak{z}}{k^2}\right) - \dfrac{hh'+1}{k}}{k\left(\dfrac{h}{k} + \dfrac{i\mathfrak{z}}{k^2}\right) - h} \\ &= \frac{h'\dfrac{i\mathfrak{z}}{k} - 1}{i\mathfrak{z}} \\ &= \frac{h'}{k} + \frac{i}{\mathfrak{z}}. \end{aligned}$$

If z is small this is big.

Now recall the transformation formula for η: if $c > 0$,

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\sqrt{\frac{c\tau + i}{i}}\,\eta(\tau)$$

In our case 156

$$\begin{aligned} f(e^{2\pi i\tau'}) &= e^{\pi i\tau'/12}(\eta(\tau'))^{-1} \\ &= e^{\pi i\tau'/12}\epsilon^{-1}\left(\frac{c\tau + d}{i}\right)^{-1/2}(\eta(\tau))^{-1} \\ & e^{\pi i\tau'/12}\epsilon^{-1}\left(\frac{c\tau + d}{i}\right)^{-1/2} e^{\pi i\tau/12} f(e^{2\pi i\tau}) \end{aligned}$$

And this is what we were after. Since

$$c\tau + d = k\tau - h = k\left(\frac{h}{k} + \frac{i\mathfrak{z}}{k^2}\right) - h = \frac{i\mathfrak{z}}{k},$$

this can be rewritten in the form:

$$f(e^{2\pi i\tau}) = f(e^{2\pi i\left(\frac{h}{k} + \frac{i\mathfrak{z}}{k^2}\right)})$$

$$= \epsilon e^{\frac{\pi i}{12}\left(\frac{h}{k}-\frac{h'}{k}\right)} e^{\frac{\pi i}{12}\left(\frac{i\mathfrak{z}}{k^2}-\frac{i}{\mathfrak{z}}\right)} \sqrt{\frac{\mathfrak{z}}{k}} f\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{\mathfrak{z}}\right)}\right)$$

and there is no doubt about the square root - it is the principal branch. We write

$$\omega_{hk} = \epsilon e^{\frac{\pi i}{12}\left(\frac{h}{k}-\frac{h'}{k}\right)}$$

So something mathematical has happened after all this long preparation; and we can make some use of our previous knowledge. We have

$$p(n) = \sum_{o\le h<k\le N}{}' \frac{i\omega_{hk}}{k^{5/2}} e^{-2\pi inh/k} \int\limits_{\mathfrak{z}'_{hk}}^{\mathfrak{z}''_{hk}} e^{\frac{\pi}{12}\left(\frac{1}{\mathfrak{z}}-\frac{\mathfrak{z}}{k^2}\right)} \sqrt{\mathfrak{z}} f\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{\mathfrak{z}}\right)}\right) e^{2\pi n\mathfrak{z}/k^2} d\mathfrak{z}$$

where $\sum'$ denotes summation over h and k, $(h,k) = 1$. Now what is the advantage we have got? Realise that **157**

$$f(x) = \sum_{n=0}^{\infty} p(n)x^n = 1 + x + \cdots$$

So for small x, $f(x)$ is close to 1. It will be a good approximation for small arguments at least to replace $f(x)$ by 1. Let us write

$$\Psi_k(\mathfrak{z}) = \sqrt{\mathfrak{z}} e^{\frac{\pi}{2}\left(\frac{1}{\mathfrak{z}}-\frac{\mathfrak{z}}{k^2}\right)}$$

Then

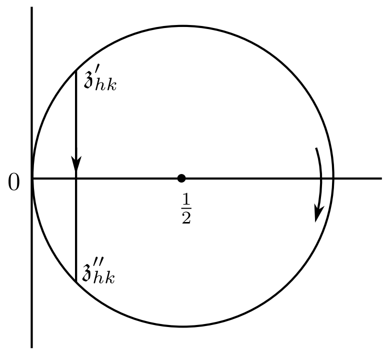
$$p(n) = \sum_{0\le h<k\le N}{}' \frac{i\omega_{hk}}{k^{5/2}} e^{-2\pi inh/k} \int\limits_{\mathfrak{z}'_{hk}}^{\mathfrak{z}''_{hk}} e^{2\pi n\mathfrak{z}/k^2} \Psi_k(\mathfrak{z}) d\mathfrak{z} +$$

$$\sum_{o\le h<k\le N} \frac{i\omega_{hk}}{k^{5/2}} e^{-2\pi inh/k} \int\limits_{\mathfrak{z}'_{hk}}^{\mathfrak{z}''_{hk}} \Psi_k(\mathfrak{z}) \left\{ f(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{\mathfrak{z}}\right)}) - 1 \right\} e^{2\pi n\mathfrak{z}/k^2} d\mathfrak{z}$$

where the second term compensates for the mistake committed on taking $f(x) = 1$. The trick will be now to use the first term as the main term and **158** to use an estimate for the small contribution from the second term. We have now to appraise this. Writing I_{hk} and I^*_{hk} for the two integrals, we have

$$p(n) = \sum_{o\leq h<k\leq N}{}' \frac{i\omega_{hk}}{k^{5/2}} e^{-2\pi inh/k} I_{hk} + \sum_{o\leq h<k\leq N}{}' \frac{i\omega_{hk}}{k^{5/2}} e^{-2\pi inh/k} I^*_{hk}$$

Here we stop for a moment and consider only I^*_{hk} and see what great advantage we got from our special path.



This is the arc of the circle $|z-\frac{1}{2}| = \frac{1}{2}$ from $\mathfrak{z}'_{hk}$ to $\mathfrak{z}''_{hk}$ described clockwise. Since $f(x)-1 = \sum_{m=1}^{\infty} p(\nu)x^\nu$, the integrand in I^*_{hk} is regular, and so for integration we can just as well run across, along the chord from $\mathfrak{z}'_{hk}$ to $\mathfrak{z}''_{hk}$. Let us see what happens on the chord. We have

$$\begin{aligned}
\left|\left(f(e^{2\pi ih'/k-2\pi\mathfrak{z}})-1\right)\Psi_k(\mathfrak{z})e^{2\pi n\mathfrak{z}/k^2}\right| &= \left|\left(f(e^{2\pi ih'/k-2\pi/\mathfrak{z}})-1\right)\sqrt{\mathfrak{z}}e^{\frac{\pi}{12\mathfrak{z}}-\frac{\pi\mathfrak{z}}{12k^2}+\frac{2\pi n\mathfrak{z}}{k^2}}\right| \\
&= \sqrt{\mathfrak{z}}e^{\mathscr{R}\frac{\pi}{12\mathfrak{z}}}e^{\mathscr{R}\mathfrak{z}\frac{\pi}{k^2}(-\frac{1}{2}+2n)} \times \left|\sum_{\nu=1}^{\infty} p(\nu)e^{(2\pi i\frac{h'}{k}-\frac{2\pi}{\mathfrak{z}})\nu}\right| \\
&\leq |\sqrt{\mathfrak{z}}| \sum_{\nu=1}^{\infty} p(\nu)e^{-\mathscr{R}\frac{1}{\mathfrak{z}}(2\pi\nu-\frac{\pi}{12})}e^{\frac{\pi}{k^2}(-\frac{1}{12}+2n)\mathscr{R}\mathfrak{z}}
\end{aligned}$$

Let us determine $\mathscr{R}\mathfrak{z}$ and $\mathscr{R}\frac{1}{\mathfrak{z}}$ on the path of integration $o<\mathscr{R}\mathfrak{z}\leq 1$ on the **159**
chord. And $\mathscr{R}\frac{1}{\mathfrak{z}}>1$; for

$$\mathscr{R}\frac{1}{\mathfrak{z}} = \mathscr{R}\frac{1}{x+iy} = \frac{x}{x^2+y^2},$$

while the equation to the circle is $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$ or $x^2 + y^2 = x$; the interior of the circle is $x^2 + y^2 < x$, and so $\mathscr{R}\frac{1}{\mathfrak{z}} \leq 1$, equality on the circle.

$|\sqrt{\mathfrak{z}}| \leq$ the longer of the distances of $\mathfrak{z}'_{hk}$, $\mathfrak{z}''_{hk}$ from 0.

We have already computed $\mathfrak{z}'_{hk}$ and $\mathfrak{z}''_{hk}$: 160

$$\mathfrak{z}'_{hk} = \frac{k^2}{k_1^2 + k^2} + i\frac{kk_1}{k_1^2 + k^2},$$

$$\mathfrak{z}''_{hk} = \frac{k^2}{k_2^2 + k^2} + i\frac{kk_2}{k_2^2 + k^2}$$

$$|\mathfrak{z}'_{hk}|^2 = \frac{k^4 + k^2k_1^2}{(k_1^2 + k^2)^2} = \frac{k^2}{k^2 + k_1^2}$$

Now we wish to appraise this in a suitable way.

$$\begin{aligned} 2(k_1^2 + k^2) &= (k_1 + k^2)^2 + (k_1 - k)^2 \\ &\geq (k_1 + k)^2 \\ &\geq N^2, \end{aligned}$$

from our discussion of adjacent fractions. So

$$|\mathfrak{z}'_{hk}|^2 \leq \frac{2k^2}{N^2}$$

or

$$|\mathfrak{z}'_{hk}| \leq \frac{\sqrt{2}\cdot k}{N};$$

also

$$|\mathfrak{z}''_{hk}| \leq \frac{\sqrt{2}\cdot k}{N}$$

So the inequality becomes 161

$$\begin{aligned} \left|\left(f(e^{2\pi i h'/k - 2\pi/\mathfrak{z}}) - 1\right)\Psi_k(\mathfrak{z})e^{2\pi n\mathfrak{z}/k^2}\right| &\leq \sqrt[4]{2}\cdot\frac{k^{1/2}}{N^{1/2}}\sum_{\nu=1}^{\infty} p(\nu)e^{(2\pi\nu - \pi/12)}e^{\pi(-\frac{1}{2}+2/n1)/k^2} \\ &\leq Ce^{2\pi|n|}\frac{k^{1/2}}{N^{1/2}} \end{aligned}$$

where C is independent of ν, since the series $\sum\limits_{\nu=1}^{\infty} p(\nu)e^{-(2\pi\nu - \pi/12)}$ is convergent. Since the length of the chord of integration $< 2\sqrt{2}\cdot k/(N+1)$m, we have

$$\left|I^*_{hk}\right| < C_1 e^{2\pi|n|}\frac{k^{3/2}}{N^{3/2}}$$

Then

$$\left|\sideset{}{'}\sum_{0\leq h<k\leq N} \frac{i\omega_{hk}}{k^{5/2}} e^{-2\pi nh/k} I^*_{hk}\right| \leq C_1 e^{2\pi|n|} \sideset{}{'}\sum_{0\leq h<k\leq N} \frac{1}{kN^{3/2}}$$

$$\leq C_1 e^{2\pi|n|} \sum_{0<k\leq N} \frac{1}{N^{3/2}},$$

Since the summation is over $h < k$ with $(h, k) = 1$, so that there are only $\varphi(k)$ terms and this is $\leq k$. So the last expression is 162

$$C_1 e^{2\pi|n|} N^{-1/2}$$

Hence

$$p(n) = \sum_{o\leq h<k\leq N} \frac{i\omega_{hk}}{k^{5/2}} e^{-2\pi inh/k} I_{hk} + R_N$$

where

$$|R_N| < C_1 e^{2\pi|n| N^{-1/2}}$$

Lecture 18

The formula that we had last time looked like this: 163

$$p(n) = \sum_{o \le h < k \le N}{}' i\omega_{hk} e^{-2\pi inh/k} k^{-5/2} I_{hk} + R_N,$$

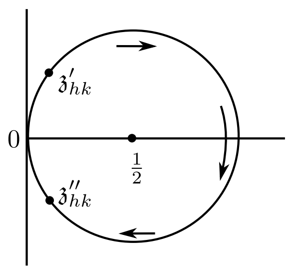
and it turned out that

$$|R_N| \le Ce^{2\pi|n|} N^{-1/2}$$

We had

$$I_{hk} = \int_{\mathfrak{z}'_{hk}}^{\mathfrak{z}''_{hk}} \Psi(\mathfrak{z}) e^{2\pi n\mathfrak{z}/k^2} d\mathfrak{z}$$

and the path of the integration was the arc from $\mathfrak{z}'_{hk}$ to $\mathfrak{z}''_{hk}$ in the sense indicated. And now what we do is this. We shall add the missing piece and take the integral over the full circle, how over excluding the origin. Now the path is taken in the negative sense, and we indicate this by writing



$$\int_{k^{(-)}} \Psi_k(\mathfrak{z}) e^{2\pi n\mathfrak{z}/k^2} d\mathfrak{z}.$$

This is an improper integral with both ends going to zero. That it exists is clear, for what do we have to compensate for that? we have to subtract $\int_0^{\mathfrak{z}'_{hk}} \cdots$

and $\int\limits_{\mathfrak{z}''_{hk}}^{0} \cdots$, and we prove that these indeed contribute very little. What is after 164
all $\Psi_k(\mathfrak{z})$?

$$\Psi_k(\mathfrak{z}) = \sqrt{\mathfrak{z}}\, e^{\frac{\pi}{12}\left(\frac{1}{\mathfrak{z}} - \frac{\mathfrak{z}}{k^2}\right)}$$

$0 < \mathscr{R}\mathfrak{z} \leq 1$ and $\mathscr{R}1/\mathfrak{z} = 1$ on the circle, so that

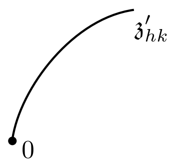
$$|\Psi_k(\mathfrak{z})| \leq |\sqrt{\mathfrak{z}}\, e^{\pi/12}|$$

and

$$|e^{2\pi n\mathfrak{z}/k^2}| \leq e^{2\pi|n|};$$

so that the integrand is bounded. Hence the limit exists. This is indeed very astonishing, for Ψ has an essential singularity at the origin; but on the circle it does not do any harm. Near the origin there are value which are as big as we want, but we can approach the origin in a suitable way. This is the advantage of this contour. The earlier treatment was very complicated.

We can now estimate the integrals. Since $|\mathfrak{z}'_{hk}| \leq \sqrt{2} \cdot k/N$, the chord can be a little longer, in fact $\dfrac{\pi}{2}$ times the chord at most. So



$$\left|\int\limits_{0}^{\mathfrak{z}'_{hk}}\right| \leq \sqrt{2} \cdot \frac{\pi}{2}\frac{k}{N} e^{2\pi|n|} \left(\frac{\sqrt{\mathfrak{z}} \cdot k}{N}\right)^{\frac{1}{2}} e^{\pi/12}$$
$$\leq C e^{2\pi|n|} k^{3/2} N^{-3/2}.$$

The same estimate holds good for $\int\limits_{\mathfrak{z}''_{hk}}^{0} \cdots$. Hence introducing.

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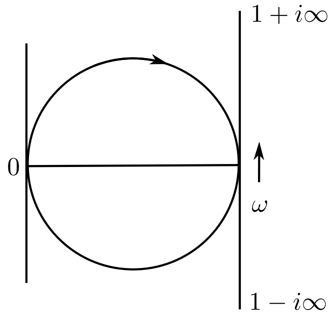
Now everything is under our control. N appeared previously tacitly in $\mathfrak{z}'_{hk}$, 166
because $\mathfrak{z}'_{hk}$ depends on the Farey arc. Now N appears in only two places. So $p(n)$ is the limit of the sum which we write symbolically as

$$p(n) = i \sum_{k=1}^{\infty} A_k(n) k^{-5/2} \int\limits_{K^{(-)}} \Psi_k(\mathfrak{z}) e^{2\pi n\mathfrak{z}/k^2} d\mathfrak{z}$$

($n \gtreqless 0$, integral, and $p(n) = 0$ for $n < 0$). So we have an exact infinite series for $p(n)$.

A thing of lesser significance is to determine the sum of this series. So we have to speak about the integral. Let us take one more step. Let us get away from the circle. Replace z by $\frac{1}{w}$. We do know what this will mean. w will now run on a line parallel to the imaginary axis, from $1 - i\infty$ to $1 + i\infty$. So



$$p(n) = -\sum_{k=1}^{\infty} A_k(n) k^{-5/2} \int\limits_{1-i\infty}^{1+i\infty} \omega^{-1/2} e^{\frac{\pi}{12}(\omega - 1/k^2\omega)} e^{\frac{2\pi n}{k^2\omega}} \cdot \frac{d\omega}{\omega^2}$$

$$= \frac{1}{i} \sum_{k=1}^{\infty} A_k(n) k^{-5/12} \int\limits_{1-i\infty}^{1+i\infty} \omega^{-5/2} e^{\frac{\pi}{2} + \frac{\pi}{12k^2\omega}(24n-1)} d\omega$$

One more step is advisable to get a little closer to the customary notation. **167**
We then get traditional integrals known in literature. Put $\dfrac{\pi\omega}{12} = s$,

$$p(n) = \frac{1}{i}\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} A_k(n) k^{-5/2} \int\limits_{\frac{\pi}{12}-i\infty}^{\frac{\pi}{12}+i\infty} s^{-5/2} e^{s + \frac{\pi^2}{12k^2 s}(24n-1)} ds$$

One could look up Watson's 'Bessel Functions' and write down this integral as a Bessel function. But since we need the series anyway we prefer to compute it directly. So we have to investigate an integral of the type

$$L(\nu) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} s^{-\nu-1} e^{s + \frac{\nu}{s}} ds$$

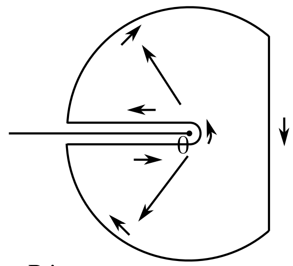
It does not matter what $c > 0$ is because it means only a shift to a parallel line, and the integrand goes to zero for large imaginary part of s. For absolute convergence it is enough to have a little more than s^{-1}. So take $\mathscr{R}\nu > 0$; in our case $\nu = 3/2$.

So let us study the integral

$$L_\nu(v) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} s^{-\nu-1} e^{s+\frac{v}{s}} ds$$

leaving it to the future what to do with v. The integration is along a 168

line parallel to the imaginary axis. We now bend the path into a loop as in the figure and push the contour out, so that along the arcs we get negligible contributions.



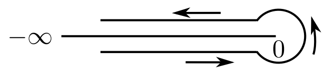
The contribution from the arc $|s| = R$ is

$$O\left(\frac{1}{R^{\nu+1}\cdot R}\right)$$

since $|e^{s+\frac{v}{s}}| \leq e^c e^{\mathscr{R}v/R}$, for a fixed v; this is $O(R^{-\nu}) \to 0$ as $R \to \infty$, since $\nu > 0$. So the integral along the ordinate becomes a 'loop integral', starting from $-\infty$ along the lower bank of the real axis, looping around the origin and proceeding along the upper bank towards $-\infty$; the loop integral is written in a fashion made popular by Watson as

$$\frac{1}{2\pi i} \int\limits_{-\infty}^{(0+)} s^{-\nu-1} e^{s+\frac{v}{s}} ds$$

For better understanding we take a specific loop. On the lower bank of the negative real axis we proceed only up to $-\epsilon$, 169



then go round a circle of radius ϵ in the positive sense and proceed thence along the upper bank, the integrand now having acquired a new value-unless ν is an integer. This we take as a standardised loop. We now prove that $L_\nu(\mathscr{V})$ is actually differentiable and that the derivative can be obtained by differentiating under the integral sign. For this we take $\{L_\nu(\mathscr{V}+h)-L_\nu(\mathscr{V})\}/h$ and compare it with what we could foresee to be $L'_\nu(\mathscr{V})$ and show that the difference goes to zero as $h \to 0$.

$$\frac{L_\nu(\mathscr{V}+h)-L_\nu(\mathscr{V})}{h} - \frac{1}{2\pi i}\int_{-\infty}^{(0+)} s^{-\nu-2} e^{s+\frac{\nu}{s}} ds$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{(0+)} s^{-\nu-1} e^{s}\left\{\frac{e^{\frac{\nu+h}{s}}-e^{\frac{\nu}{s}}}{h} - \frac{e^{\frac{\nu}{s}}}{s}\right\} ds$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{(0+)} s^{-\nu-1} e^{s+\frac{\nu}{s}}\left\{\frac{e^{\frac{h}{s}}-1}{h} - \frac{1}{s}\right\} ds$$

Now 170

$$\frac{e^{\frac{h}{s}}-1}{h} - \frac{1}{s} = \frac{\frac{h}{s}+\frac{h^2}{s^2,s!}+\cdots}{h} - \frac{1}{s}$$
$$= h\left\{\frac{1}{s^2\cdot 2!} + \frac{h}{s^3\cdot 3!} + \cdots\right\}$$

On the path of integration, $|s| \geq \epsilon > 0$; so

$$\left|\frac{e^{\frac{h}{s}-1}}{h} - \frac{1}{s}\right| \leq C|h|,$$

since we are having a quickly converging power series.

$$\therefore \quad \left|\frac{L_\nu(\nu+h)-L_\nu(\nu)}{h} - \frac{1}{2\pi i}\int_{-\infty}^{(0+)} s^{-\nu-2} e^{s+\frac{\nu}{s}} ds\right|$$
$$\leq C|h|\left\{2\int_{\epsilon}^{\infty} \frac{1}{x^{\nu+1}} e^{-x} e^{\frac{\mathscr{R}\nu}{\epsilon}} dx + 2\pi\epsilon\cdot\frac{1}{e^{\nu+1}} e^{\frac{\mathscr{R}\nu}{\epsilon}}\right\} = 0(h)$$

So the limit $\lim\limits_{h\to 0} \frac{L_\nu(\nu+h)-L_\nu(\nu)}{h}$ exists and $L_\nu(\nu)$ is differentiated uniformly in a circle of any size. Since the differential integral is of the same shape we can differentiate under the integral as often as we please.

Lecture 19

The formula for $p(n)$ looked like this: 171

$$p(n) = \frac{1}{i}\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty} A_k(n)k^{-5/2}\int\limits_{-\infty}^{(0+)} s^{-5/2}e^{s+\frac{1}{s}(\frac{\pi}{12k})^2(24n-1)}ds$$

We discussed the loop integral

$$L_\nu(\nu) = \frac{1}{2\pi i}\int\limits_{-\infty}^{(0+)} s^{-\nu-1}e^{s+\frac{\nu}{s}}ds, \mathscr{R}\nu > 0.$$

We can differentiate under the integral sign and obtain

$$L'_\nu(\nu) = \frac{1}{2\pi i}\int\limits_{-\infty}^{(0+)} s^{-\nu-2}e^{s+\frac{\nu}{s}}ds = L_{\nu+1}(\nu)$$

This integral is again of the same sort as before; so we can repeat differentiation under the integral sign. Clearly then $L_\nu(\nu)$ is an entire function of $\nu \cdot L_\nu(\nu)$ has the expansion in a Taylor series:

$$L_\nu(\nu) = \sum_{r=0}^{\infty}\frac{L_\nu^{(r)}(0)}{r!}\nu^r$$

$L_\nu^{(r)}(\nu)$ can be foreseen and is clearly

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{(0+)} s^{-\nu-1-r}e^{s+\frac{\nu}{s}}ds$$

So 172

$$L_\nu(\nu) = \sum_{r=0}^{\infty} \frac{\nu^r}{r!} \frac{1}{2\pi i} \int_{-\infty}^{(0+)} s^{-\nu-1-r} e^s ds$$

We now utilise a famous formula for the Γ-function - Hankel's formula, viz,

$$\frac{1}{\Gamma(\mu)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} s^{-\mu} e^s ds.$$

This is proved by means of the formula $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin \pi s}$ and the Euler integral. Using the Hankel formula we get L explicitly:

$$L_\nu(\nu) = \sum_{r=0}^{\infty} \frac{\nu^r}{r!\Gamma(\nu + r + 1)}$$

What we have obtained is something which we could have guessed earlier. Expanding $e^{\nu/s}$ as a power series, we have

$$\begin{aligned} L_\nu(\nu) &= \frac{1}{2\pi i} \int_{-\infty}^{(0+)} s^{-\nu-1} e^s \sum_{r=0}^{\infty} \frac{(\nu/s)^r}{r!} ds \\ &= \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \sum_{r=0}^{\infty} \frac{\nu^r}{r!} e^s s^{-\nu-1-r} ds, \end{aligned}$$

and what we have proved therefore is that we can interchange the integration and summation. We have

$$L'_\nu(\nu) = L_{\nu+1}(\nu).$$

Having this under control we can put it back into our formula and get a final statement about $p(n)$.

$$p(n) = 2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} A_k(n) k^{-5/2} L_{3/2} \left(\left(\frac{\pi}{12k}\right)^2 (24n - 1) \right)$$

This is not yet the classical formula of Hardy and Ramanujan. One trick 173
one adopts is to replace the index. Remembering that $L'_\nu(\nu) = L_{\nu+1}(\nu)$, we have

$$L_{3/2} \left(\left(\frac{\pi}{12k}\right)^2 (24n - 1) \right) = L'_{1/2} \left(\left(\frac{\pi}{12k}\right)^2 (24n - 1) \right)$$

$$= \frac{6k^2}{\pi^2}\frac{d}{dn}L_{1/2}\left(\left(\frac{\pi}{12k}\right)^2(24n-1)\right)$$

Let us write the formula for further preparation closer to the Hardy Ramanujan notation:

$$p(n) = \left(\frac{\pi}{12}\right)^{1/2}\sum_{k=1}^{\infty}A_k(n)k^{-1/2}\frac{d}{dn}L_{1/2}\left(\left(\frac{\pi}{12k}\right)^2(24n-1)\right)$$

Now it turns out that the L-functions for the subscript $\frac{1}{2}$ are elementary functions. We introduce the classical Bessel function

$$\mathcal{J}_\nu(\mathfrak{z}) = \sum_{r=0}^{\infty}\frac{(-)^r(\mathfrak{z}/2)^{2r+\nu}}{r!\Gamma(\nu+r+1)}$$

and the hyperbolic Bessel function (or the 'Bessel function with imaginary argument')

$$\mathcal{I}_\nu(\mathfrak{z}) = \sum_{r=0}^{\infty}\frac{(\mathfrak{z}/2)^{2r+\nu}}{r!\Gamma(\nu+r+1)}$$

How do they belong together? We have **174**

$$L_\nu\left(\frac{\mathfrak{z}^2}{4}\right) = \mathcal{I}_\nu(\mathfrak{z})\left(\frac{\mathfrak{z}}{2}\right)^{-\nu},$$
$$L_\nu\left(-\frac{\mathfrak{z}^2}{4}\right) = \mathcal{J}_\nu(\mathfrak{z})\left(\frac{\mathfrak{z}}{2}\right)^{-\nu},$$

connecting our function with the classical functions. In our case therefore we could write in particular

$$L_{1/2}\left(\left(\frac{\pi}{12k}\right)^2(24n-1)\right) = \mathcal{I}_{1/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right)\left(\frac{\pi}{12k}\sqrt{24n-1}\right)^{-1/2}$$

This is always good, but we would come into trouble if we have $24n-1 \leq 0$. It is better to make a case distinction; the above holds for $n \geq 1$, and for $n \leq 0$, $n = -m$, we have

$$L_{1/2}\left(\left(\frac{\pi}{12k}\right)^2(24n-1)\right) = L_{1/2}\left(-\left(\frac{\pi}{12k}\right)^2(24m+1)\right)$$
$$= J_{1/2}\left(\frac{\pi}{6k}\sqrt{24m+1}\right)\left(\frac{\pi}{12k}\sqrt{24m+1}\right)^{-1/2}$$

So we have: $n \geq 1$.

$$p(n) = \left(\frac{\pi}{12}\right)^{1/2} \sum_{k=1}^{\infty} A_k(n) k^{-1/2} \frac{d}{dn} \left(\frac{\mathcal{I}_{1/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right)}{\left(\frac{\pi}{12k}\sqrt{24n-1}\right)^{1/2}} \right)$$

$n = -m \leq 0$ **175**

$$p(n) = p(-m) = -\left(\frac{\pi}{12}\right)^{1/2} \sum_{k=1}^{\infty} A_k(-m) k^{-1/2} \cdot \frac{d}{dm} \left(\frac{J_{1/2}\left(\frac{\pi}{6k}\sqrt{24m+1}\right)}{\left(\frac{\pi}{12k}\sqrt{24m+1}\right)^{1/2}} \right)$$

We are not yet quite satisfied. It is interesting to note that the last expression is 1 for $n = 0$ and 0 for $n < 0$. We shall pursue this later.

We have now more or less standardised functions. We can even look up tables and compute the Bessel function. However $\mathcal{I}_{1/2}$ and $J_{1/2}$ are more elementary functions.

$$\begin{aligned} J_{1/2}(\mathfrak{z}) &= \sum_{r=0}^{\infty} \frac{(-)^r (\mathfrak{z}/2)^{2r+1/2}}{r!\Gamma(r+\frac{3}{2})} \\ &= \sum_{r=0}^{\infty} \frac{(-)^r (\mathfrak{z}/2)^{2r+\frac{1}{2}}}{r!\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \\ &= \left(\frac{2}{\pi\mathfrak{z}}\right)^{1/2} \sum_{r=0}^{\infty} \frac{(-)^r \mathfrak{z}^{2r+1}}{(2r+1)!} \\ &= \left(\frac{2}{\pi\mathfrak{z}}\right)^{1/2} \sin\mathfrak{z}. \end{aligned}$$

Similarly if we has abolished $(-)^r$ we should have **176**

$$I_{\frac{1}{2}}(\mathfrak{z}) = \left(\frac{2}{\pi\mathfrak{z}}\right)^{1/2} \sinh\mathfrak{z}$$

$$\frac{I_{1/2}(\mathfrak{z})}{(\mathfrak{z}/2)^{1/2}} = I_{1/2}(\mathfrak{z})\left(\frac{2}{\mathfrak{z}}\right)^{1/2} = \frac{2}{\sqrt{2}} = \frac{\sinh\mathfrak{z}}{\mathfrak{z}}$$

$$\frac{J_{1/2}(\mathfrak{z})}{(\mathfrak{z}/2)^{1/2}} = \frac{2}{\sqrt{\pi}}\frac{\sin\mathfrak{z}}{\mathfrak{z}}$$

We are now at the final step in the deduction of our formula:

$n \geq 1$

$$p(n) = \frac{1}{\sqrt{3}} \sum_{k=1}^{\infty} A_k(n) k^{-1/2} \frac{d}{dn} \left(\frac{\sinh \frac{\pi}{6k} \sqrt{24n-1}}{\frac{\pi}{6k} \sqrt{24n-1}} \right)$$

or with $\quad \frac{\pi}{6k}\sqrt{24n-1} = \frac{\pi}{k}\sqrt{\frac{2}{3}(n - \frac{1}{24})} = \frac{c}{k}\sqrt{n - \frac{1}{24}}, C = \pi\sqrt{\frac{2}{3}},$

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn} \left(\frac{\sinh \frac{c}{k} \sqrt{n - \frac{1}{24}}}{\sqrt{n - \frac{1}{24}}} \right)$$

$n = -m \leq 0$

$$p(n) = p(-m) = -\frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(-m) k^{\frac{1}{2}} \frac{d}{dm} \left(\frac{\sin \frac{c}{k} \sqrt{m + \frac{1}{24}}}{\sqrt{m + \frac{1}{24}}} \right)$$

This is the final shape of our formula -*a* convergent series for $p(n)$. **177**

The formula can be used for independent computation of $p(n)$. The terms become small. It is of interest to find what one gets if one breaks the series off, say at $k = N$

$$p(n) = \frac{\pi^{5/2}}{12\sqrt{3}} \sum_{k=1}^{N} \cdots + R_N$$

Let us appraise $R_N \cdot |A_k(n)| \leq k$, because there are only $\varphi(k)$ roots of unity. We want an estimate for $L_{3/2}$. For $n \geq 1$,

$$\begin{aligned} L_{3/2}\left(\left(\frac{\pi}{12k} \right)^2 (24n-1) \right) &\leq \sum_{r=0}^{\infty} \frac{\left(\frac{\pi^2}{6k^2} n \right)^r}{r!\Gamma\left(r + \frac{5}{2}\right)} \\ &\leq \sum_{r=0}^{\infty} \frac{\left(\frac{\pi^2}{6(N+1)^2} n \right)^r}{r!\Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2} \cdot \left(r + \frac{3}{2}\right)} \\ (\text{since} k > N \text{ in } R_N) &= \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(\frac{\pi^2}{6(N+1)^2} n \right)^r \cdot 2^{2r+1}}{(2r+1)!(r + \frac{3}{2})} \\ &\leq \frac{2}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(\frac{2\pi^2}{3(N+1)^2} n \right)^r}{(2r+1)!} \end{aligned}$$

$$\leq \frac{2}{3}\cdot\frac{2}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\left(\frac{2\pi^2}{3(N+1)^2}n\right)^r}{(2r)!}$$
$$< \frac{4}{3\sqrt{\pi}}e^{\frac{\pi}{N+1}\sqrt{\frac{3n}{3}}}$$

$$\therefore \qquad |R_N| \leq \frac{\pi^2}{9\sqrt{3}}e^{\frac{\pi}{N+1}\sqrt{\frac{2n}{3}}}\sum_{k=N+1}^{\infty}\frac{1}{k^{3/2}}$$
$$\leq \frac{\pi^2}{9\sqrt{3}}e^{\frac{\pi}{N+1}\sqrt{\frac{2n}{3}}}\int_N^{\infty}\frac{dk}{k^{3/2}}$$
$$\therefore \qquad |R_N| < \frac{2\pi^2}{9\sqrt{3}}e^{\frac{\pi}{N+1}\sqrt{2n}3}\frac{1}{N^{1/2}}$$

This tells us what we have in mind. Make N suitably large. Then one gets something of interest. Put $N = [\alpha\sqrt{n}]$, α constant. Then **178**

$$R_N = O(n^{-\frac{1}{4}})$$

And this is what Hardy and Ramanujan did. Their work still looks different. They did not have infinite series. They had replaced the hyperbolic sine by the most important part of it, the exponential. The series converges in our case since $\sinh x \sim x$ as $x \to 0$, so that $\sinh\left(\frac{c}{k}\sqrt{n-\frac{1}{24}}\right)$ behaves roughly like $\frac{c}{k}$. On differentiation we have $\frac{1}{k^2}$ so that along with $k^{1/2}$ in the numerator we get $k^{-3/2}$ and we have convergence. In the Hardy-Ramanujan paper they had

$$p(n) = \frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{[\sqrt{n}]}A_k(n)k^{1/2}\frac{d}{dn}\left(\frac{e^{\frac{c}{k}\sqrt{n-\frac{1}{24}}}}{\sqrt{n-\frac{1}{24}}}\right) + O(n^{-\frac{1}{4}}) + R_N^*$$

sinh was replaced by exp.; so they neglected **179**

$$R^* = \frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{[\sqrt{n}]}A_k(n)k^{1/2}\frac{d}{dn}\left(\frac{e^{-\frac{c}{k}\sqrt{n-\frac{1}{24}}}}{\sqrt{n-\frac{1}{24}}}\right)$$
$$|R^*| = O\left(\sum_{k=1}^{[\sqrt{n}]}k^{3/2}\left(\frac{e^{-\frac{c}{k}\sqrt{n-\frac{1}{24}}}}{n-\frac{1}{24}}\cdot\frac{c}{k} + \frac{e^{-\frac{c}{k}\sqrt{n-\frac{1}{24}}}}{(n-\frac{1}{24})}\right)\right)$$

The exponential is strongly negative if k is small; so it is best for $k = 1$. Hence

$$|R^*| = O\left(\frac{1}{n}\left(\sum_{k=1}^{[\sqrt{n}]} k^{1/2} + \frac{1}{\sqrt{n}}\sum_{k=1}^{[\sqrt{n}]} k^{3/2}\right)\right)$$

$$\sum_{k=1}^{N} k^{1/2} = O(N^{3/2})$$

$$\sum_{k=1}^{N} k^{3/2} = O(N^{5/2})$$

So

$$\begin{aligned}|R^*| &= O\left(\frac{1}{n}\left(n^{3/4} + \frac{1}{\sqrt{n}}n^{5/4}\right)\right)\\ &= O\left(n^{-\frac{1}{4}}\right)\end{aligned}$$

The constants in the O-term were not known at that time so that numerical computation was difficult. If the series was broken off at some other place the terms might have increased. Hardy and Ramanujan with good instinct broke off at the right place. **180**

We shall next resume our function-theoretic discussion and cast a glance at the generating function for $p(n)$ about which we know a good deal more now.

Lecture 20

We found a closed expression for $p(n)$; we shall now look back at the generat- 181
ing function and get some interesting results.

$$f(x) = \frac{1}{\prod_{n=1}^{\infty}(1-x^n)} = \sum_{n=0}^{\infty} p(n)x^n,$$

and we know $p(n)$. $p(n)$ in its simplest form before reduction to the traditional Bessel functions is given by

$$p(n) = 2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty} A_k(n)k^{-5/2}L_{3/2}\left(\left(\frac{\pi}{12k}\right)^2(24n-1)\right),$$

where
$$L_{3/2}\left(\frac{\pi^2}{6k^2}\left(n-\frac{1}{24}\right)\right) = \sum_{r=0}^{\infty}\frac{\frac{\pi}{6k^2}(n-\frac{1}{24})^r}{r!\Gamma\left(\frac{5}{2}+r\right)}$$

We wish first to give an appraisal of L and show that the series for $p(n)$ converges absolutely. The series is

$$f(x) = 2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{n=0}^{\infty} x^n \sum_{k=1}^{\infty} A_k(n)k^{-5/2}L_{3/2}\left(\frac{\pi^2}{6k^2}(n-\alpha)\right),$$

where we write $\frac{1}{24} = \alpha$ for abbreviation - it will be useful for some other purposes also to have a symbol there instead of a number.

We make only a crude estimate.

$$\left|L_{3/2}\left(\frac{\pi^2}{6k^2}(n-\alpha)\right)\right| \le \sum_{r=0}^{\infty}\frac{\left(\frac{\pi^2}{6}n\right)^r}{r!\Gamma\left(\frac{5}{2}+r\right)}$$

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{\pi^2}{6}n\right)^r}{r!\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\left(\frac{3}{2}+r\right)}$$

$$\frac{2^2}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(\frac{2\pi}{3}\pi n\right)^r}{(2r+1)!(3+2r)}$$

$$\leq 4 \sum_{r=0}^{\infty} \frac{(C\sqrt{n})^{2r}}{(2r)!}, C = \pi\sqrt{\frac{2}{3}},$$

$$\leq 4 \sum_{\rho=0}^{\infty} \frac{(C\sqrt{n})^{\rho}}{\rho!}$$

$$= 4e^{C\sqrt{n}}$$

So $f(x)$ is majorised by **182**

$$\text{constant } x \sum_{n=1}^{\infty} |x|^n e^{C\sqrt{n}} \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

and this is absolutely convergent for $|x| < 1$, indeed uniformly so for $|x| \leq 1-\delta$, $\delta > 0$, because $e^{C\sqrt{n}} = 0(e^{\delta n})$, $\delta > 0$, so that we need take $|xe^{\delta}| < 1$. We can therefore interchange the order of summation:

$$f(x) = 2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sum_{n=0}^{\infty} A_k(n) x^n L_{3/2}\left(\frac{\pi^2}{6k^2}(n-\alpha)\right)$$

$$= 2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sideset{}{'}\sum_{h \bmod k} \omega_{hk} \sum_{n=0}^{\infty} \left(xe^{-2\pi i\frac{h}{k}}\right)^n L_{3/2}\left(\frac{\pi^2}{6k^2}(n-\alpha)\right)$$

where the middle sum is a finite sum. This is all good for $|x| < 1$. Now call **183**

$$\Phi_k(\mathfrak{z}) = \sum_{n=0}^{\infty} L_{3/2}\left(\frac{\pi^2}{6k^2}(n-\alpha)\right)\mathfrak{z}^n$$

So in a condensed form $f(x)$ appears as

$$f(x) = 2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sideset{}{'}\sum_{h \bmod k} \omega_{hk}\Phi_k\left(xe^{-2\pi i\frac{h}{k}}\right)$$

We have now a completely new form for our function. It is of great interest to consider $\Phi_k(\mathfrak{z})$ for its own sake; it is a power series ($|\mathfrak{z}| < 1$) and the coefficients of $\mathfrak{z}^n$ are functions of $n-\alpha$.

$$L_{3/2}(\nu) = \sum_{r=0}^{\infty} \frac{\nu^r}{r!\Gamma\left(\frac{5}{2}+r\right)}$$

This is an entire function of ν, for the convergence is rapid enough in the whole plane. Looking into the Hadamard theory of entire functions, we could see that the order of this function is $\dfrac{1}{2}$. This is indeed plausible, for the denominator is roughly $(2r)!$ and $\sum \frac{\nu^r}{(2r)!} = \sum \frac{(\sqrt{\nu})^{2r}}{(2r)!} \sim e^{\sqrt{\nu}}$; or the function grows like $e^{\sqrt{\nu}}$, and this is characteristic of the growth of an entire function or order $\frac{1}{2}$. The coefficients of $\mathfrak{z}^n$ are themselves entire functions in the subscript n.

We now quote a theorem of Wigert to the following effect. Suppose that we have a power series $\Phi(\mathfrak{z}) = \sum_{n=0}^{\infty} g(n)\mathfrak{z}^n$ where $g(\nu)$ is in entire function of order **184** less than 1; then we can say something about $\Phi(\mathfrak{z})$ which has been defined so far for $|\mathfrak{z}| < 1$. This function can be continued analytically beyond the circle of convergence, and $\Phi(\mathfrak{z})$ has only $z = 1$ as a singularity; it will be an essential singularity in general, but a pole if $g(\nu)$ is a rational function. We can extract the proof of Wigert's theorem from our subsequent arguments; so we do not give a separate proof here.

$\Phi_k(\mathfrak{z})$ is a double series :

$$\Phi_k(\mathfrak{z}) = \sum_{n=0}^{\infty} \mathfrak{z}^n \sum_{r=0}^{\infty} \frac{\left(\frac{\pi^2}{6k^2}(n-d)\right)^r}{r!\Gamma\left(\frac{5}{4}+r\right)}, |\mathfrak{z}| < 1$$

This is absolutely convergent; so we can interchange summations and write

$$\Phi_k(\mathfrak{z}) = \sum_{r=0}^{\infty} \frac{\left(\frac{\pi}{k\sqrt{6}}\right)^{2r}}{r!\Gamma\left(\frac{5}{2}+r\right)} \sum_{n=0}^{\infty} (n-\alpha)^r \mathfrak{z}^n$$

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{\pi}{k\sqrt{6}}\right)^{2r}}{r!\Gamma\left(\frac{5}{2}+r\right)} \varphi_r(\mathfrak{z})$$

where $\varphi_r(\mathfrak{z})$ is the power series $\sum_{n=0}^{\infty}(n-\alpha)^r \mathfrak{z}^n$. Actually it turns out to be a rational function. $\Phi_k(\mathfrak{z})$ can be extended over the whole plane.

$$\varphi_r(\mathfrak{z}) = \sum_{n=0}^{\infty} \mathfrak{z}^n = \frac{1}{1-\mathfrak{z}}.$$

Differentiating $\varphi_r(\mathfrak{z})$, **185**

$$\varphi_r'(\mathfrak{z}) = \sum_{n=0}^{\infty} n(n-\alpha)^r \mathfrak{z}^{n-1},$$

$$\mathfrak{z}\varphi_r'(\mathfrak{z}) = \sum_{n=0}^{\infty} n(n-\alpha)^r \mathfrak{z}^{n},$$

$$\alpha\varphi_r(\mathfrak{z}) = \sum_{n=0}^{\infty} \alpha(n-\alpha)^r \mathfrak{z}^{n};$$

so,

$$\mathfrak{z}\varphi_r'(\mathfrak{z}) - \alpha\varphi_r(\mathfrak{z}) = \sum_{n=0}^{\infty}(n-\alpha)^{r+1}\mathfrak{z}^n = \varphi_{r+1}(\mathfrak{z})$$

This says that we con derive $\varphi_{r+1}(\mathfrak{z})$ from $\varphi_r(\mathfrak{z})$ by rational processes and differentiation. This will introduce no new pole; the old pole $z = 1$ (pole for $\varphi_\circ(\mathfrak{z})$) will be enhanced. So $\varphi_r(\mathfrak{z})$ is rational. Let us express the function a little more explicitly in terms of the new variable $u = \frac{1}{\mathfrak{z}-1}$ or $\frac{1}{u} + 1 = \mathfrak{z}$. Introduce $(-)^{r+1}\varphi_r(\mathfrak{z}) = (-)^{r+1}\varphi_r(1+u) = \psi_r(u)$, say he last equation which was a recursion formula now becomes

$$(-)^{r+2}\psi_{r+1}(u) = \left(\frac{1}{u}+1\right)(-)^r u^2\psi_r'(u) - \alpha(-)^{r+1}\psi_r(u)$$

because $$\psi_r'(u) = (-)^{r+1}\varphi_r'\left(1+\frac{1}{u}\right)\left(-\frac{1}{u^2}\right) = (-)^r\varphi_r'\left(1+\frac{1}{u}\right)\frac{1}{u^2}$$

$\therefore$ $$\psi_{r+1}(u) = u(u+1)\psi_r'(u) + \alpha\psi_r(u)$$

This is a simplified version of our recursion formula. We have a mind to expand about the singularity $z = 1$. Let us calculate the $\psi's$.

$$\begin{aligned}
\psi_0(u) &= u\\
\psi_1(u) &= u(u+1) + \alpha u = (1+\alpha)u + u^2\\
\psi_2(u) &= u(u+1)(2u+1+\alpha) + \alpha(1+\alpha)u + \alpha u^2\\
&= (1+\alpha)^2 u + (2\alpha+3)u^3 + 2u^3
\end{aligned}$$

$\psi_r(u)$ is a polynomial of degree $r+1$ without the constant term. The coefficients are a little complicated. If we make a few more trials we get by induction the following: **186**

Theorem.

$$\psi_r(u) = \sum_{j=0}^{r} \Delta^j (\alpha+1)^r u^{j+1},$$

where Δ^j is the j^{th} difference.

By definition,

$$\begin{aligned} \Delta f(x) &= f(x+1) - f(x), \\ \Delta^2 f(x) &= \Delta\Delta f(x) = \Delta f(x+1) - \Delta f(x) \\ &= f(x+2) - 2f(x+1) + f(x) \end{aligned}$$

The binomial coefficients appear, and

$$\Delta^k f(x) = \sum_{\ell=0}^{k} (-)^{k-\ell} \binom{k}{\ell} f(x+\ell)$$

How does the formula for ψ_r fit? For induction one has to make sure that 187
the start is good.

$$\begin{aligned} \psi_0(u) &= (\alpha+1)^\circ u = u \\ \psi_1(u) &= (\alpha+1)' u' + \Delta(\alpha+1)' u^2 = (\alpha+1)u + u^2 \\ \psi_2(u) &= (\alpha+1)^2 u' + \Delta(\alpha+1)^2 u^2 + \Delta^2(\alpha+1)^2 u^3 \\ &= (\alpha+1)^2 u + \left((\alpha+2)^2 - (\alpha+1)^2\right) u^2 + 2u^3 \\ &= (\alpha+1)^2 u + (2\alpha+3)u^2 + 2u^3 \end{aligned}$$

So the start is good. We assume the formula up to r.

$$\begin{aligned} \psi_{r+1}(u) &= \sum_{j=0}^{r} \left\{ (u^2+u)(j+1)\Delta^j(\alpha+1)^r u^j + \alpha\Delta^j(\alpha+1)^r u^{j+1} \right\} \\ &= \sum_{j=0}^{r+1} \left\{ j\Delta^{j-1}(\alpha+1)^r u^{j+1} + (j+1+\alpha)\Delta^j(\alpha+1)^r u^{j+1} \right\} \end{aligned}$$

(A Seemingly negative difference need not bother us because it is accompanied by the term $j = 0$).

$$= \sum_{j=0}^{r+1} u^{j+1} \left(j\Delta^{j-1}(\alpha+1)^r + (j+1+\alpha)\Delta^j(\alpha+1)^r \right)$$

To show that the last factor is $\Delta^j(\alpha+1)^{r+1}$, we need a side remark. Introduce a theorem corresponding to Leibnitz's theorem on the differentiation of a product. We have **188**

$$\begin{aligned}\Delta f(x)g(x) &= f(x+1)g(x+1) - f(x)g(x)\\ &= f(x+1)\Delta g(x) + f(x+1)g(x) - f(x)g(x)\\ &= f(x+1)\Delta g(x) + \Delta f(x)\cdot g(x)\end{aligned}$$

The general rule is

$$\Delta^k f(x)g(x) = \sum_{\ell=0}^{k}\binom{k}{\ell}\Delta^{k-\ell} f(x+\ell)\Delta^{\ell} g(x)$$

This is true for $k=1$. We prove it by induction,

$$\Delta^{k+1} f(x)g(x) = \Delta(\Delta^k f(x)g(x),$$

and since Δ is a linear process, this is equal to

$$\sum_{\ell=0}^{k}\binom{k}{\ell}\left\{\Delta^{k-\ell} f(x+\ell+1)\Delta^{\ell+1}g(x) + \Delta^{k+1-\ell} f(x+\ell)\Delta^{\ell} g(x)\right\},$$

which becomes, on rearranging summands,

$$\sum_{\ell=0}^{k+1}\Delta^{k+1-\ell} f(x+\ell)\Delta^{\ell} g(x)\left\{\binom{k}{\ell}+\binom{k}{\ell-1}\right\},$$

and the last factor is $\binom{k+1}{\ell}$, $\left(\binom{k}{-1}=\binom{k}{k+1}=0\right)$ This proves the rule.

Applying this to $(\alpha+1)^r$, **189**

$$(\alpha+1)^{r+1} = (\alpha+1)(\alpha+1)^r;\ \text{write } f=\alpha+1, g=(\alpha+1)^r,$$

and observe that f being linear permits only 0^{th} and 1^{st} differences;

$$\begin{aligned}\Delta^k(\alpha+1)^r &= \binom{k}{k-1}\Delta^{k-1}(\alpha+1)^r + \binom{k}{k}(\alpha+k+1)\Delta^k(\alpha+1)^r\\ &= k\Delta^{k-1}(\alpha+1)^r + (\alpha+k+1)\Delta^k(\alpha+1)^r\end{aligned}$$

$$\therefore \qquad \psi_r(u) = \sum_{j=0}^{r}\Delta^j(\alpha+1)^r u^{j+1}$$

We can now rewrite the $\varphi' s$:

$$\begin{aligned}\varphi_r(\mathfrak{z}) &= (-)^{r+1}\varphi_r(n)\\ &= (-)^{r+1}\sum_{j=0}^{r}\Delta^j(\alpha+1)^r\frac{1}{(\mathfrak{z}-1)^{j+1}}\end{aligned}$$

φ_r has now been defined in the whole plane.

Lecture 21

We have rewritten the generating function $f(x)$ as a sum consisting of certain **190**
functions which we called $\Phi_k(x)$:

$$f(x) = 2\pi\left(\frac{\pi}{12}\right)^{\frac{3}{2}} \sum_{k=1}^{\infty} k^{-5/2} \sum_{h \mod k}{}' \omega_{hk}\Phi_k\left(xe^{-2\pi i\frac{h}{k}}\right)$$

where
$$\Phi_k(\mathfrak{z}) = \sum_{n=0}^{\infty} L_{3/2}\left(\frac{\pi^2}{6k^2}(n-\alpha)\right)\mathfrak{z}^n,$$

with $\alpha = \frac{1}{24}$. $\Phi_k(\mathfrak{z})$ could also be written as

$$\Phi_k(\mathfrak{z}) = \sum_{r=0}^{\infty} \frac{\left(\frac{\pi}{k\sqrt{6}}\right)^{2r}}{r!\Gamma\left(\frac{5}{2}+r\right)}\varphi_r(\mathfrak{z}'')$$

where $\varphi_r(\mathfrak{z})$ is a rational function as we found out. We got φ explicitly by means of a certain ψ:

$$\varphi_r(\mathfrak{z}) = (-)^{r+1}\sum_{j=0}^{r} \Delta_\alpha^j(\alpha+1)^r \frac{1}{(\mathfrak{z}-1)^{j+1}}$$

What we need for questions of convergence is an estimate of φ_r; this is not difficult.

$$\Delta f(x) = f(x+1) - f(x) = f'(\xi_1),\ x < \xi_1 < x+1,$$

by the mean-value theorem; and because Δ is a finite linear process we can in-
terchange it with the operation of applying the mean value theorem and obtain **191**

$$\begin{aligned}\Delta^2 f(x) &= \Delta(\Delta f(x)) = \Delta f'(\xi_1) = f'(\xi_1+1) - f'(\xi_1)\\ &= f''(\xi_2),\ x < \xi_1 < \xi_2 < \xi_1 + 1 < x+2;\end{aligned}$$

$$\Delta^3 f(x) = \Delta(\Delta^2 f(x)) = \Delta^2 f'(\xi), x < \xi < x+1,$$
$$= f'''(\xi_3), x < \xi < \xi_3 < \xi + 2 < x + 3;$$

and in general

$$\Delta^k f(x) = f^k(\xi), x < \xi < x + k.$$

This was to be expected. Take $|1 - \mathfrak{z}| \geq \delta$, $0 < \delta < 1$ so that z is not too close to 1. $\dfrac{1}{\delta} > 1$ and $0 < \alpha < 1$

$$\begin{aligned}
|\varphi_r(\mathfrak{z}| &\leq \sum_{j=0}^{r} r(r-1)\cdots(r-j+1)(1+\alpha+j)^{r-j}\cdot\frac{1}{\delta^{j+1}}\\
&< \sum_{j=0}^{r}\frac{(\alpha+1+r)^r}{\delta^{j+1}}\\
&< (r+1)\frac{(\alpha+1+r)^r}{\delta^{r+1}}\\
&< \frac{(\alpha+1+r)^{r+1}}{\delta^{r+1}}
\end{aligned}$$

Originally we know that the formula for $f(x)$ was good for $|x| < 1$. From this point on we give a new meaning to $\varphi_r(\mathfrak{z})$ for all $\mathfrak{z} \neq 1$.

This is a new step. We prove that the series for $\Phi_k(\mathfrak{z})$ is convergent not merely for $|\mathfrak{z}| < 1$ but also elsewhere. The sum in $\Phi_k(\mathfrak{z})$ is majorised by 192

$$\frac{1}{\delta}\sum_{r=0}^{\infty}\frac{\left(\frac{\pi^2}{6}\right)^r}{r!\Gamma\left(\frac{5}{2}+r\right)}\cdot\frac{(\alpha+1+r)^{r+1}}{\delta^r}$$

This is convergent, for thought the numerator increases with r, we have by Stirling's formula

$$\frac{r^r}{r!} \sim \frac{r^r}{\sqrt{2\pi}r^{r+\frac{1}{2}}e^{-r}} = \frac{e^r}{\sqrt{2\pi r}}$$

So as far as convergence is concerned it is no worse than

$$\frac{1}{\delta}\sum_{r=1}^{\infty}\frac{\left(\frac{e\pi^2}{6\delta}\right)^r}{\Gamma\left(\frac{5}{2}+r\right)}(\alpha+1+r)\left(1+\frac{\alpha+1}{r}\right)^r$$

which is $\leq C_\delta$, the power series still being rapidly converging because of the factorial in the denominator and $\frac{e\pi^2}{6\delta}$ is fixed and $\left(1+\frac{\alpha+1}{r}\right)^r$ is bounded. So we have absolute convergence and indeed uniformly so for $|1 - \mathfrak{z}| \geq \delta$.

We have now a uniformly convergent series outside the point $z = 1$, and $\Phi_k(\mathfrak{z})$ is explained at every point except $z = 1$ which is an essential singularity. $\Phi_k(\mathfrak{z})$ is entire in $\dfrac{1}{1-\mathfrak{z}}$. From this moment if we put it back into our argument we have $f(x)$ in the whole plane if $xe^{-2\pi ih/k}$ keep away from 1. And we are sure **193**
of that; either $|x| \leq 1-\delta$ or $|x| \geq 1+\delta$. Originally x was only inside the unit circle; now it can be outside also. In both cases $f(x)$ is majorised by

$$\sum_{k=1}^{\infty} k^{-\frac{5}{2}}, k \cdot C_\delta = C_\delta \sum_{k=1}^{\infty} k^{-3/2},$$

which is absolutely convergent.

Therefore we have now a very peculiar situation. In this notation of Φ_k we have obtained a function which represents *two* analytic functions separated by a natural boundary which is full of singularities and cannot be crossed. They are not analytic continuations. The outer function is something new; it is analytic because the series is uniformly convergent in each compact subset.

Consider the circle. We state something more explicit which explains the behaviour at each point near the boundary. Since every convergence is absolute there are no difficulties and convergence prevails even if we take each piece separately.

$$f(x) = -2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-\frac{5}{2}} \sum_{h \bmod k}{}' \omega_{hk} \sum_{r=0}^{\infty} \frac{\left(-\frac{\pi^2}{6k}\right)^r}{r!\Gamma\left(\frac{5}{2}+r\right)} \sum_{j=0}^{\infty} \Delta_\alpha^j (\alpha+1)^r \cdot \frac{1}{(xe^{-2\pi ih/k}-1)^{j+1}}$$

We can now rearrange at leisure. **194**

$$f(x) = -2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-\frac{5}{2}} \sum_{h \bmod k}{}' \omega_{hk} \sum_{j=0}^{\infty} \frac{e^{-2\pi i \frac{h}{k}(j+1)}}{(x-e^{2\pi ih/k})^{j+1}} \sum_{r=j}^{\infty} \Delta_\alpha^j (\alpha+1)^r \frac{\left(-\frac{\pi^2}{6k^2}\right)^r}{r!\Gamma\left(\frac{5}{2}+r\right)}$$

However, if we replaced $\sum_{r=j}^{\infty}$ by $\sum_{r=0}^{\infty}$ it would not to any harm because the summation is applied to a polynomial of degree r and the order of the difference is one more than the power. We can therefore write, taking Δ outside,

$$f(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty}k^{-\frac{5}{2}}\sum_{h \bmod k}{}' \omega_{hk}$$
$$\sum_{j=0}^{\infty}\frac{e^{2\pi i\frac{h}{k}}(j+1)}{(x-e^{2\pi ih/k})^{j+1}}\Delta_{\alpha}^{j}\sum_{r=0}^{\infty}(\alpha+1)^r\frac{\left(-\frac{\pi^2}{6k^2}\right)^r}{r!\Gamma\left(\frac{5}{2}+r\right)}$$
$$= -2\pi\left(\frac{\pi}{12}\right)^{-5/2}\sum_{k=1}^{\infty}k^{\frac{5}{2}}\sum_{h \bmod k}{}' \omega_{hk}\sum_{\ell=1}^{\infty}\frac{e^{2\pi i\frac{h}{k}\ell}}{(x-e^{2\pi ih/k})^{\ell}}\Delta_{\alpha}^{\ell-1}L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right)$$

It is quite clear what has happened. x appears only in the denominator, a root of unity is subtracted and the difference raised to a power 1. Choose **195** specific h, x, 1; then we have a term $\dfrac{B}{(x-e^{2\pi ih/k})^{\ell}}$. We have a conglomerate of terms which look like this, a conglomerate of singularities at each root of unity. So we have a partial fraction decomposition not exactly of the Mittag-Leffler type. Here of course the singularities are not poles, and they are everywhere dense on the unit circle. Each series $\sum\limits_{\ell=1}^{\infty}$ represents one specific point $e^{2\pi ih/k}$.

Let us return to our previous statement. $f(x)$ is regular and analytic outside the unit circle. What form has it there? Inside it is $\prod\limits_{m=1}^{\infty}(1-x^m)$. We shall expand $f(x)$ about the point at infinity. We want the $\varphi's$ explicitly.

$$\varphi_0(\mathfrak{z}) = \frac{1}{1-\mathfrak{z}}$$
$$\varphi_{r+1}(\mathfrak{z}) = \mathfrak{z}\varphi_r'(\mathfrak{z}) - \alpha\varphi_r(\mathfrak{z})$$
$$\varphi_0(\mathfrak{z}) = \frac{\mathfrak{z}^{-1}}{\mathfrak{z}^{-1}-1} = -\frac{\mathfrak{z}^{-1}}{1-\mathfrak{z}^{-1}} = -\sum_{m=1}^{\infty}\mathfrak{z}^{-m}$$
$$\varphi_1(\mathfrak{z}) = \sum_{m=1}^{\infty}m\mathfrak{z}^{-m} + \alpha\sum_{m=1}^{\infty}\mathfrak{z}^{-m} = \sum_{m=1}^{\infty}(m+\alpha)\mathfrak{z}^{-m}$$

The following thing will clearly prevail

$$\varphi_r(\mathfrak{z}) = (-)^{r+1}\sum_{m=1}^{\infty}(m+\alpha)^r\mathfrak{z}^{-m}$$

This speaks for itself.

$$\varphi_{r+1}(\mathfrak{z}) = (-)^r\sum_{m=1}^{\infty}m(m+\alpha)\mathfrak{z}^{-m} + (-)^r\alpha\sum_{m=1}^{\infty}(m+\alpha)^r\mathfrak{z}^{-m}$$

So the general formula is justified by induction. 196

$$\Phi_k(\mathfrak{z}) = -\sum_{r=0}^{\infty} \frac{\left(-\frac{\pi^2}{6k^2}\right)^r}{r!\Gamma\left(\frac{5}{2}+r\right)} \sum_{m=1}^{\infty} (m+\alpha)^r \mathfrak{z}^{-m}$$

for all $|\mathfrak{z}| > 1$. Exchanging summations,

$$\begin{aligned}\Phi_k(\mathfrak{z}) &= -\sum_{m=1}^{\infty} \mathfrak{z}^{-m} \sum_{r=0}^{\infty} \frac{\left(\frac{\pi^2}{6k^2}(-m-\alpha)\right)^r}{r!\Gamma\left(\frac{5}{2}+r\right)} \\ &= -\sum_{m=1}^{\infty} \mathfrak{z}^{-m} L_{3/2}\left(\frac{\pi^2}{6k^2}(-m-\alpha)\right)\end{aligned}$$

Put this back into $f(x)$; we get for $|x| > 1$,

$$f(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-\frac{5}{2}} \sideset{}{'}\sum_{h \mod k} \omega_{hk} \sum_{m=1}^{\infty} \left(x^{-1} e^{2\pi i \frac{h}{k}}\right)^m L_{3/2}\left(\frac{\pi^2}{6k^2}(-m-\alpha)\right),$$

and since
$$A_k(n) = \sideset{}{'}\sum_{h \mod k} \omega_{hk} e^{-2\pi i h/k},$$

$$f(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sum_{m=1}^{\infty} A_k(-m) x^{-m} L_{3/2}\left(\frac{\pi^2}{6k^2}(-m-\alpha)\right)$$

Again interchanging summations, 197

$$f(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{m=1}^{\infty} x^{-m} \sum_{k=1}^{\infty} A_k(-m) k^{-5/2} L_{3/2}\left(\frac{\pi^2}{6k^2}(-m-\alpha)\right)$$

The inner sum we recognize immediately; it is exactly what we had for $p(n)$; so

$$f(x) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{m=1}^{\infty} p(-m) x^{-m}$$

And here is a surprise which could not be foreseen! By its very meaning $p(-m) = 0$. So

$$f(x) \equiv 0$$

outside the unit circle. This was first conjectured by myself and proved by H.Petersson by a completely different method. Such expressions occur in the theory of modular forms. Petersson got the outside function first and then the inner one, contrary to what we did.

The function is represented by a series inside the circle, and it is zero outside, with the circle being a natural boundary. There exist simpler examples of this type of behaviour. Consider the partial sums:

$$1+\frac{x}{1-x}=\frac{1}{1-x}$$

$$1+\frac{x}{1-x}+\frac{x^2}{(1-x)(1-x^2)}=\frac{1}{1-x}+\frac{x^2}{(1-x)(1-x^2)}=\frac{1}{(1-x)(1-x^2)}$$

$$1+\frac{x}{1-x}+\frac{x^2}{(1-x)(1-x^2)}+\frac{x^3}{(1-x)(1-x^2)(1-x^3)}+\cdots \text{to } n+1 \text{ terms}$$

$$=\frac{1}{(1-x)(1-x^2)\cdots(1-x^n)}$$

For $|x|<1$, the partial sum converges to $\dfrac{1}{\prod_{m=1}^{\infty}(1-x^m)}$. For $|x|>1$ also 198
it has a limit; the powers of x far outpace 1 and so the denominator tends to infinity and the limit is zero. The Euler series here is something just like our complicated function. Actually the two are the same. For suppose we take the partial sum $\dfrac{1}{(1-x)(1-x^2)\cdots(1-x^n)}$ and break it into partial fractions. We get the roots of unity in the denominator, so that we have a decomposition

$$\sum \frac{B_{h,k,l,n}}{\left(x-e^{2\pi i\frac{h}{k}}\right)^{\ell}}$$

$k\leq n$ and ℓ not too high. For a higher n we get a finer expression into partial fractions. Let us face one of these, keeping h, k, ℓ fixed:

$$\frac{B_{h,k,l,n}}{\left(x-e^{2\pi i\frac{h}{k}}\right)^{\ell}}$$

Let $n\to\infty$. Then I have the opinion that

$$B_{h,k,l,n}\to -2\pi\left(\frac{\pi}{12}\right)^{3/2}\omega_{hk}k^{-\frac{5}{2}}e^{2\pi i\frac{h}{k}\ell}\Delta_{\alpha}^{\ell-1}L_{3_k}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right)$$

The B's all appear from algebraic relations and so are algebraic numbers - in sufficiently high cyclotomic fields. And this is equal to something which looked highly transcendental! though we cannot vouch for this. The verification is difficult even in simple cases - and no finite number of experiments would prove the result.

$\dfrac{B_{0,1,1,n}}{x-1}$ is itself very complicated. Let us evaluate the principal formula for $f(x)$ and pick out the terms corresponding to $h = 0$, $k = l$, $\ell = 1$. 199

$L_{3/2}$ is just the sine function and terns out to be $-\dfrac{6}{25} - \dfrac{12\sqrt{3}}{75\pi}$. Since $\frac{1}{1-x} = -\frac{1}{x-1}$, -1 is the first approximation to $B_{0,1,1,n}$. If we take the partial fraction decomposition for

$$\frac{1}{(1-x)(1-x^2)}, \frac{1}{(1-x)(1-x^2)} = \frac{\cdot\cdot}{(x-1)^2} + \frac{\cdot\cdot}{(x-1)} + \frac{\cdot\cdot}{(1+x)},$$

the numerator of the second term would give the second approximation. If indeed these successive approximations converge to $B_{0,1,1,n}$ we could get a whole new approach to the theory of partitions. We could start with the Euler series and go to the partition function.

We are now more prepared to go into the structure of ω_{hk}. We shall study next time the arithmetical sum $A_k(n)$ and the discovery of A.Selberg. We shall then go back again to the η-function.

Lecture 22

We shall speak about the important sum $A_k(n)$ which appeared in the formula for $p(n)$, defined as 200

$$A_k(n) = \sum_{h \mod k}{}' \omega_{hk} e^{-2\pi inh/k}.$$

we need the explanation of the ω_{hk}; they appeared as factors in a transformation formula in the following way:

$$f\left(e^{2\pi i \frac{h+i\mathfrak{z}}{k}}\right) = \omega_{hk} \sqrt{\mathfrak{z}} e^{\frac{\pi}{12k}\left(\frac{1}{\mathfrak{z}} - \mathfrak{z}\right)} f\left(e^{2\pi i \frac{h'+i/\mathfrak{z}}{k}}\right),$$

$$hh' + 1 \equiv 0 \pmod{k}$$

Here, as we know,

$$f(x) = \frac{1}{\prod_{m=1}^{\infty}(1 - x^m)}$$

and as

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{m=1}^{\infty}(1 - e^{2\pi im\tau}),$$

$$f(e^{2\pi i\tau}) = e^{\pi i\tau/12}(\eta(\tau))^{-1}$$

We know how $\eta(\tau)$ because. ω_{hk} is something belonging to the behaviour of the modular form $\eta(\tau)$. What is ω_{hk} explicitly? We had a formula

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\sqrt{\frac{c\tau + d}{i}}\eta(\tau), c > 0,$$

and 201

epsilonn is just the question. Our procedure will be to study ϵ and η and then go back to f where ω_{hk} appeared. The trick in the discussion will be that we shall not use the product formula for $\eta(\tau)$, but the infinite series from the pentagonal numbers theorem. This was carried out at my suggestion by W.Fischer (Pacific Journal of Mathematics; vol. 1). However we shall not copy him. We shall make it shorter and dismiss for our purpose all the long and complicated discussions of Gaussian sums

$$G(h,k) = \sum_{\nu=1}^{k} e^{2\pi i\nu^2 h/k}$$

which are of great interest arithmetically, having to do with law of reciprocity to which we shall return later.

We are able to infer that a formula of the sort quoted for η should exist from the discussion of $\mathscr{V}_1'(0/\tau)$. We had the formula (see hechire 14)

$$\mathscr{V}_1\left(0\Big/\frac{a\tau+b}{c\tau+d}\right) = \cdots$$

where the right side contains a doubtful root of unity, which we could discuss in some special cases, and by iteration in all cases. We shall use as further basis of our argument that such a formula has been established with the proviso $|\epsilon| = 1$. We then make a statement about ϵ and use it directly.

After all this long talk let us go to work. We had $\tau' = (h' + i/\mathfrak{z}/k)$, $\tau = (h + i\mathfrak{z})/k$. The question is how is τ' produced from τ? It was obtained by means of the substitution 202

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}$$

We can therefore get what we are after if we specify the formula by these particular values.

$$\eta\left(\frac{h'+i\mathfrak{z}}{k}\right) = \epsilon\sqrt{\mathfrak{z}}\,\eta\left(\frac{h+i\mathfrak{z}}{k}\right)$$

with the principal value for $\sqrt{z}$. We wish to determine ϵ defined by this. We shall expand both sides and compare the results. For expansion we do not use the infinite product but the pentagonal numbers formula.

$$\begin{aligned}\eta(\tau) &= e^{\pi i\tau/12} \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda} e^{2\pi i\tau\lambda(3\lambda-1)/2} \\ &= \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda} e^{\frac{\pi i\tau}{12}(1+36\lambda^2-12\lambda)}\end{aligned}$$

$$= \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda} e^{3\pi i\tau(\lambda-1/6)^2}$$

Most determinations of $\eta(\tau)$ make use of the infinite product formula; the infinite series is simpler here

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \sum_{\lambda=-\infty}^{\infty} (-)^{\lambda} e^{3\pi i \frac{h+i\mathfrak{z}}{k}(\lambda-1/6)^2}$$

In order to get the root of unity a little more clearly exhibited, we replace λ mod $2k$. **203**

$\lambda = 2kq + j,\ j = 0, 1, \ldots, 2k-1$ and q runs from $-\infty$ to ∞. So

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \sum_{q=-\infty}^{\infty} \sum_{j=0}^{2k-1} (-)^{j} e^{3\pi i \frac{h}{k}(2kq+j-\frac{1}{6})^2} e^{-3\pi \frac{\mathfrak{z}}{k}(2kq+j-\frac{1}{6})^2}$$

The product term in the exponent $= 4kq(j-\frac{1}{6}).3\pi i\frac{h}{k}$
$= 2\pi ihq(6j-1)$
$=$ an integral multiple of $2\pi i$

(This is the reason why we used mod $2k$).

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \sum_{j=0}^{2k-1} (-)^{j} e^{3\pi i \frac{h}{k}(j-\frac{1}{6})^2} \sum_{q=-\infty}^{\infty} e^{-12\pi\mathfrak{z}k(q+\frac{j-1/6}{2k})^2}.$$

We did this purposely in order to make it comparable to what we did in the theory of $\mathscr{V}$-functions. For $\mathscr{R}t > 0$, we have

$$\sum_{q=-\infty}^{\infty} e^{-\pi t(q+\alpha)^2} = \frac{1}{\sqrt{t}} \sum_{m=-\infty}^{\infty} e^{-\frac{\pi}{t}m^2} e^{2\pi i m\alpha}$$

This is a consequence of a $\mathscr{V}$-formula we had:

$$e^{\pi i\tau\nu^2} \mathscr{V}_3(\nu\tau/\tau) = \sqrt{\frac{1}{\tau}} \mathscr{V}_3\left(\nu/-\frac{1}{\tau}\right)$$

If we write this explicitly, **204**

$$\mathscr{V}_3(\nu/\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau n^2} e^{2\pi i n\nu},$$

and put $i\tau = -t$,

$$e^{-\pi t\nu^2}\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} e^{-2\pi n t\nu} = \frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} e^{2\pi i n\nu},$$

or

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(\nu+n)^2} = \frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{t}n^2} e^{2\pi i n\nu},$$

which is the formula quoted. We now apply this deep theorem and get something completely new. Putting $t = 12zk$ and $\alpha = \dfrac{j-1/6}{k}$,

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \sum_{j=0}^{2k-1}(-)^j e^{2\pi i\frac{h}{k}\left(j-\frac{1}{6}\right)^2}\frac{1}{\sqrt{12k\mathfrak{z}}}\sum_{m=-\infty}^{\infty} e^{-\frac{\pi m^2}{12\mathfrak{z}k}} e^{\frac{\pi i m}{k}\left(j-\frac{1}{6}\right)}$$

We rewrite this, emphasizing the variable and exchanging the orders of summation. Then

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \frac{1}{2\sqrt{3k\mathfrak{z}}}\sum_{m=-\infty}^{\infty} e^{-\frac{\pi m^2}{12k\mathfrak{z}}}\sum_{j=0}^{2k-1} e^{\pi i\left(j+\frac{2h}{k}\left(j-\frac{1}{6}\right)^2+\frac{m}{12k}(6j-1)\right)}$$

205

Let us use an abbreviation.

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \frac{1}{\sqrt{2k\mathfrak{z}}}\sum_{m=-\infty}^{\infty} e^{-\frac{\pi m^2}{12k\mathfrak{z}}} T_{(m)},$$

where

$$T_{(m)} = \frac{1}{2}\sum_{j=0}^{2K-1} e^{\pi i\left(j+\frac{2h}{k}(j-\frac{1}{6})^2+\frac{m}{12k}(6j-1)\right)}.$$

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \frac{1}{\sqrt{3k\mathfrak{z}}}\left\{T_{(0)} + \sum_{m=1}^{\infty} e^{-\frac{\pi m^2}{12k\mathfrak{z}}}(T(m)+T(-m))\right\}$$

This is a function in $\frac{1}{\mathfrak{z}}$. Also

$$\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \frac{\epsilon^{-1}}{\sqrt{\mathfrak{z}}}\sum_{\lambda=-\infty}^{\infty}(-)^\lambda e^{-\frac{\pi}{12k\mathfrak{z}}(6\lambda-1)^2} e^{\frac{\pi i h'}{12k}(6\lambda-1)^2}$$

Now $\eta\left(\frac{h+i\mathfrak{z}}{k}\right)$ has been obtained in two different ways. We have in both cases a power series in $e^{-\pi/(12\mathfrak{z}k)} = x$, both for $|x| < 1$. But an analytic function has only one power series; so they are identical. This teaches us something. The

second teaches us that by no means do all sequences appear in the exponent. **206**
Only $m^2 = (6\lambda - 1)^2$ can occur. There is no constant term in the second expression. So m has the form $|6\lambda - 1| = 6\lambda \pm 1$, $\lambda > 0$. Make the comparison; the coefficients are identical. They are almost always zero. In particular $T(0) = 0$. $T(m)$ for m other than $\pm 1 \pmod 6$ also vanish. So we have the following identification.

$$\frac{1}{\sqrt{3k}}\,(T(6\lambda - 1) + T(-6\lambda + 1)) = \epsilon^{-1}(-)^{\lambda} e^{\frac{\pi i h'}{12k}}(6\lambda - 1)^2$$

Realise that we have acknowledged here that a transformation formula exists. The root of unity ϵ is independent of λ. This we can assume but W. Ruscher does not. Take in particular $\lambda = 0$. Then we have for $m = \pm 1$,

$$\frac{1}{\sqrt{3k}}(T(-) + T(1)) = \epsilon^{-1} e^{\frac{\pi i h'}{12k}}$$

This is proved by Fischer by using Gaussian sums. Therefore

$$\epsilon^{-1} = \frac{e^{-\frac{\pi i h'}{12k}}}{\sqrt{3k}} \left\{ \sum_{j=0}^{2k-1} e^{\pi i (j + \frac{3h}{k}(j - \frac{i}{6})^2) - \frac{6j-1}{6k}} + \sum_{j=0}^{2k-1} e^{\pi i \left(j + \frac{3h}{k}(j - \frac{1}{6})^2 + \frac{6j-1}{6k}\right)} \right\}$$

Now j matters only $\mod 2k$. We can beautify things slightly:

$$\epsilon^{-1} = \frac{e^{-\frac{\pi i h'}{12k} + \frac{\pi i h}{12k}}}{2\sqrt{3k}} \left\{ e^{\frac{\pi i}{6k}} \sum_{j \mod 2k} e^{\frac{\pi i}{k}(3hj^2 + j(k-h-1))} + e^{-\frac{\pi i}{6k}} \sum_{j \mod 2k} e^{\frac{\pi i}{k}\left(3hj^2 + j(k-h-1)\right)} \right\}$$

The sum appears complicated but will collapse nicely; however complicated it should be a root of unity. In $A_k(n)$ the sums are summed over h and for that purpose we shall not need to compute the sums explicitly.

Lecture 23

Last time we obtained the formula 207

$$\epsilon^{-1} = \frac{1}{2\sqrt{3k}} e^{\frac{\pi i(h-h')}{12k}} e^{-\frac{\pi i}{6k}} \sum_{j \mod 2k} e^{\frac{\pi i}{k}\left(3hj_j^2(k-h+1)\right)} + \frac{1}{2\sqrt{3k}} e^{\frac{\pi i(h-h')}{12k}} e^{\frac{\pi i}{6k}} \sum_{j \mod 2k} e^{\frac{\pi i}{k}(3hj^2+j(k-h-1))}$$

$\omega_{h,k}$ was defined by means of the equation

$$f\left(e^{2\pi i\frac{h+i\mathfrak{z}}{k}}\right) = \omega_{hk}\sqrt{\mathfrak{z}}\, e^{\frac{\pi}{12k}\left(\frac{1}{\mathfrak{z}}-\mathfrak{z}\right)} f\left(e^{2\pi i\frac{h'+i/\mathfrak{z}}{k}}\right)$$

ω_{hk} came from the ϵ in the transformation formula

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon\sqrt{\frac{c\tau+d}{i}}\,\eta(\tau)$$

In particular,

$$\eta\left(\frac{h'+1/\mathfrak{z}}{k}\right) = \epsilon\sqrt{\mathfrak{z}}\,\eta\left(\frac{h+i\mathfrak{z}}{k}\right),$$

$$f\left(e^{2\pi i\tau}\right) = e^{\pi i\tau/12}(\eta(\tau))^{-1}$$

Substituting in the previous formula,

$$e^{\frac{\pi i}{12}\frac{h+i\mathfrak{z}}{k}}\left\{\eta\left(\frac{h+i\mathfrak{z}}{k}\right)\right\}^{-1} = \omega_{hk}\sqrt{\mathfrak{z}}\, e^{\frac{\pi}{12k}\left(\frac{1}{\mathfrak{z}}-\mathfrak{z}\right)} e^{\frac{\pi i}{12}\frac{h'+i\mathfrak{z}}{k}}\left\{\eta\left(\frac{h'+i/\mathfrak{z}}{k}\right)\right\}^{-1}$$

i.e.,

$$\eta\left(\frac{h'+i/\mathfrak{z}}{k}\right) = \omega_{hk}\sqrt{\mathfrak{z}}\, e^{\frac{\pi i}{12k}(h'-h)}\eta\left(\frac{h+i\mathfrak{z}}{k}\right)$$

$$\therefore \qquad \epsilon = \omega_{hk} e^{\frac{\pi i}{12k}(h'-h)}$$

$$\text{or} \qquad \omega_{hk} = \epsilon e^{-\frac{\pi}{12k}(h'-h)}$$

In the first formula we have obtained an expression for ϵ^{-1}. However, we could make a detour and act ϵ directly instead of ϵ^{-1}. Even otherwise this could be fixed up, for after all it is a root of unity. We have $\epsilon\bar{\epsilon} = 1$ or $\epsilon = \bar{\epsilon}^{-1}$. So consistently changing the sign in the exponents, we have **208**

$$\omega_{hk} = \bar{\epsilon}^{-1} e^{\frac{\pi i}{12k}(h-h')} = \frac{1}{2\sqrt{3k}} e^{\frac{\pi i}{6k}} \sum_{j \bmod 2k} e^{-\frac{\pi i}{k}(3hj^2+j(k-h+1))} + \frac{1}{2\sqrt{3k}} e^{-\frac{\pi i}{6k}} \sum_{j \bmod 2k} e^{-\frac{\pi i}{k}(3hj^2+j(k-h-1))}$$

We now have the ω_{hk} that we need. But the ω_{hk} are only of passing interest; we put them back into $A_k(n)$;

$$A_k(n) = \sideset{}{'}\sum_{h \bmod k} \omega_{hk} e^{-2\pi i n h/k}$$

This formula has one unpleasant feature, viz. $(h,k) = 1$. But this would not do any harm. We can use a lemma from an unpublished paper by Whiteman which status that if $(h,k) = d > 1$, then

$$\sum_{j \bmod 2k} e^{-\frac{\pi i}{k}(3hj^2+j(k-h\pm 1))} = 0$$

For proving Whiteman status put $h = dh^*$, $k = dk^*$ and $j = 2k^*l + r$, **209**
$0 \le 1 \le d-1$, $0 \le r \le 2k^* - 1$. Then

$$\sum_{j \bmod 2k} e^{-\frac{\pi i}{k}(3hj^2+j(k-h\pm 1))} = \sum_{\ell=0}^{d-1}\sum_{r=0}^{2k^*-1} e^{-\frac{\pi i}{dk^*}(3dh^*(2k^*1+r)^2+(2k^*\ell+r)(dk^*-dh^*\pm 1)))}$$
$$= \sum_{r=0}^{2k^*-1} e^{-\frac{\pi i}{k}(3hr^2+r(k-h\pm 1))} \sum_{\ell=0}^{d-1} e^{\mp 2\pi i \ell/d},$$

and the inner sum $= 0$ because it is a full sum of roots of unity and $d \neq 1$.

This simplifies the matter considerably. We can now write

$$A_k(n) = \frac{1}{2\sqrt{3k}} e^{\frac{\pi i}{6k}} \sum_{h \bmod k}\ \sum_{j \bmod 2k} e^{-\frac{\pi i}{k}(3hj^2+j(k-h+1))} e^{-2\pi i n\frac{h}{k}}$$

$$+\frac{1}{2\sqrt{3k}}e^{-\frac{\pi i}{6k}}\sum_{h \mod k}\ \sum_{j \mod 2k} e^{-\frac{\pi i}{k}(3hj^2+j(k-h-1))}e^{-2\pi in\frac{h}{k}}$$

Rearranging, this gives 210

$$A_k(n)=\frac{1}{2\sqrt{3k}}e^{\frac{\pi i}{6k}}\sum_{j \mod 2k} e^{-\frac{\pi i}{k}(k+1)j}\sum_{h \mod k} e^{-\frac{2\pi i}{k}\left(n+\frac{j(3j-1)}{k}\right)h}$$
$$+\frac{1}{2\sqrt{3k}}e^{-\frac{\pi i}{6k}}\sum_{j \mod 2k} e^{-\frac{\pi i}{k}(k-1)j}\sum_{h \mod k} e^{-\frac{2\pi i}{k}\left(n+\frac{j(3j-1)}{2}\right)h}$$

The inner sum is equal to the sum of the k^{th} roots of unity, which is 0 or k, k if all the summands are separately one, i.e., if

$$n+\frac{j(3j-1)}{2}\equiv 0 \pmod k$$

Hence

$$A_k(n)=\frac{1}{2}\sqrt{\frac{k}{3}}e^{\frac{\pi i}{6k}}\sum_{\substack{j \mod 2k\\ \frac{j(3j-1)}{2}\equiv -n \pmod k}}(-)^j e^{-\frac{\pi ij}{k}}+\frac{1}{2}\sqrt{\frac{k}{3}}e^{-\frac{\pi i}{6k}}\sum_{\substack{j \mod 2k\\ \frac{j(3j-1)}{2}\equiv -n \pmod k}}(-)^j e^{\frac{\pi ij}{k}}$$

In the summation here we first take all j's modulo $2k$ (this is the first sieving out), and then retain only those j which satisfy the second condition modulo k (this is the second sieving out). Combining the terms,

$$A_k(n)=\frac{1}{2}\sqrt{\frac{k}{3}}\sum_{\substack{j \mod 2k\\ \frac{j(3j-1)}{2}\equiv -n \pmod k}}(-)^j\left\{e^{-\frac{\pi i}{6k}(6j-1)}+e^{\frac{\pi i}{6k}(6j-1)}\right\}$$
$$=\sqrt{\frac{k}{3}}\sum_{\substack{j \mod 2k\\ \frac{j(3j-1)}{2}\equiv -n \pmod k}}(-)^j\cos\frac{\pi(6j-1)}{6k}$$

This formula is due to A.Selberg. It is remarkable how simple it is. We shall change it a little, so that it could be easily computed. We shall show that the $A_k(n)$ have a certain multiplicative property, so that they can be broken up into prime parts which can be computed separately. Let us rewrite the summation condition in the following way. 211

$$12j(3j-1)\equiv -24n \pmod{24k}$$

i.e.,
$$36j^2 - 12j + 1 \equiv 1 - 24n \pmod{24k}$$
i.e.,
$$(6j-1)^2 \equiv \nu \pmod{24k}$$

where we have written $\nu = 1 - 24n$. In the formula

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}} \sum_{\substack{j \mod 2k \\ \frac{j(3j-1)}{2} \equiv -n \pmod{k}}} (-)^j \left\{e^{-\frac{\pi i}{6k}(6j-1)} + e^{\frac{\pi i}{6k}(6j-1)}\right\}$$

replace j by $2k - j$ in the popint term (where j runs through a full system of residues, so does $2k - j$). Further, observe that we have now **212**

$$(12k - 6j - 1)^2 \equiv \pmod{24k}$$
i.e.,
$$(6j+1)^2 \equiv \pmod{24k}$$

Then

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}} \left\{ \sum_{\substack{j \mod 2k \\ (6j-1)^2 \equiv \nu \pmod{24k}}} (-)^j e^{-\frac{\pi i}{6k}(6j+1)} + \sum_{\substack{j \mod 2k \\ (6j-1)^2 \equiv \nu \pmod{24k}}} (-)^j e^{\frac{\pi i}{6k}(6j-1)} \right\}$$

In both terms the range of summation is $j \mod 2k$ and there is the further condition which restricts j. So

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}} \sum_{\substack{j \mod 2k \\ (6j\pm 1)^2 \equiv \nu \pmod{24k}}} (-)^j e^{-\frac{\pi i}{6k}(6j\pm 1)}$$

Write $6j \pm 1 = \ell$. $6j \pm 1$ thus modulo $24k$. $j = \frac{\ell+1}{6}$, so it is the integer nearest to $\frac{\ell}{6}$ since $(\ell, 6) = 1$. So write $j = \left\{\dfrac{\ell}{6}\right\}$ where $\{x\}$ denotes the integer nearest to x. Then

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{3}} \sum_{\substack{\ell \mod 2k \\ (\ell,6)=1, \ell^2 \equiv \nu \pmod{24k}}} (-)^{\left\{\frac{\ell}{6}\right\}} e^{\frac{\pi i \ell}{6k}}$$

And one final touch. The ranges for ℓ in the two conditions are modulo $12k$ **213**
and modulo $24k$. Make these ranges the same. Then

$$A_k(n) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\substack{\ell \mod 24k \\ \ell^2 \equiv \nu \pmod{24k}}} (-)^{\left\{\frac{\ell}{6}\right\}} e^{\frac{\pi i \ell}{6k}}$$

We prefer the formula in this form which is much handler. We shall utilise this to get the multiplicative property of $A_k(n)$.

Lecture 24

We derived Selberg's formula, and it looked in our transformation like this: **214**

$$A_k(n) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{l^2 \equiv \gamma \pmod{24k}} (-)^{\{\frac{\ell}{6}\}} e^{\frac{\pi i \ell}{6k}},$$

where $\nu = 1 - 24n$, or $\nu \equiv 1 \pmod{24}$. We write this $B_k(\nu)$; this is defined for $\nu \equiv 1 \pmod{24}$, and we had tacitly $(\ell, 6) = 1$. We make an important remark about the symbol $(-)^{\{\frac{\ell}{6}\}}$. This repeats itself for $\ell modulo 12$. The values are

$\ell =$	1	3	7	11
$(-)^{\{\frac{\ell}{6}\}} =$	1	-1	-1	1

But $(-)^{\{\frac{\ell}{6}\}}$ can be expressed in terms of the Legendre symbol:

$$(-)^{\{\frac{\ell}{6}\}} = \left(\frac{\ell}{3}\right)\left(\frac{-1}{\ell}\right)$$

when $(\ell, 6) = 1$. We can test this, noticing that $\left(\frac{-1}{\ell}\right) = (-1)^{\frac{\ell-1}{2}}$. Since 1, 7 are quadratic residues and 5, 11 quadratic non-residues modulo 3, we have for $\ell = 1, 5, 7, 11$, $(-)^{\{\frac{\ell}{6}\}} = 1, -1, -1, 1$ respectively; this agrees with the previous list. It is sometimes simpler to write $(-)^{\{\frac{\ell}{6}\}}$ in this way, though it is an afterthought. It shows the periodicity.

Let us repeat the formula: **215**

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\ell^2 \equiv \nu \pmod{24k}} \left(\frac{\ell}{3}\right)\left(\frac{-1}{\ell}\right) e^{\frac{\pi i \ell}{6k}}$$

This depends upon how k behaves with respect to 24. It has to be done separately for 2, 3, 4, 6. For this introduce $d = (24, k^3)$. We have

$$
\begin{aligned}
d = 1 \text{ if } & \quad (k, 24) = 1,\\
3 \text{ if } & \quad 3 \mid 4, k \text{ odd},\\
8 \text{ if } & \quad k \text{ is even and } 3 \nmid k\\
24 \text{ if } & \quad 6 \mid k.
\end{aligned}
$$

Let us introduce the complementary divisor $e, de = 24$. So $e = 24, 8, 3$ or 1. $(d, e) = 1$. Also $(c, k) = 1$.

All this is a preparation for our purpose. The congruence $\ell^2 \equiv \nu \pmod{24k}$ can be re-written separately as two congruences: $\ell^2 \equiv \nu \pmod{dk}$, $\ell^2 \equiv \nu \pmod{e}$.

The latter is always fulfilled if $(\ell, 6) = 1$. Now break the condition into two subcases. Let r be a solution of the congruence

$$(er)^2 \equiv \nu \pmod{dk};$$

then we can write $\ell = er + dkj$, where j runs modulo e and moreover $(j, e) = 1$. To different pairs modulo dk and e respectively belong different ℓ modulo $24k$. $B_k(\nu)$ can then be written as

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{(er)^2 \equiv \nu \pmod{dk}} \sum_{\substack{j \mod e\\ (j,e)=1}} \left(\frac{er + dkj}{3}\right)\left(\frac{-1}{er + dkj}\right) e^{\frac{\pi i}{6k}(er+dkj)}$$

Separating the summations, this gives 216

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{(er)^2 \equiv \nu \pmod{dk}} e^{\frac{\pi i \ell k}{6k}} S_k(r),$$

where

$$S_k(r) = \sum_{j \mod e}{}' \left(\frac{er + dhj}{3}\right)\left(\frac{-1}{er + dkj}\right) e^{\frac{\pi i d j}{6k}}$$

We compute this now in the four different cases implied in the possibilities $d = 1, 3, 8, 24$.

Case 1. $d = 1$, $e = 24$

$$
\begin{aligned}
S_k(r) &= \sum_{j \mod 24}{}' \left(\frac{kj}{3}\right)\left(\frac{-1}{kj}\right) e^{\frac{\pi i j}{6}}\\
&= \left(\frac{k}{3}\right)\left(\frac{-1}{k}\right) \sum_{j \mod 24}{}' \left(\frac{j}{3}\right)(-)^{\frac{j-1}{2}} e^{\frac{\pi i j}{6}}
\end{aligned}
$$

There are eight summands, but effectively only four, because they can be folded together.

$$S_k(r) = 2\left(\frac{k}{3}\right)\left(\frac{-1}{k}\right) \sum_{j \mod 12}{}' \left(\frac{j}{3}\right)(-)^{\frac{\pi-1}{2}} e^{\pi i j}$$
$$= 2\left(\frac{h}{3}\right)\left(\frac{-1}{k}\right)\left\{e^{\frac{\pi i}{6}} - e^{\frac{5\pi i}{6}} - e^{\frac{7\pi i}{6}} + e^{\frac{11\pi i}{6}}\right\}$$

(We replaced the nice symbol $(-)^{\{\frac{\ell}{6}\}}$ by the Legendre symbol because we did not know a factorisation law for the former. So we make use of one special character that we know).

$$S_k(r) = 4\left(\frac{k}{3}\right)\left(\frac{-1}{k}\right)\left(\cos\frac{\pi}{6} - \cos\frac{5\pi}{6}\right)$$
$$= 4\left(\frac{k}{3}\right)\left(\frac{-1}{k}\right)\sqrt{3}$$

and since $\left(\frac{k}{3}\right)\left(\frac{3}{k}\right) = (-)^{\frac{k-1}{2}\cdot 1} = \left(\frac{-1}{k}\right)$, this gives gives 217

$$S_k(r) = 4\sqrt{3}\left(\frac{3}{k}\right)$$

Case 2. $d = 3$, $e = 8$.

$$S_k(r) = \sum_{j \mod 8}{}' \left(\frac{8r}{3}\right)\left(\frac{-1}{3kj}\right) e^{\frac{\pi i j}{2}}$$
$$= \left(\frac{-r}{3}\right)\left(\frac{-1}{3k}\right) \sum_{j \mod 8}{}' \left(\frac{-1}{j}\right) e^{\frac{\pi i j}{2}}$$
$$= 2\left(\frac{r}{3}\right)\left(\frac{-1}{k}\right) \sum_{j \mod 4}{}' \left(\frac{-1}{j}\right) e^{\frac{\pi i j}{2}}$$
$$= 2\left(\frac{r}{3}\right)\left(\frac{-1}{k}\right)(i + i)$$
$$= 4i\left(\frac{r}{3}\right)\left(\frac{-1}{k}\right).$$

Case 3. $d = 8$, $e = 3$.

$$S_k(r) = \sum_{j \mod 3}{}' \left(\frac{8kj}{3}\right)\left(\frac{-1}{3r}\right) e^{\frac{4\pi i j}{3}}$$

$$
\begin{aligned}
&= \left(\frac{k}{3}\right)\left(\frac{-1}{r}\right)_j \sum_{\bmod 3}{}' \left(\frac{j}{3}\right) e^{\frac{4\pi i}{3}}\\
&= \left(\frac{k}{3}\right)\left(\frac{-1}{r}\right)\left(e^{\frac{4\pi i}{3}} - e^{\frac{8\pi i}{3}}\right)\\
&= -2i\left(\frac{k}{3}\right)\left(\frac{-1}{r}\right)\sin\frac{2\pi}{3}\\
&= \frac{1}{i}\sqrt{3}\left(\frac{k}{3}\right)\left(\frac{-1}{r}\right)
\end{aligned}
$$

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Case 4. $d = 24$, $e = 1$.

$$S_k(r) = \left(\frac{k}{3}\right)\left(\frac{-1}{r}\right) = \left(\frac{3}{r}\right)$$

Now utilise these; we get a handier definition for $A_k(n)$.

Case 1.

$$B_k(\nu) = \left(\frac{3}{k}\right)^{\sqrt{k}} \sum_{(24r)^2 \equiv \nu \pmod{k}} e^{\frac{4\pi i r}{k}}$$

Case 2.

$$B_k(\nu) = i\sqrt{\frac{k}{3}}\left(\frac{-1}{k}\right) \sum_{(8k)^2 \equiv \nu \pmod{3k}} \left(\frac{r}{3}\right) e^{\frac{4\pi i r}{3k}}$$

The i should not bother us because r and $-r$ are solutions together, so they combine to give a real number.

$$B_k(\nu) = -\sqrt{\frac{k}{3}}\left(\frac{-1}{k}\right) \sum_{(8r)^2 \equiv \nu \pmod{3k}} \left(\frac{r}{3}\right) \sin\frac{4\pi r}{3k}$$

Case 3.

$$
\begin{aligned}
B_k(\nu) &= \frac{1}{4i}\sqrt{k}\left(\frac{k}{3}\right) \sum_{(3k)^2 \equiv \nu \pmod{8k}} \left(\frac{-1}{r}\right) e^{\frac{\pi i r}{3k}}\\
&= \frac{1}{4}\sqrt{k}\left(\frac{k}{3}\right) \sum_{(3r)^2 \equiv \nu \pmod{8k}} \left(\frac{-1}{r}\right) \sin\frac{\pi r}{2k}
\end{aligned}
$$

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Case 4.

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{r^2 \equiv \nu \pmod{24k}} \left(\frac{3}{r}\right) e^{\frac{\pi i r}{6k}}$$

This is the same as the old definition.

This makes it possible to compute $A_k(n)$. We break k into prime factors and because of the multiplicative property which we shall prove, have to face only the task of computing for prime powers.

Lecture 25

We wish to utilise the formula for $B_k(\nu)$ that we had: **220**

$$A_k(n) = B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\ell^2 \equiv \nu \pmod{24k}} \left(\frac{\ell}{3}\right)\left(\frac{-1}{\ell}\right) e^{\frac{\pi i \ell}{6k}},$$

with $\nu = 1 - 24n$ (and so $\equiv 1$ modulo 24). Some cases were considerably simpler. Writing $d = (24, k^3)$, $de = 24$, we have four cases: $d = 1, 3, 8, 24$.

$d = 1$

$$B_k(\nu) = \left(\frac{3}{k}\right)\sqrt{k} \sum_{(24r)^2 \equiv \nu \pmod{k}} e^{4\pi i r/k}$$

$d = 3$

$$B_k(\nu) = 2i\left(\frac{-1}{k}\right)\sqrt{\frac{k}{3}} \sum_{(8r)^2 \equiv \nu \pmod{3k}} \left(\frac{r}{3}\right) e^{4\pi i r/3k}$$

$d = 8$

$$B_k(\nu) = \frac{1}{4i}\left(\frac{k}{3}\right)\sqrt{k} \sum_{(8r)^2 \equiv \nu \pmod{8k}} \left(\frac{-1}{r}\right) e^{\pi i r/2k}$$

$d = 24$

There is nothing new; we get the old formula back.

We wish first to anticipate what we shall use later and get $A_n(n)$ for prime powers which will be the ultimate elements. Again we have to discuss several cases.

First take $k = p^\lambda$, p a prime exceeding 3. Then, by case 1 above (since $(24, k^3) = 1$), **221**

$$B_k(\nu) = \left(\frac{3}{p}\right)^\lambda p^{\lambda/2} \sum_{(24r)^2 \equiv \nu \pmod{p^\lambda}} e^{4\pi i r/p^\lambda}$$

Look into the condition of summation. It is quite clear that this implies $(24r)^2 \equiv \nu \pmod{p}$ i.e., ν is a quadratic residue modulo p. Hence

$$B_p\lambda(\nu) = 0 \text{ if } \left(\frac{\nu}{p}\right) = -1. \tag{1}$$

On the other hand, if $x^2 \equiv \nu \pmod{p}$ is solvable, then $x^2 \equiv \nu \pmod{p^\lambda}$ is also solvable (we take for granted the structure of the cyclic residue group). $x^2 \equiv \nu \pmod{p^\lambda}$ has two solutions, and now we want only $x = 24r \pmod{p^\lambda}$. Let r be a solution, $-r$ is the other solution: $(24r)^2 \equiv \pmod{p^\lambda}$. Then

$$\begin{aligned} B_k(\nu) &= \left(\frac{3}{p}\right)^\lambda p^{\lambda/2}\left\{e^{4\pi ir/p^\lambda} + e^{-4\pi ir/p^\lambda}\right\} \\ &= 2\left(\frac{3}{p}\right)^\lambda p^{\lambda/2}\cos\frac{4\pi r}{p^\lambda} \end{aligned} \tag{2}$$

This is roughly of the order of $\sqrt{p^\lambda}$

Next, suppose that p/ν. This is a special case of p^λ/ν. Then $(24r)^2 \equiv 0 \pmod{p^\lambda}$, and the solutions are

$$\begin{gathered} r = p^{\left[\frac{\lambda+1}{2}\right]}\cdot j, \\ j = 0, 1, 2, \ldots, p^{\lambda-\left[\frac{\lambda+1}{2}\right]} - 1. \end{gathered}$$

when $\lambda = 1$, $\left[\frac{\lambda+1}{2}\right] = \lambda$ and we have only one summand. Hence **222**

$$B_k(\nu) = \left(\frac{3}{p}\right)p^{1/2} \tag{3}$$

Now let $\lambda > 1$. Then

$$B_k(\nu) = \left(\frac{3}{p}\right)^\lambda p^{\frac{\lambda}{2}} \sum_{j=1}^{p^{\lambda-\left[\frac{\lambda+1}{2}\right]}} e^{4\pi ij/p^{\left[\frac{\lambda+1}{2}\right]}}$$

This again involves two cases, λ even and λ odd. If λ is even, $\lambda = 2\mu$ and the sum becomes

$$\sum_{j=1}^{p^\mu} e^{4\pi ij/p^\mu}$$

and this is 0, being a full sum of roots of unity. Hence in this case

$$B_k(\nu) = 0 \tag{4}$$

Now let λ be odd: $\lambda = 2\mu + 1$.

$$r = p^{\mu+1} \cdot j, j = 0, 1, \ldots, p^{\mu} - 1.$$

Then the sum becomes

$$\sum_{j=1}^{p^{\mu}} e^{4\pi i j/p^{\mu}}$$

which is again zero; hence

$$B_k(\nu) = 0 \tag{5}$$

Now suppose that $p^{\mu} \mid \nu, \mu < \lambda$ and $p^{\lambda} \nmid \nu$. $r^2 \equiv \nu \pmod{p^{\lambda}} \nu = p^{\mu}\nu, p + \nu$, **223**
or $\nu^1 \equiv p^{\mu}\nu \pmod{p^{\lambda}}$ $\nu = p^{\mu}\nu_1$, $p \nmid \nu_1$; or $\nu^2 \equiv p^{\mu}\nu_1 \pmod{p^{\lambda}}$. If is odd, $\mu < \lambda$, then $p^{\mu} \mid \nu$; and again

$$B_k(\nu) = 0 \tag{6}$$

There remain the case in which μ is even, $\mu = 2\rho$. Then $r^2 \equiv p^{2\rho}\nu$, $\pmod{p^{\lambda}}$. Writing $r = p^{\rho} j$, $p^{2\ell} j^2 \equiv p^{2\rho}\nu_1 \pmod{p^{\lambda}}$, or $j^2 \equiv \nu_1 \pmod{p^{\lambda-2\rho}}$

If $\left(\frac{\nu_1}{p}\right) = -1$, then again

$$B_k(\nu) = 0 \tag{7}$$

However $\left(\frac{\nu_1}{p}\right) = 1$ implies $j^2 \equiv \nu_1 \pmod{p^{\lambda-2\rho}}$ has two solutions, j and $-j$. Then

$$r \equiv p^{\rho}\left(j + \ell p^{\lambda-2\rho}\right) \pmod{p^{\lambda}}$$

or τ $\qquad r \equiv p^{\rho} j + \ell p^{\lambda-\rho} \pmod{p^{\lambda}}$

where $\qquad \ell = 0, 1, \ldots, p^{\rho} - 1.$

Then the sum becomes

$$\sum_{\ell=0}^{p^{\rho}-1} e^{\frac{4\pi i}{p^{\lambda}}}(\pm p^{\rho} j + \ell p^{\lambda-\rho}) = e^{\pm\frac{4\pi i}{p^{\lambda-\rho}} j} \sum_{\ell=0}^{p^{\rho}-1} e^{\frac{4\pi i}{p^{\rho}}\ell}$$
$$= 0$$

Again

$$B_k(\nu) = 0 \tag{8}$$

We now take up the case $p = 3$. This corresponds to $p = 3$. If $k = p^{\lambda} = 3^{\lambda}$, **224**

$$B_3\lambda(\nu) = i(-)^{\lambda} 3^{\frac{\lambda-1}{2}} \sum_{(8r)^2 \equiv \pmod{3^{\lambda+1}}} \left(\frac{r}{3}\right) e^{4\pi i r/3^{\lambda+1}}$$

$\nu \equiv 1 \pmod{24}$ or $\nu \equiv 1 \pmod 3$. So $\left(\frac{\nu}{3}\right) = 1$. There are two solutions, r and $-r$ for the congruence $(8r)^2 \equiv \nu \pmod{3^{\lambda+1}}$. Since $\left(\frac{-r}{3}\right) = -\left(\frac{r}{3}\right)$,

$$\begin{aligned} B_{3\lambda}(\nu) &= i(-)^{\lambda}\left(\frac{r}{3}\right)3^{\frac{\lambda-1}{2}}\left(e^{\frac{4\pi ir}{3^{\lambda+1}}} - e^{-\frac{4\pi ir}{3^{\lambda+1}}}\right) \\ &= 2(-)^{\lambda+1}\left(\frac{r}{3}\right)3^{\frac{\lambda-1}{2}} \sin\frac{4\pi r}{3^{\lambda+1}} \end{aligned} \tag{9}$$

Finally, we take $p = 2$; then d is 8. Let $k = 2^{\lambda}$. Then

$$B_2\lambda(\nu) = \frac{1}{4i}(-)^{\lambda}2^{\lambda/2} \sum_{(3r)^2 \equiv \nu \pmod{2^{\lambda+3}}} \left(\frac{-1}{r}\right) e^{4\pi ir/2^{\lambda+1}}$$

$\nu \equiv 1 \pmod 8$ implies that $(3r^2) \equiv \nu \pmod 8$ has four solutions, and these solutions are inherited by the higher powers of the modulus. The solutions are $r \equiv 1, 3, 5, 7 \pmod 8$. In general the congruence $x^2 \equiv \nu \pmod{2^{\mu}}$, $\mu \geq 3$ has four solutions

$$\pm r + h2^{\mu-1}, h = 0, 1$$

Then 225

$$B_{2^{\lambda}}(\nu) = \frac{1}{4i}(-)^{\lambda}2^{\lambda/2}\left\{e^{4\pi ir/2^{\lambda+1}} - e^{-4\pi ir/2^{\lambda+1}} + e^{4\pi ir/2^{\lambda+1}} - e^{-4\pi ir/2^{\lambda+1}}\right\}\left(\frac{-1}{r}\right)$$

and since $\left(\frac{-1}{r}\right) = (-)^{\frac{r-1}{2}}$,

$$B_{2^{\lambda}}(\nu) = (-)^{\lambda}e^{\lambda/2}\left(\frac{-1}{r}\right)\sin\frac{4\pi r}{2^{\lambda+!}} \tag{10}$$

We have thus computed the fundamental cases explicitly.

Lecture 26

We had the formula for $B_k(\nu)$: 226

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{\ell^2 \equiv \nu \pmod{24k}} \left(\frac{\ell}{3}\right)\left(\frac{-1}{3}\right) e^{\pi i \ell/6k},$$

with $\nu \equiv 1 \pmod{24}$. Writing $d = (24, k^3)$, we had the following cases:

1) $d = 1$

$$B_k(\nu) = \left(\frac{3}{k}\right)\sqrt{k} \sum_{(24\pi)^2 \equiv \nu \pmod{k}} e^{4\pi i r/k}$$

2) $d = 3$

$$B_k(\nu) = i\left(\frac{-1}{k}\right)\sqrt{\frac{k}{3}} \sum_{(24r)^2 \equiv \nu \pmod{3k}} \left(\frac{-1}{r}\right) e^{\pi i r/2k}$$

3) $d = 8$

$$B_k(\nu) = \frac{1}{4i}\left(\frac{k}{3}\right)\sqrt{k} \sum_{(3r)^2 \equiv \nu \pmod{8k}} \left(\frac{-1}{r}\right) e^{\pi i r/2k}$$

4) $d = 24$. We do not get anything new.

Assume $k = k_1k_2$, $(k_1, k_2) = 1$. We desire to write $B_k(\nu_1)$. $B_{k_2}(\nu_2) = B_k(\nu)$, with a suitable ν to be found out from ν_1 and ν_2. It cannot be foreseen. It is a multiplication of a peculiar sort. Two cases arise.

(i) At least one of k_1, k_2 is prime to 24 and therefore to 6, say $(k_1, 6) = 1$. 227

(ii) None is prime to 6. But since $(k_1, k_2) = 1, 2/k_1, 3/k_1$. Under the circumstances prevailing these are the two mutually exclusive cases.

Case 1. *Utilise* $d = 1$.

$$B_{k_1}(\nu_1)\cdot B_{k_2}(\nu_2) = \left(\frac{3}{k_1}\right)\sqrt{k_1}\cdot\frac{1}{4}\sqrt{\frac{k_2}{3}}\sum_{(24r)^2\equiv\nu_1 \pmod{k_1}} e^{4\pi ir/k_1}$$
$$\cdot\sum_{\ell^2\equiv\nu_2 \pmod{24k_2}}\left(\frac{\ell}{3}\right)\left(\frac{-1}{\ell}\right)e^{\pi i\ell/6k_2}$$
$$= \frac{1}{4}\left(\frac{3}{k_1}\right)\sqrt{\frac{k_1k_2}{3}}\sum_{\substack{(24r)^2\equiv\nu_1 \pmod{k_1}\\ \ell^2\equiv\nu_2 \pmod{24k_2}}}\sum e^{\frac{\pi i}{6k_1k_2}(24k_2r+k_1l)}\left(\frac{l}{3}\right)\left(\frac{-1}{l}\right)$$

k_1 and $24k_2$ are coprime moduli. If r runs modulo k_1 and ℓ runs modulo $24k_2$, $24k_2r + k_1\ell$ would then run modulo $24k_1k_2$.

Write

$$24k_2r + k, \ell \equiv t \pmod{24k_1k_2}$$

Then

$$t^2 = (24k_2 + k_1\ell)^2 \equiv (24k_2r)^2 \pmod{k_1}$$
$$\equiv k_2^2\nu_1 \pmod{k_1}, \text{ since } (24r)^2 \equiv \nu_1 \pmod{k_1}$$

Similarly

$$t^2 \equiv (k_1\ell)^2 \pmod{24k_2}$$
$$\equiv k_1^2\nu_2 \pmod{24k_2}, \quad \text{since } \ell^2 \equiv \nu_2 \pmod{24k_2}.$$

So in order to get both conditions of summation, we need only choose 228
$t^2 \equiv \nu \pmod{24k_1k_2}$; and this can be done by the Chinese remainder theorem. So

$$B_{k_1}(\nu_1)B_{k_2}(\nu_2) = \frac{1}{4}\left(\frac{3}{k},\right)\sqrt{\frac{k}{3}}\sum_{t^2\equiv \pmod{24k,k_2}}\left(\frac{\ell}{3}\right)\left(\frac{-1}{\ell}\right)e^{\pi it/6k}$$

This already looks very much like the first formula though not quite. What we have in mind is to compare it with

$$B_k(\nu) = \frac{1}{4}\sqrt{\frac{k}{3}}\sum_{t^2\equiv\nu \pmod{24k}}\left(\frac{t}{3}\right)\left(\frac{-1}{t}\right)e^{\pi it/6k}$$

So find out

$$\left(\frac{t}{3}\right)\left(\frac{-1}{t}\right) = \left(\frac{24k_2r + k_1\ell}{3}\right)\left(\frac{-1}{24k_2r + k_1\ell}\right)$$

$$= \left(\frac{k_1\ell}{3}\right)\left(\frac{-1}{k_1\ell}\right)$$
$$= \left(\frac{k_1}{3}\right)\left(\frac{-1}{k_1}\right)\left(\frac{\ell}{3}\right)\left(\frac{-1}{\ell}\right)$$
$$= \left(\frac{3}{k_1}\right)\left(\frac{\ell}{3}\right)\left(\frac{-1}{\ell}\right),$$

by the reciprocity law. So the formulas agree: $B_{k_1}(\nu_1)B_{k_2}(\nu_2) = B_k(\nu)$; and we have settled the affair in this case by

Theorem 1. *If* $k_2^2\nu_1 \equiv \nu \pmod{k_1}$ *and* $k_1^2\nu_2 \equiv \nu \pmod{24k_2}$, $(k, 6) = 1$, *then* **229**

$$B_{k_1}(\nu_1)B_{k_2}(\nu_2) = B_{k_1k_2}(\nu)$$

Case 2. *This corresponds to* $d = d_1 = 8$ *and* $d = d_2 = 3$.

$$B_{k_1}(\nu_1)\cdot B_{k_2}(\nu_2) = \frac{1}{4}\left(\frac{k_1}{3}\right)\sqrt{k_1}\left(\frac{-1}{k_2}\right)\sqrt{\frac{k_2}{3}}$$
$$\sum_{(3r)^2\equiv\nu_1 \pmod{8k_1}}\left(\frac{-1}{r}\right)e^{\pi ir/2k_1}\sum_{(8r)^2\equiv\nu_2 \pmod{3k_2}} e^{\pi is/3k_2}.$$
$$= \frac{1}{4}\left(\frac{k_1}{3}\right)\left(\frac{-1}{k_2}\right)\sqrt{\frac{k_1k_2}{3}}$$
$$\sum_{\substack{(3r)^2\equiv\nu_1 \pmod{8k_1}\\ (8r)^2\equiv\nu_2 \pmod{3k_2}}}\sum\left(\frac{-1}{3}\right)\left(\frac{s}{3}\right)e^{\frac{\pi i}{6k_1k_2}(3k_2r+8k_1s)}$$

Since $(k_1, k_2) = 1$, $(8k_1, 3k_2) = 1$ and so $3k_2r + 8k_1s = t$ runs through a full system of residues modulo $24k_1k_2$. So

$$B_{k_1}(\nu_1)B_{k_2}(\nu_2) = \frac{1}{4}\left(\frac{k_1}{3}\right)\left(\frac{-1}{k_2}\right)\sqrt{\frac{k}{3}}\sum_{t^2\equiv\nu \pmod{24k_1k_2}}\left(\frac{-1}{r}\right)\left(\frac{s}{3}\right)e^{\pi it/(6k_1k_2)}$$

As before **230**

$$t^2 = (3k_2r + 8k_1s)^2 \equiv (3k_2r)^2 \equiv (3k_2r)^2 \equiv k_2^2\nu_1 \pmod{8k_1}$$
$$t^2 = (8k_1s)^2 \equiv k_1^2\nu_2 \pmod{3k_2}$$

Now determine ν such that $\nu \equiv k_2^2\nu_1 \pmod{8k_1}$ and $\nu \equiv k_1^2\nu_2 \pmod{3k_2}$, again by the Chinese remainder theorem. So $t^2 \equiv \pmod{24k_1k_2}$. Now

$$\left(\frac{t}{3}\right)\left(\frac{-1}{t}\right) = \left(\frac{8k_1s}{3}\right)\left(\frac{-1}{3k_1r}\right)$$

$$= \left(\frac{k_1}{3}\right)\left(\frac{-1}{k_2}\right)\left(\frac{s}{3}\right)\left(\frac{-1}{r}\right)$$

(since 8 and -1 are quadratic non-residues modulo 3). So

$$B_{k_1}(\nu_1)B_{k_2}(\nu_2) = \frac{1}{4}\sqrt{\frac{k}{3}} \sum_{t^2 \equiv \nu \pmod{24k}} \left(\frac{t}{3}\right)\left(\frac{-1}{t}\right) e^{\pi i t/6k}$$
$$= B_k(\nu)$$

where ν is given. Hence

Theorem 2. *If* $k_2^2\nu_1 \equiv \nu \pmod{8k_1}$ *and* $k_1^2\nu_2 \equiv \nu \pmod{3k_2}$, *then*

$$B_{k_1}(\nu_1)B_{k_2}(\nu_2) = B_{k_1k_2}(\nu)$$

Let us give an example of what this is good for. Calculate $A_{10}(26)$. Since we can reduce modulo 10, $A_{10}(26) = A_{10}(6)$.

$$\nu = 1 - 24n = -143.$$
$$A_{10}(26) = A_{10}(6) = B_{10}(-143) = B_{10}(-23)$$
$$= B_5(\nu_1)B_2(\nu_2)$$

where ν_1, ν_2 are determined by the conditions **231**

$$4\nu_1 \equiv -23 \pmod 5 \text{ or } -\nu_1 \equiv -3 \pmod 5$$

and $$25\nu_2 \equiv -23 \pmod{48} \text{ or } \nu_2 \equiv 1 \pmod{48}$$

So $A_{10}(26) = B_5(3)B_2(1)$, and these are explicitly known. Since $\left(\frac{3}{5}\right) = -1$, $B_5(3) = 0$. It is actually not necessary now to calculate $B_2(1)$.

$$B_2(1) = (-)^\lambda \left(\frac{-1}{r}\right) 2^{\lambda/2} \sin \frac{\pi r}{2^{\lambda+1}}$$

where $$(3r)^2 \equiv \nu \pmod{2^{\lambda+3}}, (3r)^2 \equiv 1 \pmod{16},$$

or $3r \equiv 1 \pmod{16}$, $r \equiv 11 \pmod{16}$. (there being four solutions). Then

$$B_2(1) = (-)(-)\sqrt{2}\sin\frac{11\pi}{4} = 1 \times \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$$
$$A_{10}(26) = 0.$$

One more thing can be established now. We have the inequalities:

$$|B_{2^\lambda}(\nu)| \leq 2^{\lambda/2},$$

$$|B_{3^\lambda}(\nu)| \leq 3^{\frac{\lambda}{2}} 2\sqrt{3},$$
$$|B_{p^\lambda}(\nu)| \leq 2p^{\frac{\lambda}{2}}, p > 3.$$

By the multiplicative property, **232**

$$|B_k(\nu)| = |A_k(\nu)| \leq \sqrt{k}(2\sqrt{3})^{\lambda(k)}$$

where
$$\lambda(k) = \sum_{p|k} 1.$$

This is a rough appraisal, but $\lambda(k)$ is in any case a small number. So

$$|B_k(\nu) < C\sqrt{k} \cdot k^{\epsilon}, \epsilon > 0, C = C_{\epsilon}.$$

We see that although $A_n(n)$ has $\varphi(k)$ summands and in general all that one knows is that $\varphi(k) \leq k - 1$, because of strong mutual cancellations among the roots of unity, the order is brought down to that of $k^{\frac{1}{2}+\epsilon}$. This reminds us of other arithmetical sums like the Gaussian sums and the Kloosterman sums.

Lecture 27

We now give a proof of the transformation formula for $\eta(\tau)$. $\eta(\tau)$ we first introduced by Dedekind in his commentary on a fragment on modular functions by Riemann; it is natural in the theory of elliptic functions. 233

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau})$$

We want to replace τ by $\tau' = \dfrac{a\tau + b}{c\tau + d}$. Actually in the whole literature there is no full account except in a paper by W.Fischer (Pacific Journal of Mathematics, Vol. 1). We know what happens in the special cases $-\frac{1}{\tau}$ and $\tau + 1$. We get the explicit form in which the root of unity appears in the transformation formula if we put together some things from the theory of modular functions. There some discussion in Tannery-Molk; they write $h(\tau)$ instead of $\eta(\tau)$. $(\eta(\tau))^3$ is up to a factor $\mathscr{V}_1^1(o/\tau)$. It turns out for quite other reasons that $(\eta(\tau))^8$ can be discussed too; it has to do with the modular invariant $J(\tau)$. Dedekind did something more than what is needed here. He studied $\log\eta(\tau)$. For $\operatorname{Im}\tau > 0$, $\eta(\tau)$ is a function in the interior of the unit circle (if we set $x = e^{2\pi i\tau}$) free from zeros and poles. So the logarithm has no branch points and is fully defined without ambiguity.

$$\log\eta(\tau) = \frac{\pi i\tau}{12} + \sum_{m-1}^{\infty}\log(1 - e^{2\pi i m\tau})$$

(For purely imaginary τ, the logarithms on the right side are real).

The multiplicative root of unity now appear as something additive. This is what Dedékind investigated. Recently (Mathematika, vol.1, 1954) Siegel published a proof for the particular case $-\dfrac{1}{\tau}$, using logarithms. Actually Siegel 234 proves much more than the functional equation for $\eta(\tau)$. He proves that

$$\log\eta(-\tau^{-1}) = \log\eta(\tau) + \frac{1}{2}\log\frac{\tau}{i}$$

We shall extend his proof to the more general case. The interesting case where a root of unity appears explicitly has not been dealt with by Siegel.

We write the general modular transformation in the form

$$\tau = \frac{h+i\mathfrak{z}}{k}, \tau' = \frac{h'+i/\mathfrak{z}}{k}, hh' \equiv -1 \pmod{k}$$

We wish to prove that

$$\log\eta\left(\frac{h'+i/\mathfrak{z}}{k}\right) = \log\eta\left(\frac{h+i\mathfrak{z}}{k}\right) + \frac{1}{2}\log\mathfrak{z} + \pi i C(h,k) \qquad (*)$$

where $C(h,k)$ is a real constant.

From the definition of $\eta(\tau)$,

$$\begin{aligned}\log\eta\left(\frac{h+i\mathfrak{z}}{k}\right) &= \frac{\pi(h+i\mathfrak{z})}{12k} - \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i m r(h+i\mathfrak{z})/k}\\ &= \frac{\pi i h}{12k} - \frac{\pi\mathfrak{z}}{12k} - \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i m r h/k}e^{-2\pi m r\mathfrak{z}/k}\end{aligned}$$

$e^{2\pi i m r h/k}$ is periodic with period k; we emphasize this and write 235

$$m = qk+\mu; \mu = 1,\ldots,k; q = 0,1,2,\ldots.$$

Then

$$\log\eta\left(\frac{h+i\mathfrak{z}}{k}\right) = \frac{\pi i h}{12k} - \frac{\pi\mathfrak{z}}{12k} - \sum_{q=0}^{\infty}\sum_{\nu=1}^{k}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\mu\frac{rh}{k}}e^{-2\pi(1k+\mu)\frac{r\mathfrak{z}}{k}},$$

and taking the summation over q inside, this becomes

$$\begin{aligned}&\frac{\pi i h}{12k} - \frac{\pi\mathfrak{z}}{12k} - \sum_{\mu=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\mu\frac{rh}{k}}e^{-2\pi\mu\frac{r\mathfrak{z}}{k}}\sum_{q=0}^{\infty}e^{-2\pi q r\mathfrak{z}}\\ &\qquad = \frac{\pi i h}{12k} - \frac{\pi\mathfrak{z}}{12k} - \sum_{\mu=1}^{k}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\mu\frac{rh}{k}}\frac{e^{-2\pi\mu r\mathfrak{z}/k}}{1-e^{-2\pi r\mathfrak{z}}}\end{aligned}$$

Substituting in (*), with similar expansion for $\eta\left(\frac{h'+i/\mathfrak{z}}{k}\right)$, we have

$$\frac{\pi ih}{12k} - \frac{\pi}{12k\mathfrak{z}} - \sum_{\nu=1}^{k}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\nu r\frac{h'}{k}}\frac{e^{-\frac{2\pi\nu r}{k\mathfrak{z}}}}{1-e^{-2\pi r/\mathfrak{z}}}$$
$$= \frac{1}{2}\log\mathfrak{z} + \pi iC(h,k) + \frac{\pi ih}{12k} - \frac{\pi\mathfrak{z}}{12k} - \sum_{\mu=1}^{k}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\mu\frac{rh}{k}}\frac{e^{-2\pi\mu\frac{r\mathfrak{z}}{k}}}{1-e^{-2\pi r\mathfrak{z}}}$$

Rearranging this, we get 236

$$\sum_{\nu=1}^{k}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\nu r\frac{h'}{k}}\cdot\frac{e^{-2\pi\nu r/k\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}} - \sum_{\mu=1}^{k}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\mu rh/k}\frac{e^{-2\pi\mu r\mathfrak{z}/k}}{1-e^{-2\pi r/\mathfrak{z}}}$$
$$+ \frac{\pi}{12k}\left(\frac{1}{\mathfrak{z}} - \mathfrak{z}\right) + \frac{\pi i}{12k}(h-h') + \pi iC(h,k) = -\frac{1}{2}\log\mathfrak{z}.$$

We now follow Siegel's idea to get the whole thing as a sum of residues of a certain function. Clearly there is r in it. Being integers r can be produced by something like $\dfrac{1}{1-e^{2\pi ix}}$ which has poles with residue $-\frac{1}{2\pi i}$ at every integral valued x. So let us study a function like

$$\frac{1}{x}\frac{1}{1-e^{2\pi ix}}e^{2\pi i\mu xh/k}\frac{e^{-2\pi\mu x\mathfrak{z}/k}}{1-e^{-2\pi x\mathfrak{z}}}$$

We may have to sum this from $\mu = 1$ to $\mu = k$. This should somehow be the form of the function that we wish to integrate. We do not want it in the whole plane. In fact, we can either take a wider and wider path of integration, or multiply the function by a factor and magnify it; we prefer to do the latter. 237
We shall put xN for x, keep the path fixed and take $N = n + \frac{1}{2}$, n integer, to avoid integral points, and then make $n \to \infty$. The term corresponding to $\mu = k$ should be treated separately, as otherwise the factor $e^{-2\pi x\mathfrak{z}}$ would stop convergence. Also μ^h and μ should appear symmetrically for reasons which we shall see. So introduce $\mu^* \equiv \mu^h \pmod{k}$, $\mu = 1, 2, \ldots, k-1$, and choose $1 \leq \mu^* \leq k-1$. It turns out, taking all this together, that the following thing will do. Write

$$F_n(x) = -\frac{1}{4ix}\cot h\pi Nx\cot\frac{\pi Nx}{\mathfrak{z}} + \sum_{\mu=1}^{k-1}\frac{1}{x}\cdot\frac{e^{2\pi\mu Nx/k}}{1-e^{2\pi Nx}}\cdot\frac{e^{-2\pi i\mu Nx/k\mathfrak{z}}}{1-e^{-2\pi iNx/\mathfrak{z}}}$$

The first term is a consequence of the term for $\mu = k$:

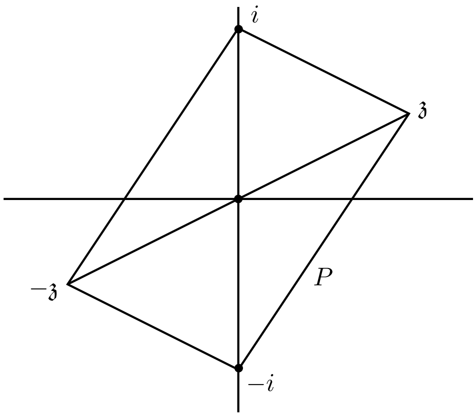
$$\frac{1}{x}\times\frac{e^{2\pi Nxi}}{1-e^{2\pi ixN}}\times\frac{e^{-2\pi xN\mathfrak{z}}}{1-e^{-2\pi xN\mathfrak{z}}}$$

The poles will not change if we write this as

$$\frac{1}{x}\left(\frac{e^{2\pi Nxi}}{1-e^{2\pi iNx}}+\frac{1}{2}\right)\left(\frac{e^{-2\pi xN\mathfrak{z}}}{1-e^{-2\pi xN\mathfrak{z}}}+\frac{1}{2}\right)=\frac{1}{x}\frac{1+e^{2\pi iNx}}{2(1-e^{2\pi iNx})}\cdot\frac{1+e^{-2\pi xN\mathfrak{z}}}{2(1-e^{-2\pi xN\mathfrak{z}})}$$
$$=\frac{1}{4xi}\cot\pi xN\cdot\cot h\pi xN\mathfrak{z}.$$

We integrate $F_n(x)$ along a certain parallelogram P, a little different from Siegel's. P has vertices at $\pm z$, $\pm i$ (since $\mathcal{J}_m\tau>0$, $\mathcal{R}_e\mathfrak{z}>0$). Then **238**

$$\frac{1}{2\pi i}\int_p F_n(x)dx=\sum(\text{Residues}).$$



We then let $n\to\infty$.

The poles of $F_n(x)$ are indicated by the denominators and the cotangent factors. These are

$$x=0,\quad x=-\frac{r\mathfrak{z}}{N},\quad x=\frac{ir}{N},\quad r\quad\text{integer.}$$

$x=0$ is a triple pole for the first summand.

$$-\frac{1}{4ix}\cot h\pi Nx\cot\frac{\pi Nx}{\mathfrak{z}}=-\frac{1}{4ix}\cdot\frac{1}{\pi Nx}\frac{\mathfrak{z}}{\pi Nx}$$
$$\left\{1+\frac{(\pi Nx)^2}{3}+\cdot\right\}\times\left\{1-\frac{(\pi Nx/\mathfrak{z})^2}{3}+\cdots\right\}$$

Residue for this term at $x = 0$ **239**

$$= \frac{i\mathfrak{z}}{4\pi^2 N^2} \cdot \frac{1}{3}\left(\pi^2 N^2 - \frac{\pi^2 N^2}{\mathfrak{z}}\right)$$
$$= \frac{i}{12}\left(\mathfrak{z} - \frac{1}{\mathfrak{z}}\right).$$

which had been foreshadowed already.

Lecture 28

We had **240**

$$F_n(x) = -\frac{1}{4ix}\cot h\pi Nx \cot\frac{\pi Nx}{\mathfrak{z}} + \sum_{\mu=1}^{k-1}\frac{1}{x}\cdot\frac{e^{2\pi\mu Nx/k}}{1-e^{2\pi Nx}}\times\frac{e^{-2\pi i\mu^* Nx/k\mathfrak{z}}}{1-e^{-2\pi iNx}/\mathfrak{z}},$$

$N = n + \frac{1}{2}$, n integer > 0, $\mu^* \equiv h\mu \pmod{k}$ and $1 \leq \mu^* \leq k-1$. At the triple pole $x = 0$ the residue from the first summand $= -\frac{1}{12i}\left(\mathfrak{z} - \frac{1}{\mathfrak{z}}\right)$- Let us find the residues from the more interesting pieces of the sum. The general term on the right has in the neighbourhood of $x = 0$ the expansion

$$\begin{aligned}
&\frac{1}{x}\left\{1 + \frac{2\pi\mu Nx}{k} + \frac{(2\pi\mu N_2/k)^2}{2!} + \cdots\right\}\\
&\qquad\times\frac{-1}{2\pi Nx}\left\{1 + \frac{2\pi Nx}{2} + \frac{(2\pi Nx)^2}{6} + \cdots\right\}^{-1}\\
&\qquad\times\left\{1 - \frac{2\pi i\mu^* Nx}{k\mathfrak{z}} - \frac{(2\pi\mu^* Nx/k\mathfrak{z})^2}{2} + \cdots\right\}\\
&\qquad\times\frac{1}{2\pi iNx/\mathfrak{z}}\left\{1 - \frac{2\pi iNx}{2\mathfrak{z}} - \frac{(2\pi Nx/\mathfrak{z})^2}{6} + \cdots\right\}^{-1}\\
&= \frac{-\mathfrak{z}}{4\pi^2 iN^2x^3}\left\{1 + \frac{2\pi\mu Nx}{k} + \frac{1}{2}\left(\frac{2\pi\mu Nx}{k}\right)^2 - \cdots\right\}\\
&\qquad\times\left\{1 - \left(\frac{2\pi Nx}{2} + \frac{(2\pi Nx)^2}{6} + \cdots\right) + (\cdots)^2 + \cdots\right\}\\
&\qquad\times\left\{1 - \frac{2\pi i\mu^* Nx}{k\mathfrak{z}} - \frac{1}{2}\left(\frac{2\pi\mu^* Nx}{k\mathfrak{z}}\right)^2 + \cdots\right\}
\end{aligned}$$

$$\times \left\{ 1 + \frac{2\pi i N z}{\mathfrak{z}} + \frac{(2\pi N^2/\mathfrak{z})^2}{2} + \cdots \right\}$$

Fishing out the term in $\dfrac{1}{x}$, the residue at $x = 0$ from this summand becomes **241**

$$\frac{i\mathfrak{z}}{4\pi^2 N^2} \left\{ \frac{1}{2} \left(\frac{2\pi\mu N}{k} \right)^2 + \frac{1}{12}(2\pi N)^2 - \frac{1}{2} \left(\frac{2\pi\mu^* N}{k\mathfrak{z}} \right)^2 - \frac{1}{12} \left(\frac{2\pi N}{\mathfrak{z}} \right)^2 - 2\pi\mu \frac{N}{k} \pi N \right.$$
$$\left. -4\pi^2 \mu\mu^* \frac{N^2 i}{k^2 \mathfrak{z}} + \frac{2\pi^2 i\mu N^2}{k\mathfrak{z}} + \frac{2\pi^2 i\mu^* N^2}{k\mathfrak{z}} - \frac{\pi^2 i N^2}{\mathfrak{z}} + \frac{2\pi^2 \mu^* N^2}{k\mathfrak{z}^2} \right\}$$
$$= \frac{i\mathfrak{z}}{4} \left\{ \frac{2}{\mu^2} k^2 + \frac{1}{3} - \frac{2\mu}{k} \right\} + \frac{i}{4\mathfrak{z}} \left\{ \frac{-2\mu^{*^2}}{k^2} - \frac{1}{3} + \frac{2\mu^*}{k} \right\}$$
$$+ \frac{i}{4} \left\{ -\frac{4i\mu\mu^*}{k^2} + \frac{2i\mu}{k} + \frac{2i\mu^*}{k} - i \right\}$$
$$= i\mathfrak{z} \left\{ \frac{\mu^2}{2k^2} - \frac{\mu}{2k} + \frac{1}{12} \right\} + \frac{1}{i\mathfrak{z}} \left\{ \frac{\mu^{*^2}}{2k^2} - \frac{\mu^*}{2k} + \frac{1}{12} \right\} + \left(\frac{\mu}{k} - \frac{1}{2} \right) \left(\frac{\mu^*}{k} - \frac{1}{2} \right) \qquad (*)$$

We have to sum this up from $\mu = 1$ to $\mu = k - 1$. Let us prepare a few things. **242**

Let us remark that

$$\sum_{\mu=1}^{k-1} \mu = \frac{(k-1)k}{2}; \quad \sum_{\mu=1}^{k-1} \mu^2 = \frac{(k-1)k(2k-1)}{6}$$

Also if μ runs through a full system of residues, so would μ^* because $(h, k) = 1$. Further $0 < \frac{\mu^*}{k} < 1$, and $\frac{\mu^*}{k}$ and $\frac{h\mu}{k}$ differ only by an integer, so that $\frac{\mu^*}{k} = \frac{h\mu}{k} - \left[\frac{h\mu}{k}\right]$. Hence summing up the last expression $(*)$ from $\mu = 1$ to $\mu = k - 1$, we have

$$\imath\mathfrak{z} \left\{ \frac{(k-1)(2k-1)}{12k} - \frac{k-1}{4} + \frac{k-1}{12} \right\}$$
$$+ \frac{1}{i\mathfrak{z}} \left\{ \frac{(k-1)(2k-1)}{12k} - \frac{k-1}{4} + \frac{k-1}{12} \right\} + \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \frac{1}{2} \right) \left(\frac{h\mu}{k} - \left[\frac{h\mu}{k} \right] - \frac{1}{2} \right)$$
$$= (k-1) \left(\frac{2k-1}{12k} - \frac{1}{6} \right) \left(i\mathfrak{z} + \frac{1}{i\mathfrak{z}} \right) + s(h, k)$$

where $s(h,k)$ stands for the arithmetical sum

$$\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right)$$

which appears here very simply as a sum of residues. The last expression becomes

$$-\frac{k-1}{12k}\left(i\mathfrak{z}+\frac{1}{i\mathfrak{z}}\right)+s(h,k)$$

So the total residue at $x=0$ is 243

$$\frac{1}{12}\left(i\mathfrak{z}+\frac{1}{i\mathfrak{z}}\right)-\frac{k-1}{12k}\left(i\mathfrak{z}+\frac{1}{i\mathfrak{z}}\right)+s(h,k)=\frac{1}{12k}\left(i\mathfrak{z}+\frac{1}{i\mathfrak{z}}\right)+s(h,k)$$

Next, we consider the simple poles of $F_n(x)$ at the points $x=\frac{ir}{N}(r\neq 0)$. The coth factor is periodic and so the residue at any of these poles is the same as that at the origin, which is $\frac{1}{\pi}$. Hence the residue of $F_n(x)$ at $x=\frac{ir}{N}(r\neq 0)$ becomes

$$\frac{N}{4r}\cdot\frac{1}{\pi N}\cot\frac{\pi ir}{\mathfrak{z}}+\sum_{\mu=1}^{k-1}\frac{N}{ir}\frac{-1}{2\pi N}e^{2\pi i\mu\frac{r}{k}}\frac{e^{2\pi\mu^* r/k\mathfrak{z}}}{1-e^{2\pi r/\mathfrak{z}}}$$

(There is a very interesting juxtaposition of an arithmetical term and a function theoretic term in the last part; this gets reversed for the next set of poles)

$$=\frac{1}{4\pi ir}\coth\frac{\pi r}{\mathfrak{z}}-\frac{1}{2\pi i}\sum_{\mu=1}^{k-1}\frac{1}{r}e^{2\pi i\frac{\mu r}{k}}\frac{e^{2\pi\mu^* r/k\mathfrak{z}}}{1-e^{2\pi r/\mathfrak{z}}}$$

x remains between $\pm i$ on the imaginary axis. So $\left|\frac{r}{N}\right|<1$; so we need consider only $r=\pm 1,\pm 2,\ldots,\pm n$. Again,

$$\coth\ y=\frac{e^y+e^{-y}}{e^y-e^{-y}}=1+\frac{2e^{-y}}{e^y-e^{-y}}$$
$$=1+\frac{2e^{-2y}}{1-e^{-2y}}$$

coth y is an odd function so that $\frac{1}{y}\cot h\,y$ is even. Hence summing up over all 244
the poles corresponding to $r=\pm 1,\ldots,\pm n$, we get the sum of the residues

$$=\frac{1}{2\pi i}\sum_{r=1}^{n}\frac{1}{r}\left\{1+\frac{2e^{-2\pi r/\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}}\right\}+\frac{1}{2\pi i}\sum_{\mu^*=1}^{k-1}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi ih'\mu^* r/k}\frac{e^{-2\pi\mu^* r/k\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}}$$

$$-\frac{1}{2\pi i}\sum_{\mu^*=1}^{k-1}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h'(k-\mu^*)r/k}\frac{e^{2\pi\mu^* r/k\mathfrak{z}}}{1-e^{2\pi r/\mathfrak{z}}},$$

where we have made use of the fact that $hh' \equiv -1 \pmod k$, so $h'\mu^* \equiv hh'\mu \equiv -\mu \pmod k$, or $\mu \equiv -h'\mu^* \pmod k$. In the last sum replace μ^* by $k-\mu^*$; then the previous sum is duplicated and we get

$$\frac{1}{2\pi i}\sum_{r=1}^{n}\frac{1}{r}\left\{1+\frac{2e^{-2\pi r/\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}}+\frac{1}{\pi i}\sum_{\mu^*=1}^{k-1}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h'\mu^* r/k}\frac{e^{-2\pi\mu^* r/k\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}}\right\}$$
$$=\frac{1}{2\pi i}\sum_{r=1}^{n}\frac{1}{r}+\frac{1}{\pi i}\sum_{\nu=1}^{k}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h'\nu r/k}\frac{e^{-2\pi\nu n/k\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}}$$

This accounts for all the poles on the imaginary axis (except the origin **245**
which has been considered separately before).

Finally we have poles $x = \frac{rz}{N}(e \neq 0)$ on the other diagonal of the parallelogram. The same calculation goes through verbatim and we get the sum of the residues at these poles to be

$$\frac{i}{2\pi}\sum_{r=1}^{n}\frac{1}{r}+\frac{i}{\pi}\sum_{\nu=1}^{k}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h\nu r/k}\frac{e^{-2\pi\nu r\mathfrak{z}/k}}{1-e^{-2\pi r\mathfrak{z}}}$$

Lecture 29

We had 246

$$F_n(x) = -\frac{1}{4ix}\cot h\pi Nx \cot\frac{\pi Nx}{\mathfrak{z}} + \sum_{\mu=1}^{k-1}\frac{1}{x}\frac{e^{2\pi\mu Nx/k}}{1-e^{2\pi Nx}}\frac{e^{-2\pi i\mu^* Nx/k\mathfrak{z}}}{1-e^{-2\pi Nx/\mathfrak{z}}}$$

The residue at $x = 0$ is

$$\frac{1}{12k}\left(i\mathfrak{z}+\frac{1}{i\mathfrak{z}}\right)+s(h,k),$$

$s(h,k)$, which will interest us for some time, being

$$\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right).$$

The residues at the points $x = \frac{ir}{N} (r \neq 0)$ amount to

$$\frac{1}{2\pi i}\sum_{r=1}^{n}\frac{1}{r} + \frac{1}{\pi i}\sum_{\nu=1}^{k}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h'\nu\frac{r}{k}}\frac{e^{-2\pi\nu r/k\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}};$$

and the residues at the points $x = \frac{\mathfrak{z}r}{N} (r \neq 0)$

$$\frac{i}{2\pi}\sum_{r=1}^{n}\frac{1}{r} + \frac{i}{\pi}\sum_{\mu=1}^{k}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h\mu\frac{r}{k}}\frac{e^{-2\pi\mu r\mathfrak{z}/k}}{1-e^{-2\pi r\mathfrak{z}}}$$

When we add up, the sums $\sum_{r=1}^{n}\frac{1}{r}$, the disagreeable ones which would have 247
gone to infinity, fortunately destroy each other; so the sum of the residues of $F_n(x)$ at all its poles becomes

$$\frac{1}{12ki}\left(\frac{1}{\mathfrak{z}} - \mathfrak{z}\right) + s(h,k) + \frac{1}{\pi i}\sum_{\nu=1}^{k}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h'\nu r/k}\frac{e^{-2\pi\nu r/k\mathfrak{z}}}{1-e^{-2\pi r/\mathfrak{z}}}$$
$$- \frac{1}{\pi i}\sum_{\mu=1}^{k}\sum_{r=1}^{n}\frac{1}{r}e^{2\pi i h\mu r/k}\frac{e^{-2\pi\mu r\mathfrak{z}/h}}{1-e^{-2\pi r\mathfrak{z}}}$$

We had prepared in advance what we were going to obtain. $s(h,k)$ is what we had called $C(h,k) + (h-h')/12k$. We have to prove that the sum of the residues above, with $C(h,k) = s(h,k) - \frac{h-h'}{12k}$, is equal to $-\frac{1}{2\pi i}\log z$, as $n \to \infty$. But there is one difference. The sums we have earlier were sums from $r = 1$ to $r = \infty$; whereas here they are sums from $r = 1$ to $r = n$. But this does not matter as convergence is guaranteed since we have an exponential factor e^{-z} with $\mathcal{R}_e\mathfrak{z} > 0$. We have to see what becomes of our sum when we evaluate it in another way. We have to consider $\lim\limits_{n\to\infty}\int_p F_n(x)dx$. So in effect we have to prove that

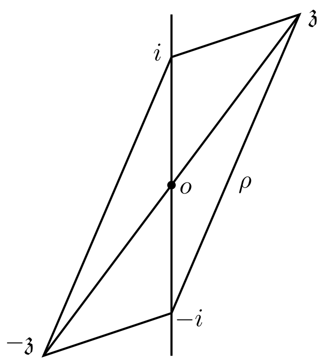
$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_p F_n(x)dx = -\frac{1}{2\pi i}\log\mathfrak{z}.$$ **248**

Now this is a question of direct computation. Let us look at the path of integration. $F_n(x)$ will be seen to have simple limits on the sides of the parallelogram. We consider $xF_n(x)$ broken into pieces. Take the first piece

$$\frac{1}{4i}\cot h\pi Nx\cot\frac{\pi Nx}{\mathfrak{z}}$$

On the side from $x = i$ to $x = z$,

$$x = \rho_i + \sigma\mathfrak{z}; \rho,\sigma \leq 0, \rho+\sigma = 1.$$



Actually we take only $\rho,\sigma > 0$; we shall exclude the points i and z themselves. Then this becomes

$$-\frac{1}{4i}\frac{e^{\pi N(\rho i+\sigma\mathfrak{z})} + e^{-\pi N(\rho i+\sigma\mathfrak{z})}}{e^{\pi N(\rho i+\sigma\mathfrak{z})} - e^{-\pi N(\rho i+\sigma\mathfrak{z})}} \times i \times \frac{e^{\pi i N(\rho i+\sigma\mathfrak{z})/\mathfrak{z}} + e^{-\pi i N(\rho i+\sigma\mathfrak{z})/\mathfrak{z}}}{e^{\pi i N(\rho i+\sigma\mathfrak{z})/\mathfrak{z}} - e^{-\pi i N(\rho i+\sigma\mathfrak{z})/\mathfrak{z}}}$$

The size of the first factor is determined by the terms $e^{\pi N\sigma\mathfrak{z}}$ and $e^{-N\pi\sigma\mathfrak{z}}$ in the numerator; the first term becomes big and the other goes to zero as $N \to \infty (\sigma > 0$ and $\mathcal{R}_e\mathfrak{z} > 0)$. So we divide by the first term. Similarly for the second factor. We therefore get

$$-\frac{1}{4}\frac{1+e^{-2\pi N(\rho i+\sigma\mathfrak{z})}}{1-e^{-2\pi N(\rho i+\sigma\mathfrak{z})}}\cdot\frac{e^{-2\pi N\left(\frac{\rho}{\mathfrak{z}}+\frac{\sigma}{i}\right)}+1}{e^{-2\pi N\left(\frac{\rho}{\mathfrak{z}}+\frac{\sigma}{i}\right)}-1}$$

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As $N \to \infty$ the exponential factors go to zero; so the whole expression tends to $\frac{1}{4}$. It will further remain on its way bounded, because the numerators in either factor are at most equal to 2, while the denominators remain away from zero by a fixed amount, as we shall be showing in a moment - and for this it is essential to have $N = n + \frac{1}{2}$.

Since the functions concerned are even functions, what was good here would also be good on the apposite side, from $x = -i$ to $x = -z$. So on this side also the expression will tend to $\frac{1}{4}$. We cannot say uniformly; indeed if $\sigma = 0$, here is no convergence in the first factor, and if $\rho = 0$ none in the second factor, though there is boundedness: the thing would oscillate finitely.

Now take the other pieces of $xF_n(x)$ on the same sides of ρ. We have to consider

$$\frac{e^{2\pi\mu\frac{N}{k}(\rho i+\sigma\mathfrak{z})}}{1-e^{2\pi N(\rho i+\sigma\mathfrak{z})}}\times\frac{e^{-2\pi i\mu^{*}\frac{N}{k\mathfrak{z}}(\rho i+\sigma\mathfrak{z})}}{1-e^{-2\pi i\frac{N}{\mathfrak{z}}(\rho i+\sigma\mathfrak{z})}}$$

Remember, what is now important, that $0 < \mu < k$, but neither 0 nor k. The denominator in the first factors goes more strongly to infinity as $N \to \infty$ than the numerator because $\frac{\mu}{k}$ is a proper fraction; so too in the second factor because $\mu > 1$. So the whole function tends to zero. Hence on these two sides $xF_n(x) \to \frac{1}{4}$.

Now consider the other two sides; it looks different here and has got to be **250**
inspected. On the side from $x = -i$ to $x = z$, $x = -\rho i + \sigma\mathfrak{z}$; $\sigma, \rho > 0$, $\sigma + \rho = 1$, and the first part of $xF_n(x)$ is

$$-\frac{1}{4i}\cot h\pi Nx\cot\frac{\pi Nx}{\mathfrak{z}} = -\frac{1}{4}\frac{e^{\pi N(-\rho i+\sigma\mathfrak{z})}+-e^{-\pi N(-\rho i+\sigma\mathfrak{z})}}{e^{\pi N(-\rho i+-\sigma\mathfrak{z})}-e^{-\pi N(-\rho i+\sigma-\mathfrak{z})}}$$

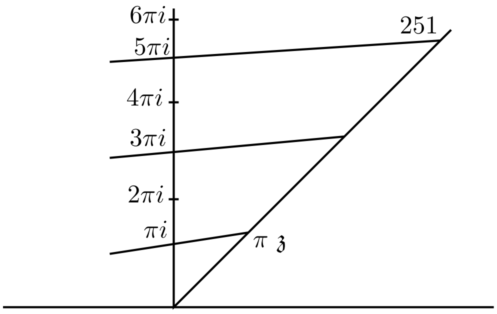
$$\times\frac{e^{\pi iN\left(\frac{-\rho i}{\mathfrak{z}}+\sigma\right)}+e^{-\pi iN\left(\frac{-\rho i}{\mathfrak{z}}+-\sigma\right)}}{e^{\pi iN\left(\frac{-\rho}{\mathfrak{z}}i+\sigma\right)}-e^{-\pi iN\left(-\frac{\rho i}{\mathfrak{z}}+\sigma\right)}}$$

$$= -\frac{1}{4}\frac{1+e^{-2\pi N(-\rho i+\sigma\mathfrak{z})}}{1-e^{-2\pi N(-\rho i+\sigma\mathfrak{z})}} - \times\frac{1+e^{-2\pi iN\left(-\frac{\rho i}{\mathfrak{z}}+\sigma\right)}}{1-e^{-2\pi iN\left(-\frac{\rho-i}{\mathfrak{z}}+\sigma\right)}}$$

Let $N \to \infty$. Assuming that the denominator is going to behave decently, this goes to $-\frac{1}{4}$. The other pieces go to zero for the same reason as before. And all this is good for the opposite side too.

We now have got to show that the convergence it nice and the denominators do not make any fuse. This we can clarify in the following way. Consider the denominator $1 - e^{-2\pi N(\rho i + \sigma \mathfrak{z})}$.

Difficulties will arise if the exponent comes close to an even multiple of πi. So we should see that it stays safely away from these points.



And actually it stays away from the danger spots by the same distance, for 251
the exponent is $-2N(\pi i\rho + \pi \mathfrak{z}\sigma)$ i.e., a point on the segment joining $(2r + 1)\pi i$ and $(2r + 1)\pi\mathfrak{z}$. Since $e^{\mathfrak{z}}$ is periodic there is a minimal amount by which it stays away from 1. The second denominator looks a little different. We have $\frac{\pi}{\mathfrak{z}}$ instead of $\pi\mathfrak{z}$. But we have only to turn the whole thing around. We see how essential it was to take $N = n + \frac{1}{2} = (2n + 1)\frac{1}{2}$ = on odd multiple of $\frac{1}{2}$.

So the convergence is nice, but not uniform. We can nevertheless say that $xF_n(x) \to \pm\frac{1}{4}$ boundedly on the sides of ρ except for the vertices where it does not converge but oscillates finitely. But bounded convergence is enough for interchanging integration and summation. $F_n(x) \to \pm\frac{1}{4x}$ and the x does not ruin anything because it stays away from zero everywhere on ρ. Hence

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_p F_n(x)dx$$

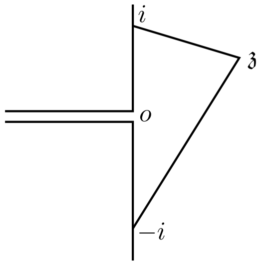
exists and we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{2\pi i}\int_p F_n(x)dx &= \frac{1}{2\pi i}\int_p \pm\frac{1}{4x}dx \\
&= \frac{1}{2\pi i}\left\{\int_{\mathfrak{z}}^{i} \frac{dx}{4x} - \int_{i}^{-\mathfrak{z}} \frac{dx}{4x} + \int_{-\mathfrak{z}}^{-i} \frac{dx}{4x} - \int_{-i}^{\mathfrak{z}} \frac{dx}{4x}\right\} \\
&= \frac{1}{8\pi i}\left\{\int_{\mathfrak{z}}^{i} \frac{dx}{x} - \int_{-i}^{\mathfrak{z}} \frac{dx}{x} + \int_{\mathfrak{z}}^{i} \frac{dx}{x} - \int_{-i}^{\mathfrak{z}} \frac{dx}{x}\right\}
\end{aligned}$$

$$= \frac{1}{4\pi i}\left\{\int_{\mathfrak{z}}^{i}\frac{dx}{x} - \int_{i}^{\mathfrak{z}}\frac{dx}{x}\right\}$$

z is in the positive half-plane; we can take the principal branch of the logarithm, **252**
so that we get on integration, since $\log i$ is completely determined,

$$\frac{1}{4\pi i}\left\{\frac{\pi}{2} - \log\mathfrak{z} - \left(\log\mathfrak{z} + \frac{\pi i}{2}\right)\right\} = -\frac{1}{2\pi i}\log\mathfrak{z}$$



So we have proved the foreseen formula with the particular substitution $C(h,k) = s(h,k) - \frac{h-h'}{12k}$:

$$\log\eta\left(\frac{h' + i/\mathfrak{z}}{k}\right) = \log\eta\left(\frac{h + i\mathfrak{z}}{k}\right) + \frac{1}{2}\log\mathfrak{z} + \pi i s(h,k) + \pi i\frac{h'-h}{12k},$$

which is the complete formula in all its details. The mysterious $s(h,k)$ enjoys **253**
certain properties. It has the group properties of the modular group behind it and so must participate in them.

Lecture 30

Last time we had the formula of transformation of $\log\eta$ in the following shape: **254**

$$\log\eta\left(\frac{h'+i/\mathfrak{z}}{k}\right)=\log\eta\left(\frac{h+i\mathfrak{z}}{k}\right)+\frac{1}{2}\log z+\frac{\pi i}{12k}(h'-h)+\pi i s(h,k),$$

where $s(h,k)$ is the Dedekind sum, which, by direct computation of residues, was seen to be

$$\sum_{\mu=0}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right).$$

We use the abbreviation: for real x,

$$((x))=\begin{cases}x-\quad [x]-\frac{1}{2}, & \text{if } x \text{ is not an integer,}\\ 0 & \text{, if } x \text{ is an integer.}\end{cases}$$

Then

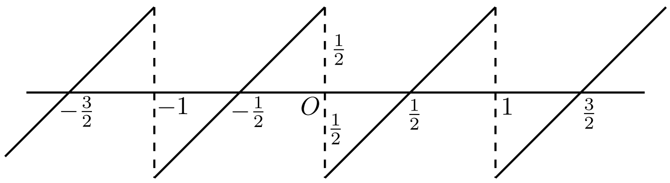
$$s(h,k)=\sum_{\mu=1}^{k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Now $((x))$ is an odd function; for x integer, trivially $((-x))=-((x))$, and for x not an integer,

$$\begin{aligned}((-x)) &= -x-[-x]-\frac{1}{2}\\ &= -x+[x]+1-\frac{1}{2},\ \text{since } [-x]=-[x]-1,\\ &= -((x)).\end{aligned}$$

* *

$((x))$ is the familiar function whose graph is as indicated. **255**



We now prove that

$$\sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right) = 0$$

Because of periodicity we can write

$$\begin{aligned}\sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right) &= \sum_{\mu \mod k} \left(\left(\frac{\mu}{k}\right)\right)\\ &= \sum_{\mu \mod k} \left(\left(\frac{\mu}{k}\right)\right)\\ &= -\sum_{\mu \mod k} \left(\left(\frac{\mu}{k}\right)\right)\end{aligned}$$

$$\therefore \quad \sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right) = 0$$

We can also write $s(h,k)$ in the form

$$\begin{aligned} s(h,k) &= \sum_{\mu=1}^{k} \left(\frac{\mu}{k} - \frac{1}{2}\right)\left(\left(\frac{h\mu}{k}\right)\right)\\ &= \sum_{mu=1}^{k} \frac{\mu}{k}\left(\left(\frac{h\mu}{k}\right)\right) - \frac{1}{2}\sum_{\mu=1}^{k}\left(\left(\frac{h\mu}{k}\right)\right),\end{aligned}$$

and since $h\mu$ also runs through a full system of residues $\mod k$ when μ does **256**
so, as $(h,k) = 1$, the second sum is zero, and we can therefore write

$$s(h,k) = \sum_{\mu=1}^{k} \frac{\mu}{k}\left(\left(\frac{h\mu}{k}\right)\right)$$

Let us now rewrite this in a form in which the modular substitution comes into play

$$\tau' = \frac{h' + i/\mathfrak{z}}{k}, \quad \tau = \frac{h + i\mathfrak{z}}{k};$$

so $k\tau - h = i\mathfrak{z}$, and

$$\tau' = \frac{h' - 1/(k\tau - h)}{k} = \frac{h'k\tau - hh' - 1}{k(k\tau - h)}$$
$$= \frac{h'\tau - (hh' + 1)/k}{k\tau - h}$$

($\frac{hh'+1}{k}$ is necessarily integral for $hh' \equiv -1 \mod k$). So the modular substitution is

$$\begin{pmatrix} h' & \frac{-hh'+1}{k} \\ k & -h \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, c > 0.$$

The transformation formula for $\log\eta$ now reads

$$\log\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \log\eta(\tau) + \frac{1}{2}\log\frac{c\tau + d}{i} + \frac{\pi i}{12c}(a + d) - \pi i s(d, c),$$

since $s(-d, c) = -- s(d, c)$.

Let us take in particular 257

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

then we obtain

$$\log\eta\left(\frac{1}{\tau}\right) = \log\eta(\tau) + \frac{1}{2}\log\frac{\tau}{i},$$

the special case discussed by Siegel.

Let us now make two substitutions in succession:

$$\tau'' = \frac{a\tau' + b}{c\tau' + d}, \quad \tau' = -\frac{1}{\tau}.$$

Then

$$\tau'' = \frac{-a/\tau + b}{-c/\tau + d} = \frac{b\tau - a}{d\tau - c}$$

We suppose $c > 0$, $d > 0$; $(c, d) = 1$. Then

$$\log\eta(\tau'') = \log\eta(\tau') + \frac{1}{2}\log\frac{c\tau' + d}{i} + \frac{\pi i}{12c}(a + d) - \pi i s(d, c);$$

$$\log \eta(\tau'') = \log \eta(\tau) + \frac{1}{2}\log\frac{d\tau - c}{i} + \frac{\pi i}{12d}(b-c) - \pi i s(-c,d).$$

Sub tracting, and observing that

$$\log \eta(\tau') - \log \eta n(\tau) = \frac{1}{2}\log\frac{\tau}{i},$$

we have 258

$$0 = \frac{1}{2}\log\frac{\tau}{i} + \frac{1}{2}\log\frac{c\tau' + d}{i} - \frac{1}{2}\log\frac{d\tau - c}{i} + \frac{\pi i}{12}\left(\frac{a+d}{c} - \frac{b-c}{d}\right) - \pi i(s(d,c) - s(c,d))$$

The sum of the logarithms on the right side is determinate only up to a multiple of $2\pi i$:

$$\begin{aligned}\log\frac{\tau}{1} + \log\frac{c\tau' + d}{i} - \log\frac{d\tau - c}{i} &= \log\frac{\tau}{i}\frac{(-c/\tau + d)/i}{(d\tau - c)/i} + 2\pi ik\\ &= \log\left(\frac{1}{i}\right) + 2\pi ik\\ &= -\frac{\pi i}{2} + 2\pi ik\end{aligned}$$

Now each logarithm above has an imaginary part which is strictly less than $\frac{\pi}{2}$ in absolute value; so

$$\left| Im\left\{\log\frac{\tau}{i} + \log\frac{c\tau' + d}{i} - \log\frac{d\tau - c}{i}\right\}\right| < \frac{3\pi}{2}$$

So the only admissible value of k is zero.

Hence we have 259

$$0 = -\frac{\pi i}{4} + \frac{\pi i}{12}\left(\frac{a+d}{c} - \frac{b-c}{d} - \pi i(s(d,c) + s(c,d))\right),$$

or since $ad - bc = 1$,

$$s(d,c) + s(c,d) = -\frac{1}{4} + \frac{1}{12}\left(\frac{d}{c} + \frac{c}{d} + \frac{1}{cd}\right).$$

This is the reciprocity law for Dedekind sums. It is a purely arithmetical formula for which I have given several proofs; here I reproduce the proof that I gave originally, by lattice-point enumeration.

We have to prove that

$$\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left\{\frac{h\mu}{k}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right\}+\sum_{\mu=1}^{h-1}\frac{\nu}{h}\left\{\frac{k\nu}{h}-\left[\frac{h\nu}{k}\right]-\frac{1}{2}\right\}$$
$$=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right),$$

or

$$\frac{h}{k^2}\sum_{\mu-1}^{k-1}\mu^2-\frac{1}{2k}\sum_{\mu-1}^{k-1}\mu+\frac{k}{h^2}\sum_{\nu-1}^{h-1}\nu^2-\frac{1}{2h}\sum_{\nu=1}^{h-1}\nu-\frac{1}{k}\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right]-\frac{1}{h}\sum_{\nu=1}^{h-1}\nu\left[\frac{k\nu}{h}\right]$$
$$=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{h}{h}+\frac{1}{hk}\right);$$

or **260**

$$\frac{h^2(k-1)(2k-1)}{6}-\frac{h}{2}\frac{k(k-1)}{2}+\frac{k^2(h-1)(2h-1)}{6}-\frac{k}{2}\frac{h(h-1)}{2}$$
$$-h\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right]-k\sum_{\nu=1}^{h-1}\nu\left[\frac{k\nu}{h}\right]$$
$$=\frac{-3hk+h^2+k^2+1}{12}$$

i.e., $$12h\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right]+12k\sum_{\nu=1}^{h-1}\nu\left[\frac{k\nu}{h}\right]$$

$= h(k-1)(2h(2k-1)-3k)+k(h-1)(2k(2h-1)-3h)+3hk-h^2-k^2-1$

$= 8h^2k^2-9h^2k-9hk^2+h^2+k^2+9hk-1$

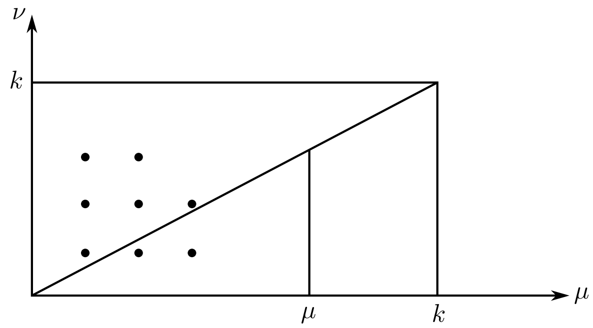
$= (h-1)(k-1)(8hk-h-k-1)$

So the whole thing is equivalent to proving that

$$12h\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right]+12k\sum_{\nu=1}^{h-1}\nu\left[\frac{k\nu}{h}\right]=(h-1)(k-1)(8hk-h-k-1).$$

This reduces to something that looks familiar; indeed the square brackets appear in lattice-point enumeration. Here $(h,k)=1$, but in a paper with Whiteman I have also discussed the case where h,k are not coprime.

Enumerating by rows and columns parallel to the $\mu-$ and $\nu-$ axes, the number of lattice-points in the integer a the rectangle **261**



with sides of length k, h along the axes of μ and ν respectively is seen to be $(h-1)(k-1)$. This can be enumerated in another way also. The number of lattice points in the interior, with abscissa μ and lying below the diagonal through the origin is the full integer in $\frac{h\mu}{k}$. So we have $\sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right]$ lattice points below the diagonal. Similarly there are $\sum_{\nu=1}^{h-1}\left[\frac{k\nu}{h}\right]$ points above the diagonal. Since $(h,k)=1$ there are no points on the diagonal. Hence

$$(h-1)(k-1)=\sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right]+\sum_{\nu=1}^{h-1}\left[\frac{k\nu}{h}\right]$$

In out case we have quadratic summands; but something which goes so well here in the plane should go well in space also.

Lecture 31

We want to prove directly the reciprocity formula 262

$$s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right)$$

with $$s(h,k)=\sum_{\mu=1}^{k}\frac{\mu}{k}\left(\left(\frac{h\mu}{k}\right)\right)$$

The reciprocity formula is equivalent to proving that

$$12h\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right]+12k\sum_{\nu=1}^{h-1}\nu\left[\frac{k\nu}{h}\right]=(h-1)(k-1)(8hk-h-k-1)$$

We made a little digression and spoke of similar sums which occur in lattice-point summations:

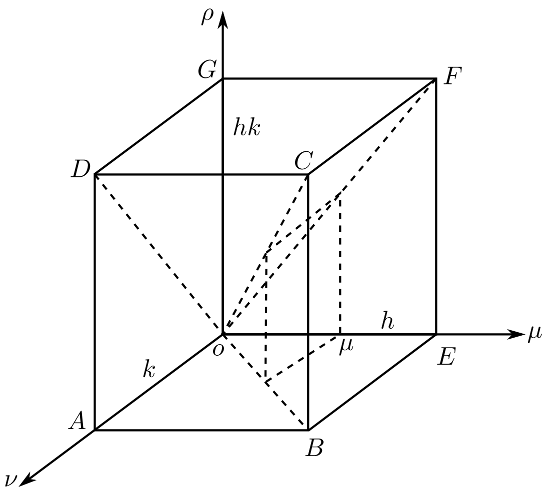
$$\sum_{\mu=1}^{k-1}\left[\frac{h\mu}{k}\right]+\sum_{\nu=1}^{h-1}\left[\frac{h\nu}{h}\right]=(h-1)(k-1)$$

If we use a rectangle of sides $\frac{h}{2}$, $\frac{k}{2}$, $(h,k$ odd) we obtain

$$\sum_{\mu=1}^{\frac{k-1}{2}}\left[\frac{h\mu}{k}\right]+\sum_{\nu=1}^{\frac{h-1}{2}}\left[\frac{h\nu}{h}\right]=\frac{1}{4}(h-1)(k-1).$$

This is made use of the theory of quadratic residues.

The summands in our case are 'quadratic' in μ and ν.



Consider the rectangular parallelopiped with three concurrent edges along the axes of μ, ν and ρ, the lengths of these edges being h, k, hk respectively. Dissect the parallelopiped into three pyramids having a common apex at the origin and having for bases the three rectangular faces which do not pass through the origin, viz. $ABCD$, $BCFE$ and $CDGF$. We now compute the number of lattice points in each pyramid. Take for example the pyramid $O(BEFC)$. Consider a section parallel to the (ρ,ν)-plane at a distance μ along the μ-axis. The lattice points lie in such sheets. The edges of this section are $h\mu$ and $\mu\frac{h}{k}$. The number of lattice points on this sheet (including possibly those on the edges) is $h\mu\left[\frac{\mu h}{k}\right]$. So for the whole pyramid the number $= \sum_{\mu=1}^{k-1} h\mu\left[\frac{\mu h}{k}\right]$. For the pyramid $O(ABCD)$, the one facing us, the number is $\sum_{\nu=1}^{h-1} k\nu\left[\frac{\nu k}{h}\right]$ **263**

Of course are some points on the common edge. Finally there is a pyramid of exceptional sort which lies upside down. Consider a section at a height h parallel to the (μ,ν) plane the numberr of lattice points on and inside this pyramid is seen to be **264**

$$\sum_{\rho=1}^{hk-1}\left[\frac{\rho}{h}\right]\left[\frac{\rho}{k}\right].$$

So altogether we have

$$\sum_{\mu=1}^{k-1} h\mu\left[\frac{\mu h}{k}\right]+\sum_{\nu=1}^{h-1} k\nu\left[\frac{\nu k}{k}\right]+\sum_{\rho=1}^{hk-1}\left[\frac{\rho}{h}\right]\left[\frac{\rho}{k}\right]$$

points, including some points which have been counted twice over. But the number of lattice points *inside*: the parallelopiped is equal to $(h-1)(k-1)(hk-1)$. Hence making a correction for the lattice points on the cleaving surfaces through the edges CF and CD which have been counted twice (the surface along BC has no points on it because $(h,k)=1$), we have

$$\sum_{\mu=1}^{k-1} h\mu\left[\frac{\mu h}{k}\right] + \sum_{\nu=1}^{h-1} k\nu\left[\frac{\nu k}{h}\right] + \sum_{\rho=1}^{hk-1}\left[\frac{\rho}{k}\right]\left[\frac{\rho}{h}\right]$$
$$= (h-1)(k-1)(hk-1) + (h-1)(k-1)$$
$$= hk(h-1)(k-1)$$

Now write

$$S = \sum_{\rho=1}^{hk-1}\left[\frac{\rho}{h}\right]\left[\frac{\rho}{k}\right]$$
$$\left[\frac{\rho}{h}\right] = \frac{\rho}{h} - \frac{1}{2} - \left(\left(\frac{\rho}{h}\right)\right), \text{ if } h\nmid\rho;\ \frac{\rho}{h} - \left(\left(\frac{\rho}{h}\right)\right), \text{ if } h\nmid\rho.$$

So **265**

$$S = \sum_{\rho=1}^{hk-1}\left\{\frac{\rho}{h} - \frac{1}{2} - \left(\left(\frac{\rho}{h}\right)\right)\right\}\left\{\frac{\rho}{2} - \frac{1}{2} - \left(\left(\frac{\rho}{k}\right)\right)\right\}$$

With some correction. Indeed $h\mid\rho$, $k\mid\rho$ do not happen together: Let $\rho = h\sigma$, $\rho = k\tau$. In the first case. i.e., $h\mid\rho$, we have to correct the above by an amount

$$\sum_{\sigma=1}^{k-1}\frac{1}{2}\left\{\frac{h\sigma}{k} - \frac{1}{2} - \left(\left(\frac{h\sigma}{k}\right)\right)\right\},$$

and in the second case, $k\mid\rho$, by

$$\sum_{\tau=1}^{h-1}\frac{1}{2}\left\{\frac{k\tau}{h} - \frac{1}{2} - \left(\left(\frac{k\tau}{h}\right)\right)\right\}$$

So

$$S = \sum_{\rho=1}^{hk-1}\left\{\frac{\rho}{h} - \frac{1}{2}\right\}\left\{\frac{\rho}{k} - \frac{1}{2}\right\} - \sum_{\rho=1}^{hk}\left(\left(\frac{\rho}{h}\right)\right)\left(\frac{\rho}{k} - \frac{1}{2}\right) - \sum_{\rho=1}^{hk}\left(\left(\frac{\rho}{k}\right)\right)\left(\frac{\rho}{h} - \frac{1}{2}\right)$$
$$+ \sum_{\rho=1}^{hk}\left(\left(\frac{\rho}{h}\right)\right)\left(\left(\frac{\rho}{k}\right)\right) + \frac{1}{2}\sum_{\sigma=1}^{k-1}\left\{\frac{h\sigma}{k} - \frac{1}{2}\right\} + \frac{1}{2}\sum_{\tau=1}^{h-1}\frac{1}{2}\left\{\frac{k\tau}{h} - \frac{1}{2}\right\}$$

Since $\sum\limits_{\mu \bmod k} \left(\left(\frac{\mu}{k}\right)\right) = 0,$ this becomes

$$S = \sum_{\rho=1}^{hk-1} \left\{ \frac{\rho^2}{hk} - \frac{1}{2}\left(\frac{\rho}{h} + \frac{\rho}{k}\right) + \frac{1}{4} \right\} - \frac{1}{4} \sum_{\rho=1}^{hk} \rho \left(\left(\frac{\rho}{k}\right)\right) - \frac{1}{h} \sum_{\rho=1}^{hk} \rho \left(\left(\frac{\rho}{k}\right)\right)$$
$$+ \sum_{\rho=1}^{hk} \left(\left(\frac{\rho}{h}\right)\right)\left(\left(\frac{\rho}{k}\right)\right) + \frac{1}{2}\left(\frac{h(k-1)}{2} - \frac{k-1}{2}\right) + \frac{1}{2}\left(\frac{k(h-1)}{2} - \frac{h-1}{2}\right)$$

we use the periodicity in the non-elementary pieces; so write **266**

$$\rho = hr + s; r = 0, 1, \ldots, k-1; s = 1, \ldots, h.$$

$$\sum_{\rho=1}^{hk} \rho \left(\left(\frac{\rho}{h}\right)\right) = \sum_{r=0}^{k-1} \sum_{s=1}^{h} (hr + s) \left(\left(\frac{hr+s}{h}\right)\right)$$
$$= \sum_{r=0}^{k-1} \sum_{s=1}^{h} hr \left(\left(\frac{s}{h}\right)\right) + \sum_{r=0}^{k-1} \sum_{s=1}^{h} s \left(\left(\frac{s}{h}\right)\right)$$
$$= k \sum_{s=1}^{h} s \left(\left(\frac{s}{h}\right)\right)$$

(since the first sum is zero, as we see by summing over s first)

$$= k \sum_{s=1}^{h-1} s \left(\frac{s}{h} - \frac{1}{2}\right)$$
$$= k \left\{ \frac{(h-1)(2h-1)}{6} - \frac{1}{4} h(h-1) \right\}$$
$$= \frac{k(h-1)(h-2)}{12}$$

Similarly **267**

$$\sum_{\rho=1}^{hk} \rho \left(\left(\frac{\rho}{k}\right)\right) = \frac{h(k-1)(k-2)}{12}$$

next, consider

$$\sum_{\rho=1}^{ik} \left(\left(\frac{\rho}{h}\right)\right)\left(\left(\frac{\rho}{k}\right)\right)$$

Write $\rho = h\alpha + k\beta$; when α, β run through complete systems of residues modulo h, k respectively, $h\alpha + k\beta$ runs through a complete system modulo hk, by the Chinese remainder theorem. Then

$$\begin{aligned}
\sum_{\rho=1}^{hk}\left(\left(\frac{\rho}{h}\right)\right)\left(\left(\frac{\rho}{k}\right)\right) &= \sum_{\alpha \mod k}\ \sum_{\beta \mod h}\left(\left(\frac{h\alpha+k\beta}{h}\right)\right)\left(\left(\frac{h\alpha+k\beta}{k}\right)\right)\\
&= \sum_{\alpha \mod k}\ \sum_{\beta \mod h}\left(\left(\frac{k\beta}{h}\right)\right)\left(\left(\frac{h\alpha}{k}\right)\right)\\
&= \sum_{\alpha \mod k}\left(\left(\frac{h\alpha}{k}\right)\right)\sum_{\beta \mod h}\left(\left(\frac{k\beta}{h}\right)\right)\\
&= 0
\end{aligned}$$

since each sum is separately zero. Hence

$$\begin{aligned}
S &= \frac{1}{6}(hk-1)(2hk-1)-\frac{1}{4}(hk-1)(k+h)+\frac{1}{4}(hk-1)\\
&\quad -\frac{1}{12}(h-1)(h-2)-\frac{1}{12}(k-1)(k-2)+\frac{1}{2}(k-1)(h-1)\\
&= \frac{1}{12}(hk-1)(4hk-3h-3k+1)-\frac{1}{12}(h-1)(h-2)\\
&\qquad -\frac{1}{12}(k-1)(k-2)+\frac{1}{2}(k-1)(h-1)\\
&= \frac{1}{12}(h-1)(k-1)(4hk+h+k+1)
\end{aligned}$$

Thus **268**

$$\begin{aligned}
h\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right]+k\sum_{\nu=1}^{h-1}\nu\left[\frac{k\nu}{h}\right]+\frac{1}{12}(h-1)(k-1)(4hk+h+k+1)&\\
= (h-1)(k-1)hk&
\end{aligned}$$

$$\begin{aligned}
\therefore\quad 12h\sum_{\mu=1}^{k-1}\mu\left[\frac{h\mu}{k}\right]+12k\sum_{\nu=1}^{h-1}\nu\left[\frac{k\mu}{h}\right]&\\
= (h-1)(k-1)(8hk-h-k-1)&
\end{aligned}$$

We make some elementary remarks about quadratic residues. The reciprocity formula gives, on multiplication by $12h^2k$.

$$12h^2ks(h,k)+12h^2ks(k,h) == -3h^2k+h^3+k^2h$$

Look at the denominator of $s(h,k)$. At worst it can have for factors 2 and **269**

k^2. So $2k^2 s(h,k)$ is integral. $2h^2 s(k,h)$ is also integral.

$$\begin{aligned} 12h^2 ks(h,k) &\equiv h^3 + k^2 + h \pmod{3k} \\ &\equiv h(h^2+1) \pmod{k}, \end{aligned}$$

and since h^2 cannot help to make an integer of the left side,

$$12hks(h,k) \equiv h^2 + 1 \pmod{k}.$$

Sp $12ks(h,k)$ is an integer. The highest possible denominator for $s(h,k)$ is $(2k^2, 12k) = 2k(k,6)$. So the denominator which at first glance could conceivably be as big as $2k^2$ is actually at most only $2k(k,6)$. This is achieved, for instance, in $s(1,3) = 1/18$, where $6(6,3) = 18$. In fact $s(1,3)$ can be computed from the reciprocity formula:

$$s(1,3) + s(3,1) = -\frac{1}{4} + \frac{1}{12}\left(\frac{1}{3} + \frac{3}{1} + \frac{1}{3}\right) s(3,1) = 0$$

since an integer is involved and so $s(1,3) = \frac{1}{18}$. In general,

$$\begin{aligned} s(1,k) &= -\frac{1}{4} + \frac{1}{12}\left(\frac{1}{k} + \frac{k}{1} + \frac{1}{k}\right) \\ &= \frac{(k-1)(k-2)}{12k} \end{aligned}$$

$s(2,k)$ is also easily obtained. k is odd; so we have

$$s(2,k) + s(1,2) = -\frac{1}{4} + \frac{1}{12}\left(\frac{2}{k} + \frac{k}{2} + \frac{1}{2k}\right)$$

and as $s(1,2) = 0$ (by direct computation), we get **270**

$$s(2,k) = \frac{(k-1)(k-5)}{24k}$$

Let us calculate $s(5,27)$.

$$\begin{aligned} s(5,27) + s(27,5) &= -\frac{1}{4} + \frac{1^2 + 5^2 + 27^2}{12 \times 5 \times 27} \\ s(2,5) + s(5,2) &= -\frac{1}{4} + \frac{1^2 + 2^2 + 5^2}{12 \times 2 \times 5} \\ s(5,2) = 0 &= s(1,2), \quad \text{and on sub traction,} \\ s(5,27) &= 35/(6 \times 27); \quad \text{and we know that} \end{aligned}$$

the denominator could be at most $2.27(27,6) = 6 \times 27$.

Lecture 32

We shall study a few more properties of Dedekind sums. We had the reciprocity law 271

$$s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right).$$

From this we deduced as a consequence

$$12hk\, s(h,k)\equiv h^2+1 \pmod k \tag{*}$$

Now when do the Dedekind sums vanish? Let us write $s(h,k)$ in the more flexible form:

$$s(h,k)=\sum_{\mu \mod k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{\mu h}{k}\right)\right)$$

Let $hh^* \equiv 1 \pmod k$. Since $(h^*,k)=1$, $h^*\mu$ runs through a full residue system modulo k, and so

$$\begin{aligned} s(h,k) &= \sum_{\mu \mod k}\left(\left(\frac{\mu h^*}{k}\right)\right)\left(\left(\frac{\mu h h^*}{k}\right)\right)\\ &= \sum_{\mu \mod k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{\mu h^*}{k}\right)\right)\\ &= s(h^*,k) \end{aligned}$$

This is of some significance. We came to s from the substitution $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ and since $ad\equiv 1 \pmod c$, $s(d,c)=s(a,c)$. $hh'\equiv -1 \pmod k$, and

$$s(h,k)=\sum_{\mu \mod k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{\mu h}{k}\right)\right)$$

$$
\begin{aligned}
&= \sum_{\mu \mod k} \left(\left(\frac{h'\mu}{k}\right)\right)\left(\left(\frac{\mu h h'}{k}\right)\right)\\
&= \sum_{\mu \mod k} \left(\left(-\frac{\mu}{k}\right)\right)\left(\left(\frac{h'\mu}{k}\right)\right)\\
&= -s(h', k)
\end{aligned}
$$

When $h = h'$ i.e., $h^2 \equiv - \pmod k$ (cg. $2^2 \equiv - \pmod 5$)) 272

$$s(h,k) = -s(h,k)$$

or $$s(h,k) = 0 \qquad if \qquad h^2 \equiv -1 \pmod k$$

(in particular if $h^2 + 1 = k$. I have a conjecture that $s(h,k) \geq 0$ if $h^2 < k$). In fact we can say more. We have the

Theorem. $12\, s(h,k)$ *is en integer only for* $h^2 \equiv -1(\pmod k)$ *and is then equal to zero.*

For assume that $12s(h,k) =$ integer; this implies, because of (*), that $0 \equiv h^2 + 1 \pmod k$

In such cases, therefore, we can make a direct statement about the value of $s(h,k)$ without going through the rigmarole of the Euclidean algorithm. Thus $s(2,5) = 0$, $s(5,26) = 0$.

In a recent issue of the Duke Mathematical Journal (1954), I gave a generalisation of the reciprocity formula for Dedekind sums. It takes into account three summands. The formula is very elegant and throws some light on the reciprocity relation itself. We quote it without proof.

Theorem. *If a, b, c are pairwise coprime and* $aa^* \equiv 1 \pmod{bc}$, $bb^* \equiv 1 \pmod{ca}$, $cc^* \equiv 1 \pmod{ab}$, *then*

$$
\begin{aligned}
T &\equiv s(bc^*, a) + s(ca^*, b) + s(ab^*, c)\\
&= -\frac{1}{4} + \frac{1}{2}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)
\end{aligned}
$$

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The proof is by an algebraic method due to Rédel. The formula is very gratifying as a generalisation of the reciprocity formula is which latter there is some non-homogeneity. Put $c = 1$; then $c^* = 1$, and $s(ab^*, c) = 0$; so we get the reciprocity formula. The right side above is $\frac{1}{12\ abc} - 3abc + a^2 + b^2 + c^2$. Hence $T = 0$ if and only if $a^2 + b^2 + c^2 = 3abc$. This combination of three integers plays some role the theory of quadratic forms; it is called a Markoff triple. It has reappeared in literature in connection with the geometry of numbers. It has to do with the existence if certain quadratic forms with minimum values close to zero for integers. 1, 1, 2 is a Markoff triple. If we keep two of them fixed, for the third we get a quadratic equation of which one root we know to be rational. So the other root is rational too. For instance if $a, b = 1$ are fixed, we have $c^2 - 3c + 2 = 0$ or $(c-1)(c-2) = 0$; – the triples pre 1, 1, 1 and 1,1,2. If we take the triple a, 1, 2, then $a^2 + 5 = 6a$ or $a = 1, 5$; we have the triples b, 1, 2; 1, 1, 2. $T = 0$ only if a, b, c being to a Markoff triple. For such a triple,

$$b^2 + c^2 \equiv 0 \pmod a, c^2 + a^2 \equiv 0 \pmod b, a^2 + b^2 \equiv 0 \pmod c$$

So
$$b^2 \equiv -c^2 \pmod a, \text{ or } (c^* b)^2 \equiv -1 \pmod a, etc.$$

Then $s(bc^*, a) = 0$, and each summand in T is zero.

Dedekind sums have something to do with Farey fractions. Let us suppose that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1, c, d > 0$.

$$s(c,d) + s(d,c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{c}{d} + \frac{d}{c} + \frac{1}{cd}\right)$$

$$cb \equiv -1 \pmod d \text{ and } ad \equiv 1 \pmod c$$

so
$$s(c,d) = -s(-b,d) \text{ and } s(d,c) = s(a,c).$$

So
$$s(a,c) - s(b,d) = -4 + \frac{1}{12}\left(\frac{c}{d} + \frac{d}{c} + \frac{1}{cd}\right)$$

Now if $\frac{h_1}{k_1}$, $\frac{h_2}{k_2}$ the adjacent Farey Fractions, then $\begin{vmatrix} h_1 & h_2 \\ k_1 & k_2 \end{vmatrix} = -1$ **274**
so

$$s(h_1,k_1) - s(h_2,k_2) = \frac{1}{4} - \frac{1}{12}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1} + \frac{1}{k_1 k_2}\right)$$

Write the left side as $s\left(\frac{h_1}{k_1}\right) - s\left(\frac{h_2}{k_2}\right)$.

Suppose $\frac{h_2}{k_2}$ is fixed. Let us look at all possible adjacent fractions $\frac{h_1}{k_1}$. They are obtainable by forming mediants; replace $\frac{h_1}{k_1}$ successively by $\frac{h_1+\lambda h_2}{k_1+\lambda k_2}$. Make k_1 larger and larger. Then $\frac{k_2}{k_1}$ and $\frac{1}{k_1 k_2} \to 0$. So $\frac{k_1}{k_2} \to \infty$. Thus $s\left(\frac{h_1}{k_1}\right) - s\left(\frac{h_2}{k_2}\right)$ goes unboundedly by $-\infty$, and so $s\left(\frac{h_1}{k_1}\right) \to -\infty$. Therefore only on the left side of $\frac{h_2}{k_2}$ can we get a sequence of rational fractions for which the Dedekind sums tend to $-\infty$.

We now give another proof of the reciprocity law, by the method of finite Fourier series, $\left(\left(\frac{\mu}{k}\right)\right)$ is a number- theoretic periodic function. It has a finite Fourier expansion:

$$\left(\left(\frac{\mu}{k}\right)\right) = \sum_{j=1}^{k} c_j e^{2\pi i j \frac{\mu}{k}}$$

In fact this is always solvable for $c_1, c_2, \ldots, c_k$. For writing down $\mu = 1, 2, 3, \ldots, k$ in succession, we have a system of k linear equations whose determinant is a Vandermonde determinant which is non-zero since the roots of unity are different. We have 275

$$\begin{aligned}\sum_{\mu=1}^{k}\left(\left(\frac{\mu}{k}\right)\right)e^{-2\pi i\mu\frac{\ell}{k}} &= \sum_{j=1}^{k} c_j \sum_{\mu=1}^{k} e^{2\pi i\mu\frac{(j-\ell)}{k}}\\ &= kc_l,\end{aligned}$$

i.e.,

$$c_l = \frac{1}{k}\sum_{\mu=1}^{k}\left(\left(\frac{\mu}{k}\right)\right)e^{-2\pi i\mu\frac{\ell}{k}}$$

This was done by Eisenstein, We can also write

$$\begin{aligned}c_l &= \frac{1}{k}\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)e^{-2\pi i\mu\frac{l}{k}}\\ &= \frac{1}{k^2}\sum_{\mu=1}^{k-1}\mu e^{-2\pi i\mu\frac{l}{k}} - \begin{cases}\frac{k-1}{2k}, & \text{if } k \mid l;\\ \frac{1}{2k}, & \text{if } k \nmid l.\end{cases}\end{aligned}$$

So if $k \mid l$, then

$$c_\ell = \frac{1}{k^2}\frac{k(k-1)}{2} - \frac{k-1}{2k} = 0.$$

In particular

$$c_k = 0.$$

If $k \nmid l$, then writing

$$\begin{aligned}S &= \sum_{\mu=1}^{k-1}\mu e^{-2\pi i\mu\frac{\ell}{k}},\\ S\,e^{-2\pi i\frac{\ell}{k}} &= \sum_{\mu=1}^{k-1}\mu e^{-2\pi i\frac{\mu+1}{k}\ell}\\ &= \sum_{\nu=2}^{k}(\nu-1)e^{-2\pi i\nu\frac{\ell}{k}}\\ &= S - e^{-2\pi i\frac{l}{k}} + k - \sum_{\nu=1}^{k} e^{-2\pi i\nu\frac{l}{k}} + e^{-2\pi i\frac{l}{k}}\end{aligned}$$

So 276

$$S = \frac{k}{e^{-2\pi i\ell/k} - 1}$$

Hence, if $k \nmid l$, then

$$\begin{aligned} c_\ell &= \frac{-1}{k(1-e^{2\pi i\ell/k})} + \frac{1}{2k} \\ &= \frac{-2+1-e^{-2\pi i\ell/k}}{2k(1-e^{-2\pi i\ell/k})} \\ &= -\frac{1}{2k}\cdot\frac{1+e^{-2\pi i\frac{\ell}{k}}}{1-e^{-2\pi i\frac{\ell}{k}}} \\ &= \frac{i}{2k}\cot\frac{\pi\ell}{k} \end{aligned}$$

So we have what is essentially Eisenstein's formula:

$$\left(\left(\frac{\mu}{k}\right)\right) = \frac{i}{2k}\sum_{j=1}^{k-1}\cot\frac{\pi\ell}{k}e^{2\pi ij\frac{\mu}{k}}$$

This is an explicit formula for $\left(\left(\frac{\mu}{k}\right)\right)$ as a finite Fourier series. We utilise it for Dedekind sums.

$$\begin{aligned} s(h,k) &= \sum_{\mu \bmod k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right) \\ &= -\frac{1}{4k^2}\sum_{\mu \bmod k}\sum_{j=1}^{k-1}\cot\frac{\pi j}{k}e^{2\pi ij\frac{\mu}{k}} \times \sum_{\ell=1}^{k-1}\cot\frac{\pi\ell}{k}e^{2\pi i\ell h\frac{\mu}{k}} \\ &= -\frac{1}{4k^2}\sum_{j=1}^{k-1}\sum_{\ell=1}^{k-1}\cot\frac{\pi j}{k}\cot\frac{\pi l}{k}\sum_{\mu \bmod k}e^{2\pi i\frac{\mu}{k}(j+h\ell)} \\ &= -\frac{1}{4k}\sum_{\ell=1}^{k-1}\cot\frac{\pi\ell}{k}\cot\frac{-\pi h\ell}{k}, \end{aligned}$$

since in the summation with respect to μ only those terms remain for which 277
$j + h\ell \equiv \pmod{k}$. Then

$$s(h,k) = \frac{1}{4k}\sum_{\ell=1}^{k-1}\cot\frac{\pi\ell}{k}\cot\frac{\pi h\ell}{k}.$$

The reciprocity formula can be tackled immediately by the powerful method of residues. We have to construct the proper function for which these become the residues. Take

$$f(\mathfrak{z}) = \cot \pi\mathfrak{z} \cot \frac{\pi\mathfrak{z}}{k} \cot \frac{\pi h\mathfrak{z}}{k}$$

and integrate over a rectangle with vertices $\pm i\Omega$, $\pm i(k + i\Omega)$, indented at o and k. The poles of the first factor all in the contour one $0, 1, \ldots, k-1$, for the second 0; and for the third $0, k/h, 2k/h, \ldots, (h-i)12/h$. We have

$$\cot \omega = \frac{1}{\omega}\left(1 - \frac{\omega^2}{3} - \cdots\right)$$

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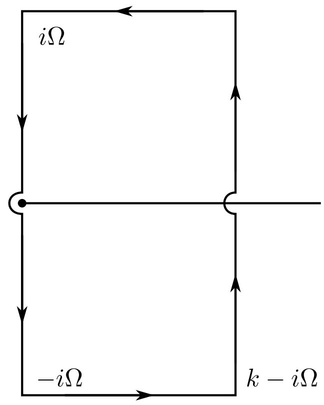
About the triple pole $\mathfrak{z} = 0$,

$$f(\mathfrak{z}) = \frac{1}{\pi\mathfrak{z}} \cdot \frac{k}{\pi\mathfrak{z}} \cdot \frac{1}{\pi h\mathfrak{z}}\left(1 - \frac{\pi^2\mathfrak{z}^2}{3} + \cdots\right)\left(1 - \frac{\pi\mathfrak{z}^2}{3} + \cdots\right)\left(1 - \frac{\pi^2 h^2\mathfrak{z}^2}{3k^2} + \cdots\right)$$



So the residue at $\mathfrak{z} = 0$ is

$$\frac{k^2}{\pi^2 h}\left(-\frac{\pi^2}{3} - \frac{\pi^2}{3k^2} - \frac{\pi^2 h^2}{3k^2}\right) = -\frac{k}{3\pi}\left(\frac{k}{h} + \frac{1}{hk} + \frac{h}{k}\right)$$

So

$$\sum(\text{Res}) = -\frac{k}{3\pi}\left(\frac{k}{h} + \frac{h}{k} + \frac{1}{hk}\right) + \frac{1}{\pi}\sum_{\ell=1}^{k-1} \cot \frac{\pi\ell}{k} \cot \frac{\pi h\ell}{k}$$

$$+\frac{k}{\pi h}\sum_{k=1}^{h-1}\cot\frac{\pi r h}{h}\cot\frac{\pi h}{h}$$
$$=\frac{k}{3\pi}\left(-\left(\frac{k}{h}+\frac{h}{k}+\frac{1}{hk}\right)+12s(h,k)+12s(k,h)\right)$$

And this is equal to **279**

$$\frac{1}{2\pi i}\int_R f(\mathfrak{z})d\mathfrak{z}$$

where R is the rectangle. On the vertical lines the function the same value (by periodicity) end so the integrals cancel out. Hence

$$\frac{1}{2\pi i}\int_R f(\mathfrak{z})d\mathfrak{z}=\frac{1}{2\pi i}\left\{\int_{-i\Omega}^{-i\Omega+k}-\int_{i\Omega}^{i\Omega+k}\right\}$$

Now

$$\cot\omega = i\frac{e^{i\omega}+e^{-i\omega}}{e^{i\omega}-e^{-i\omega}}, \omega = x+iy,$$
$$= i\frac{e^{ix-y}+e^{-ix+y}}{e^{ix-y}-e^{-ix+y}};$$

x varies from o to k and $y=\pm\Omega$, for this

$$\to\begin{cases}-i, & \text{as } y=\Omega\to\infty\\ i, & \text{as } y=-\Omega\to-\infty\end{cases}\qquad\text{uniformly}$$

Therefore

$$\lim_{\Omega\to\infty}\frac{1}{2\pi i}\int_R f(\mathfrak{z})d\mathfrak{z}=\frac{1}{2\pi i}\left\{i^3k-(-i)^3k\right\}$$
$$=\frac{2ki^3}{2\pi i}=-\frac{k}{\pi}$$

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$$\therefore\qquad \frac{k}{3\pi}\left(-\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right)+12s(h,k)+12s(k,h)\right)$$
$$=-\frac{k}{\pi}$$

or
$$12s(h,k)+12s(k,h)=-3+\left(\frac{k}{h}+\frac{h}{k}+\frac{1}{hk}\right),$$

which is the reciprocity formula.

Part IV

Representation by squares

Lecture 33

We wish to begin the study of the representation of a number as the sum of squares: **281**

$$n = n_1^2 + n_2^2 + \cdots + n_r^2$$

We shall develop in this connection the Hardy-Littlewood circle method. Historically it is an off shoot of the Hardly-Ramanujan method in partition-theory, though we did not develop the latter in its original form in our treatment. The circle method has been applied to very many cases, and the problem of squares is a very instructive one for finding out the general thread. We shall later replace the problem by that of the representation of n by a positive quadratic form. This would involve only the general Poisson summation formula. In the case of representation as the sum of squares there is some simplification, because the generating ing function is the r^{th} power of a simple $\mathscr{V}$ function. We shall deal with the asymptotic theory. Later we may go into Siegel's theory of quadratic forms.

Let us write

$$\Theta(x) = \sum_{n=-\infty}^{\infty} x^{n^2} = 1 + 2\sum_{n=1}^{\infty} x^{n^2},$$

$|x| < 1$. For r at least equal to 4, we consider

$$\Theta^r(x) = \left(\sum_{n=-\infty}^{\infty} x^{n^2}\right)^r = \sum_{n_j=-\infty}^{\infty} x^{n_1^2+n_2^2+\cdot\cdot+n_r^2}$$
$$= \sum_{n=0}^{\infty} A_r(n) x^n,$$

on collecting the terms with exponent n, where $A_r(n)$ is the number of times n **282** appears as the sum of r square:

$$A_r(n) = \sum_{n_1^2+\cdots+n_r^2=n} 1$$

It is clear that n_i can be positive or negative. The more serious thing is that we have to count the representations differently when the summands are interchanged, in contradiction to the situation in the case of partitions. The problem of partition into squares would be a more complicated problem; the generating function would be more complicated, and what is worse, all the help one gets in partition theory from the theory of modular forms would break down here.

$A_n(n)$ is the n^{th} coefficient of a power-series;

$$A_r(n) = \frac{1}{2\pi i}\int_C \frac{\Theta^r(x)}{x^{n+1}}dx$$

where C is a suitable circle inside and close to the unit circle. The trick of Hardy and Littlewood was to break the circle $|x| = e^{-2\pi\delta_N}$ where N is the order of a certain Farey dissection, into Farey arcs and write

$$A_r(n) = \frac{1}{2\pi i}\sum_{o\leq h<k\leq N}{}' \int_{\xi_{hk}} \frac{\Theta^n(x)}{x^{n+1}}dx,$$

where ξ_{hk} are the arcs over which one integration piecemeal the prime denoting that $(h,k) = 1$. Consider on each piece ξ_{hk} the neighbourhood of a root of unity: 283

$$x = e^{2\pi i\frac{h}{k}-2\pi\xi}$$

$\mathscr{R}_e\mathfrak{z} < 0$, and set $\mathfrak{z} = \delta_N - i\varphi$, so chat we have a little freedom along both real and imaginary axes.

$$x = e^{2\pi i\frac{h}{k}-2\pi\delta_N+2\pi i\varphi}.$$

The choice of the little arc φ is also classical. $\frac{h}{k}$ is a certain Farey fraction, with adjacents $\frac{h_1}{k_1}$ and $\frac{h_2}{k_2}$, say. $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$. We limit φ on the seperate arcs. Introduce the mediants:

$$\frac{h_1}{k_1} < \frac{h_1+h}{k_1+k} < \frac{h}{k} < \frac{h_2+h}{k_2+k} < \frac{h_2}{k_2},$$

So that the interval $\left(\frac{h_1+h}{k_1+k}, \frac{h_2+h}{k_2+2}\right)$ gives the movement of $\frac{h}{k}+\varphi$. So φ runs between

$$-\mathscr{V}'_{hk} = \frac{h_1+h}{k_1+k} - \frac{h}{k} \leq \varphi \leq \frac{h_2+h}{h_2+k} - \frac{h}{k} = \mathscr{V}''_{hk}$$

$$-\mathscr{V}'_{hk} = -\frac{1}{(k_1+k)k}; \mathscr{V}''_{hk} = \frac{1}{(k_2+k)k};$$

and since $2N > \frac{k_1+k}{k_2+k} > N$, we have necessarily

$$\frac{1}{2Nk} \leq |\mathscr{V}_{hk}| \leq \frac{1}{Nk}$$

Now changing the variable of integration to φ, we can write

$$A_r(n) = \sum_{0\leq h<k\leq N}{}' e^{-2\pi i \frac{h}{k} n} \int\limits_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} \Theta^r \left(e^{2\pi i \frac{h}{k} - 2\pi \mathfrak{z}}\right) e^{2\pi n \mathfrak{z}} d\varphi$$

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The trick is to overcome the difficulty in the integral by replacing on each arc the highly transcendental function by a simpler function. Here we stop for a moment to see what we can do with the integrand.

$$\begin{aligned}
\Theta \cdot \left(e^{2\pi i \frac{h}{k} - 2\pi \mathfrak{z}}\right) &= \sum_{n=-\infty}^{\infty} e^{\left(2\pi i \frac{h}{k} - 2\pi \mathfrak{z}\right) n^2} \\
&= \sum_{j=0}^{k-1} e^{2\pi i \frac{h}{k} j^2} \sum_{n\equiv j \pmod{k}} e^{-2\pi \mathfrak{z} n^2} \\
&= \sum_{j=0}^{k-1} e^{2\pi i \frac{h}{k} j^2} \sum_{q=-\infty}^{\infty} e^{2\pi \mathfrak{z} k^2 (q+\frac{j}{k})^2},
\end{aligned}$$

where we have written $n = kq + j$. We can now handle this from our $\mathscr{V}$-series formula. We proved (Lecture 12) that

$$\frac{C(\tau)}{i} \mathscr{V}_3(\mathscr{V}/-\frac{1}{\tau}) = e^{\pi i \tau \mathscr{V}^2} \mathscr{V}_3(\mathscr{V}\tau/\tau)$$

and $$\frac{C(\tau)}{i} = \sqrt{\frac{i}{\tau}}, Im\tau > 0.$$

Since **285**

$$\mathscr{V}_3(\mathscr{V}/\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n \mathscr{V}},$$

writing $\tau = it$, $\mathscr{R}_e t > 0$, we have from the above,

$$\frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi \frac{n^2}{t}} e^{2\pi i n \mathscr{V}} = e^{-\pi t \mathscr{V}^2} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} e^{-2\pi \mathscr{V} n t}$$

$$= \sum_{n=-\infty}^{\infty} e^{-\pi t(n+\mathscr{V})^2}$$

Replacing n by q, $\mathscr{V}$ by $\frac{j}{k}$ and t by $2\mathfrak{z}k^2$, we have

$$\Theta\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right) = \sum_{j=0}^{k-1} e^{2\pi i\frac{h}{k}j^2} \frac{i}{\sqrt{2\mathfrak{z}k^2}} \sum_{q=-\infty}^{\infty} e^{-\frac{\pi q^2}{2\mathfrak{z}k^2}} e^{2\pi iq\frac{j}{k}}$$

$$= \frac{1}{k\sqrt{2\mathfrak{z}}} \sum_{q=-\infty}^{\infty} e^{-\frac{\pi q^2}{2\mathfrak{z}k^2}} T_q(h,k)$$

where $$T_q(h,k) = \sum_{j=0}^{k-1} e^{2\pi i\frac{hj^2+qj}{k}}$$

This is already a good reduction. $T_q(h,k)$ depends on q modulo k, so it is periodic. We shall approximate to it in general.

One special case, however, is of interest: for $q = 0$, **286**

$$T_0(h,k) = \sum_{j=0}^{k-1} e^{2\pi i\frac{h}{k}j^2} = G(h,k),$$

where $G(h,k)$ are the so-called Gaussian sums which we shall study in detail. They are sums of roots of unity raised to a square power, Θ is actually a $\mathscr{V}_3$, and when we evaluate T_q we get some other $\mathscr{V}$.

We now write

$$\Theta\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right) = \frac{1}{k\sqrt{2\mathfrak{z}}}\{G(h,k) + H(h,k;\mathfrak{z})\}$$

where $$H(h,k;\mathfrak{z}) = \sum_{\substack{q=-\infty \\ q\neq 0}}^{\infty} T_q(h,k)e^{-\frac{\pi q^2}{2k^2\mathfrak{z}}}$$

We shall throw H into the error term. Let us appraise $T_q(h,k)$, not explicitly: that will take us into Gaussian sums.

$$T_q(h,k) = \sum_{j=0}^{k-1} e^{2\pi i\frac{hj^2+qj}{k}}$$

$$|T_q(h,k)|^2 = \sum_{j=0}^{k-1}\sum_{\ell=0}^{k-1} e^{\frac{2\pi i}{k}(hj^2+qj)} e^{-2\pi\frac{i}{k}(h\ell^2+q\ell)}$$

$$= \sum_{j \mod k} \sum_{\ell \mod k} e^{2\pi \frac{i}{k}(h(\ell j^2 - \ell^2) + q(j-\ell))}$$
$$= \sum_{j \mod k} \sum_{\ell \mod k} e^{2\pi \frac{i}{k}(j-\ell)(h(j+\ell)+q)},$$

which, an rearranging according to the difference $j - \ell$, becomes **287**

$$= \sum_{a \mod k} \sum_{j-\ell \equiv a \pmod k} e^{2\pi i \frac{a}{k}(h(j+\ell)+q)}$$
$$= \sum_{a \mod k} \sum_{\ell \mod k} e^{2\pi i \frac{a}{k}(h(a+2\ell)+q)}$$
$$= \sum_{a \mod k} e^{2\pi \frac{i}{k}(ha^2+aq)} \sum_{\ell \mod k} e^{4\pi i a \frac{h}{k}\ell}$$

The inner sum is a sum of the roots of unity. Two cases arises, according as $k \mid 2a$ or $k \nmid 2a$. k odd implies that $a = 0$ and k even implies that $a = 0$ or $k \mid 2$. In case $k \mid 2a$, the sum is zero. We then have

$$|T_q(h,k)|^2 = k, \text{ if } k \text{ is odd}; k\left(1 + e^{2\pi \frac{i}{k}\left(h\frac{k^2}{4} + \frac{k}{2}q\right)}\right), \text{ if } k \text{ is even}$$
$$= k\left(1 + e^{\pi i\left(\frac{hk}{2}+q\right)}\right), \quad \text{if } k \text{ is even}$$
$$= o \text{ or } 2k \qquad \text{if } k \text{ is even}$$

It is of interest to notice that $T_q = 0$ only if k is even and $\frac{hk}{2} + q$ is an odd integer. In any case,

$$|T_q(h,k)| \leq \sqrt{2k},$$

and this cannot be improved. We then have

$$|H(h,k;k;\mathfrak{z})| \leq 2\sum_{q=1}^{\infty} \sqrt{2k} e^{-\frac{\pi q^2}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}} (q = 0 \text{ is not involved here}).$$
$$= 2\sqrt{2k} e^{-\frac{\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}} \sum_{q=1}^{\infty} e^{-\pi\frac{(q^2-1)}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}}$$
$$= 2\sqrt{2k} e^{-\frac{\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}} \sum_{m=0}^{\infty} e^{-\frac{3\pi m}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}}$$
$$= 2\sqrt{2k} e^{-\frac{\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}} \frac{1}{1 - e^{-\frac{3\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}}}$$

Since $\mathfrak{z} = \delta_N - i\varphi$, **288**

$$\frac{1}{k^2}\mathscr{R}\frac{1}{\mathfrak{z}} = \mathscr{R}\frac{1}{k^2\mathfrak{z}} = \mathscr{R}\frac{1}{k^2(\delta_N - i\varphi)} = \frac{\delta_N}{k^2(\delta_N^2 + \varphi^2)}$$

$$\therefore \quad \frac{1}{k^2}\mathscr{R}\frac{1}{\mathfrak{z}} \geq \frac{\delta_N}{k^2\delta_N^2 + \frac{1}{N^2}}, \quad \text{since } |\mathscr{V}_{hk}| \leq \frac{1}{kN},$$

$$\geq \frac{\delta_N}{N^2\delta_N^2 + \frac{1}{N^2}} = \frac{1}{N^2\delta_N + \frac{1}{N^2\delta_N}}$$

We want to make this keep away from 0 as far as possible. This gives a desirable choice of δ_N. Make the denominator as small as possible. Since $x + \frac{1}{x}$ is minimised when $x = 1$, we have

$$\frac{1}{k^2}\mathscr{R}\frac{1}{\mathfrak{z}} \geq \frac{1}{2},$$

this minimum corresponding to $N^2\delta_N = 1$. So if we choose the radius of the circle in terms of the Farey order, we shall have secured the best that we can:

$$|H(h, k; \mathfrak{z})| \leq 2\sqrt{2k}e^{-\frac{\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}} \times C$$

It would be unwise to appraise the remaining exponential now.

Lecture 34

We had discussed the sum $\Theta\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right)$ and written it equal to **289**

$$\frac{1}{k\sqrt{2\mathfrak{z}}}\{G(h,k)+H(h,k;\mathfrak{z})\}$$

where
$$|H(h,k;\mathfrak{z})| < C\sqrt{k}e^{-\frac{\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}}$$

If we apply this to the integral in which Θ^r appears,

$$\Theta\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right)^r = \frac{1}{k^r(2\mathfrak{z})^{\frac{k}{2}}}\sum_{\lambda=0}^{r}\binom{r}{\lambda}G(h,k)^{r-\lambda}H(h,k;\mathfrak{z}),$$

or, keeping the piece corresponding to $\lambda = 0$ apart,

$$\Theta\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right)^r - \frac{1}{k^r(2\mathfrak{z})^{\frac{r}{2}}}G(h,k)^r = \frac{1}{k^r(2\mathfrak{z})^{\frac{r}{2}}}\sum_{\lambda=1}^{r}\binom{r}{\lambda}G(h,k)^{r-\lambda}H(h,k;\mathfrak{z})^{\lambda}$$

Let us appraise this. Since

$$\left|\Theta\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right)^r - \frac{1}{k^r(2\mathfrak{z})^{\frac{r}{2}}}G(h,k)^r\right| < C\frac{1}{k^r|\mathfrak{z}|^{\frac{r}{2}}}\sum_{\lambda=1}^{r}(\sqrt{k})^{r-\lambda}k^{\frac{\lambda}{2}}e^{-\frac{2\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}}$$
$$< C\cdot\frac{1}{(k|\mathfrak{z}|)^{\frac{r}{2}}}e^{-\frac{\pi}{2}}e^{-\frac{\pi}{2k^2}\mathscr{R}\frac{1}{\mathfrak{z}}}$$

Now **290**

$$A_r(n) = \sum_{0\le h<k\le N}{}' e^{-2\pi i\frac{h}{k}n}\int_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} 2\pi\mathfrak{z}n\Theta\left(e^{2\pi i\frac{h}{k}n-2\pi\mathfrak{z}}\right)$$

where, of course, $\mathfrak{z} = \delta_N - i\varphi$. Hence

$$\left| A_r(n) - \sum_{0\le h<k\le N}{}' e^{-2\pi i \frac{h}{k} n} \int_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} \frac{e^{2\pi n \mathfrak{z}}}{k^r (2\mathfrak{z})^{\frac{r}{2}}} G(h,k)^r d\varphi \right|$$

$$\le C \sum_{0\le h<k\le N}{}' \int_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} e^{2\pi n \mathscr{R}\mathfrak{z}} \frac{e^{-\frac{2\pi}{k^2}\mathscr{R}\frac{1}{\mathfrak{z}}}}{k^{r/2}|\mathfrak{z}|^{r/2}} d\varphi$$

$$\le C \sum_{0\le h<k\le N}{}' e^{2\pi n \delta_N} \int_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} \frac{e^{-\frac{\pi}{2k^2}\frac{\delta\nu}{\delta_N^2+\varphi^2}}}{\left[h^2(\delta_N^2+\varphi^2)\right]^{\frac{r}{4}}} d\varphi$$

$$= C \sum_{0\le h<k\le N}{}' e^{2\pi n \delta_N} \delta_N^{-\frac{r}{4}} \int_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} \left(\frac{\delta_N}{k^2(\delta_N^2+\varphi^2)}\right)^{\frac{r}{4}} e^{-\frac{\pi}{2k^2}\frac{\delta_N}{\delta_N^2+\varphi^2}} d\varphi$$

Now $\frac{1}{2kN} \le \mathscr{V}_{hk} \le \frac{1}{kN}$ and $\mathscr{V}'_{hk} \le \varphi \le \mathscr{V}''_{hk}$, while $\delta_N = \frac{1}{N^2}$. Putting **291**
$X = \frac{\delta_N}{k^2(\delta_N^2+\varphi^2)}$, the integrand becomes $X^{\frac{r}{4}} e^{-\frac{\pi}{2}X}$ which remains bounded. (It was for this purpose that in our estimate of $H(h,k;\mathfrak{z})$ earlier we retained the factor $e^{-\pi/(2k^2)\cdot\mathscr{R}\frac{1}{\mathfrak{z}}}$). Hence the last expression is less than or equal to

$$C \sum_{o\le h<k\le N} e^{2\pi\frac{n}{N^2}} N^{\frac{r}{2}} \int_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} d\varphi = C e^{2\pi\frac{n}{N^2}} N^{\frac{r}{2}},$$

since the whole Farey dissection exactly fills the interval $(0,1)$.

In the next stage of our argument we take the integral

$$\int_{\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} \frac{e^{2\pi n \mathfrak{z}}}{\mathfrak{z}^{\frac{k}{4}}}$$

and write it as

$$\left(\int_{-\infty}^{\infty} - \int_{\mathscr{V}''_{hk}}^{\infty} - \int_{-\infty}^{-\mathscr{V}'_{hk}} \right) \frac{e^{2\pi n \mathfrak{z}}}{\mathfrak{z}^{r/2}} d\varphi$$

The infinite integrals are conditionally convergent if $r > 0$ (because the numerator is essentially trigonometric), and absolutely convergent for $r > 2$, **292** so that we take r at least equal to 3. Then

$$\left|\int\limits_{\mathscr{V}''_{hk}}^{\infty} \frac{e^{2\pi n\mathfrak{z}}}{\mathfrak{z}^{r/2}} d\varphi\right| \le e^{2\pi\frac{n}{N^2}} \int\limits_{\mathscr{V}''_{hk}}^{\infty} \frac{d\varphi}{(\delta_N^2+\varphi^2)^{\frac{r}{2}}}$$

$$\le e^{2\pi\frac{n}{N^2}} \int\limits_{\frac{1}{2kN}}^{\infty} \frac{d\varphi}{(\delta_N^2+\varphi^2)^{\frac{r}{4}}}$$

(Here and in the estimate of the other integral $\int\limits_{-\infty}^{-\mathscr{V}'_{hk}}$, we make use of the fact that the interval from $\mathscr{V}'_{hk}$ to $\mathscr{V}''_{hk}$ is neither too long nor too short. This argument arises also in Goldbach's problem and Waring's problem). The right side is equal to

$$N^{r-2}e^{2\pi\frac{n}{N^2}} \int\limits_{-\frac{1}{2kN}}^{\infty} \frac{N^2 d\varphi}{(1+N^4\varphi^2)^{\frac{r}{4}}} = e^{2\pi\frac{n}{N^2}} N^{r-2} \int\limits_{\frac{N}{2k}}^{\infty} \frac{d\psi}{(1+\psi^2)^{r/4}}$$

$$< e^{2\pi\frac{n}{N^2}} N^{r-2} \int\limits_{\frac{N}{2k}}^{\infty} \frac{d\psi}{\psi^{r/2}}$$

This appears crude but is nevertheless good since φ never comes near 0; $N/2k > \frac{1}{2}$, and the ratio of ψ^2 to $1+\psi^2$ is at least $\frac{1}{3}$ and so we lose no essential order of magnitude. The last integral is equal to

$$Ce^{2\pi\frac{n}{N^2}} N^{r-2} \left(\frac{N}{2k}\right)^{-\frac{r}{2}+1}, r \ge 3,$$
$$= Ce^{2\pi\frac{n}{N^2}} N^{\frac{r}{2}-1} k^{\frac{r}{2}-1}$$

A similar estimate holds for $\int\limits_{-\infty}^{-\mathscr{V}'_{hk}}$ also. So, 293

$$\left|A_r(n) - \sum_{0\le h<k\le N}{}' e^{-2\pi i\frac{h}{k}n} \left(\frac{G(h,k)}{k\sqrt{2}}\right)^r \int\limits_{-\infty}^{\infty} \frac{e^{2\pi n\mathfrak{z}}}{\mathfrak{z}^{r/2}} d\varphi\right|$$
$$< Ce^{2\pi\frac{n}{N^2}} N^{r/2} + C \sum_{0\le h<k\le N}{}' \frac{1}{k^{r/2}} e^{2\pi\frac{n}{N^2}} N^{r/2-1} k^{r/2-1}$$

$$< Ce^{2\pi\frac{n}{N^2}}N^{r/2} + Ce^{2\pi\frac{h}{N^2}}N^{r/2-1}\sum_{0\le k\le N}{}' $$
$$= Ce^{2\pi\frac{n}{N^2}}N^{r/2}.$$

This, however, does not go to zero as $N\to\infty$; we have no good luck here as we had in partitions. So we make the best of it, and obtain an asymptotic result. Let n also tend to infinity. We shall keep n/N^2 bounded, without lotting; it go to zero, as in the latter case the exponential factor would become 1. We have to see to it that $n\le CN^2$ i.e., N is at least $\sqrt{n}$. Otherwise the error term would increase fast. Making N bigger would not help in the first factor and would **294** make the second worse. So the optical choice for N would be $N=[\sqrt{N}]$. The error would now be

$$O\left(n^{\frac{r}{4}}\right)$$

We next evaluate the integral

$$\int_{-\infty}^{\infty}\frac{e^{2\pi n\mathfrak{z}}}{\mathfrak{z}^{r/2}}d\varphi$$

This is the some as

$$\int_{-\infty}^{\infty}\frac{e^{2\pi n(\delta_N-i\varphi)}}{(\delta_N-i\varphi)^{r/2}}d\varphi = -\int_{\infty}^{-\infty}\frac{e^{2\pi n(\delta_N+i\alpha)}}{(\delta_N+i\alpha)^{r/2}}d\varphi$$
$$= \frac{1}{i}\int_{\delta_N-i\infty}^{\delta_N+i\infty}\frac{e^{2\pi ns}}{s^{r/2}}ds$$

After a little embellishment this becomes a well-known integral. It is equal to

$$\frac{(2\pi n)^{r/2}}{i}\int_{2\pi n\delta_N-i\infty}^{2\pi n\delta_N+i\infty}\frac{e^{\omega}}{\omega^{r/2}}d\omega$$

which exists for $r>2$, and is actually the Hankel loop integral, and hence equal to

$$\frac{2\pi(2\pi n)^{r/2}-1}{\Gamma(r/2)}$$

Hence, for $f\ge 3$. We hence the number of representations of n as the sum **295** of r squares:

$$A_r(n) = \frac{(2\pi)^{r/2}}{\Gamma(r/2)}\cdot\frac{n^{\frac{r}{2}-1}}{2^{r/2}}\cdot\sum_{0\le h<k\le N}{}'\frac{G(h,k)^r}{k^r}e^{-2ri\frac{h}{k}} + O(n^{r/4}).$$

One final step. Let us improve this a little further. Write

$$\sum_{h \mod k} \frac{G(h,k)^r}{k^r} e^{-2\pi i \frac{h}{k} n} = V_k^{(r)}(n) = V_k(n)$$

We have to sum $V_k(n)$ from $k = 1$ to $k = N$. However, we sum from $k = 1$ to $k = \infty$, thereby incurring an error

$$\left| \sum_{k=N+1}^{\infty} V_k(n) \right| \leq \sum_{k=N+1}^{\infty} k^{-\frac{r}{2}+1},$$

and this converging absolutely for $r \geq 5$ is

$$O\left(N^{-\frac{r}{2}+2}\right) = O\left(n^{-\frac{r}{4}+1}\right)$$

This along with the factor $n^{\frac{r}{2}-1}$ would give exactly $O(n^{r/4})$. (We could have saved this for $r = 4$ also if we had been a little more careful). Thus, for $r \geq 5$, we have 296

$$A_r(n) = \frac{\pi^{r/2}}{\Gamma(r/2)} n^{\frac{r}{2}-1} S_r(n) + O(n^{r/4}),$$

where

$$S_r(n) = \sum_{k=1}^{\infty} V_k(n)$$

$S_r(n)$ is the singular series. We shall show that $S_r(n)$ remains bounded at least for $r \geq 5$.

Lecture 35

After we reduced our problem to the singular series in which the Gaussian sums appear conspicuously, we have to do something about them before we proceed further. The Gaussian sums are defined as 297

$$G(h,k)=\sum_{\ell \mod k} e^{2\pi i\frac{h}{k}\ell^2}, (h,k)=1$$

They obey a simple multiplication rule: if $k=k_1k_2$, $(k_1,k_2)=1$, then

$$G(h,k_1k_2)=G(hk_1,k_2)\cdot G(hk_2,k_1).$$

For, put $\ell = rk_1+sk_2$; when r runs modulo k_2 and s modulo k_1, ℓ runs through a full residue system modulo k_1k_2. Hence

$$\begin{aligned} G(h,k_1k_2) &= \sum_{k \mod k_2}\sum_{s \mod k_1} e^{2\pi i\frac{h}{k_1k_2}(k_1r+k_2s)^2}\\ &= \sum_{r \mod k_2}\sum_{s \mod k_1} e^{2\pi i\frac{h}{k_1k_2}(k_1^2r^2+k_2^2s^2)}\\ &= \sum_{r \mod k_2} e^{2\pi i\frac{hk_1}{k_2}r^2}\sum_{s \mod k_1} e^{2\pi i\frac{hk_2}{k_1}s^2}\\ &= G(hk_1,k_2)G(hk_2,k_1). \end{aligned}$$

Ultimately, therefore, only prime powers have to be considered to denominators. We have to distinguish the cases $p=2$ and $p>2$, p prime.

1) Let $p\geq 3$, $k=p^\alpha$ with $\alpha>1$ 298

$$G(h,p^\alpha)=\sum_{l \mod p^r} e^{2\pi i\frac{h}{p^\alpha}\ell^2}$$

write $$\ell = mp^{\alpha-1} + r;$$

$m = 0, 1, \ldots, p-1$; $r = 0, 1, \ldots p^{\alpha-1} - 1$. Then this becomes

$$\sum_{m=0}^{p-1}\sum_{r=0}^{p^{\alpha-1}-1} e^{2\pi i \frac{h}{p^\alpha}(mp^{\alpha-1}+r)^2} = \sum_{m=0}^{p-1}\sum_{r=0}^{p^{\alpha-1-1}} e^{2\pi i \frac{h}{p^\alpha}(m^2p^{2\alpha-1}+2mrp^{\alpha-1}+r^2)}$$

Since $\alpha \geq 2$, $2\alpha - 2 \geq \alpha$ and so the first term in the exponent may be omitted. This gives

$$\sum_{r=0}^{p^{\alpha-1}-1} e^{2\pi i \frac{h}{p^\alpha} r^2} \sum_{m=0}^{p-1} e^{2\pi i \frac{h}{p} 2mr}$$

The inner sum is a sum of p^{th} roots of unity; so it depends on whether p divides $2rh$ or not. But $(h, p) = 1$ and $p \nmid 2$. So we need consider only the cases: $p \mid r$ and $p \nmid r$. However in the latter case this sum is 0 while in the former it is p. We therefore get, when $p \mid r$, $r = ps$,

$$\begin{aligned} p\sum_{r=0, p|r}^{p^{\alpha-1}-1} e^{2\pi i \frac{h}{p^\alpha} r^2} &= p \sum_{s=0}^{p^{\alpha-2}-1} e^{2\pi i \frac{h}{p^\alpha} p^2 s} \\ &= p \sum_{s=0}^{p^{\alpha-2}-1} e^{2\pi i \frac{h}{p^{\alpha-2}} s} \\ &= pG(h, p^{\alpha-2}) \end{aligned}$$

We have therefore reduced the never of the denominator by 2. We can repeat the process and proceed as long as we end with either the 0^{th} or the 1^{st} power. So we have two chances. In the former case, evidently $G(h, 1) = 1$. So for α even, **299**

$$G(h, p^\alpha) = p^{\alpha/2}$$

On the other hand, if α is odd, we have

$$G(h, p^\alpha) = p^{\frac{\alpha-1}{2}} G(h, p).$$

These may be combined into the single formula

$$G(h, p^\alpha) = p^{[\frac{\alpha}{2}]} G\left(h, p^{\alpha - 2[\frac{\alpha}{2}]}\right) \tag{1}$$

2) $p = 2^\lambda$, $\lambda \geq 2$. h is now odd. Write

$$\ell = m2^{\lambda-1} + r;\ m = 0, 1;\ r = 0, 1, \ldots, 2^{\lambda-1} - 1$$

$$G(h,2^\lambda) = \sum_{r=0}^{2^{\lambda-1}-1} e^{2\pi i \frac{h}{2^\lambda} r^2} + \sum_{r=0}^{2^{\lambda-1}-1} e^{2\pi i \frac{h}{2^\lambda}(2^{\lambda-1}+r)^2}$$

since $\lambda \geq 2$, $2\lambda - 2 \geq \lambda$, in the second sum it is only the exponent r^2 that contributes a non-zero term; and this is then the same the first. Altogether we **300**
have then

$$2\sum_{r=0}^{2^{\lambda-1}-1} e^{\pi i \frac{h}{2^{\lambda-1}} r^2} \qquad (*)$$

This, however is not a Gaussian sum. The substitution for ℓ does not work; to be effective, then we take

$$\ell = m2^{\lambda-2} + r; m = 0,1,2,3;\ r = 0,1,\ldots,2^{\lambda-2}-1.$$

Now take $\lambda \geq 4$ and start again all over.

$$\begin{aligned}
G(h,2^\lambda) &= \sum_{m=0}^{3}\sum_{r=0}^{2^{\lambda-2}-1} e^{2\pi i \frac{h}{2^\lambda}(m2^{\lambda-2}+r)^2}\\
&= \sum_{m=0}^{3}\sum_{r=0}^{2^{\lambda-2}-1} e^{2\pi i \frac{h}{2^\lambda}(2^{\lambda-1}mr+r^2)}, \quad \text{(for } \lambda \geq 4 \text{ i.e., } 2\lambda-4\geq\lambda).\\
&= \sum_{r=0}^{2^{\lambda-2}-1} e^{2\pi i \frac{h}{2^\lambda} r^2} \sum_{m=0}^{3} e^{\pi i h m r}\\
&= \sum_{r=0}^{2^{\lambda-2}-1} e^{2\pi i \frac{h}{2^\lambda} r^2} \sum_{m=0}^{3} (-)^{mn}\\
&= 2\sum_{r=0}^{2^{\lambda-2}-1} (-)^r e^{2\pi i \frac{h}{2^\lambda} r^2} + 2\sum_{r=0}^{2^{\lambda-2}-1} e^{2\pi i \frac{h}{2^\lambda} r^2}\\
&= 4\sum_{s=0}^{2^{\lambda-3}-1} e^{\pi i \frac{h}{2^{\lambda-3}} s^2}
\end{aligned}$$

This is not Gaussian sum either. But is is of the form (*). We therefore **301**
have, for $\lambda \geq 4$, $G(h,2^\lambda) = 2G(h,2^{\lambda-2})$. If $\lambda = 4$, we need go down to only $2^2 = 4$ and if $\lambda = 5$ to $2^3 = 8$. So we need separately $G(h,8)$ and $G(h,4)$; and of course $G(h,2)$. These cases escape us, while formerly only $G(h,p)$ did. For $\lambda \geq 4$, we may write

$$G(h,2^\lambda) = 2^{\left[\frac{\lambda}{2}\right]-1} G\left(h, 2^{\lambda-2\left[\frac{\lambda}{2}+2\right]}\right) \qquad (2)$$

This supplements formula (1).

We now consider the special cases, $k = 2, 4, 8$. Here h is odd.

$$\begin{aligned}
G(h,2) &= 1 + e^{2\pi i \frac{h}{2}} = 0\\
G(h,4) &= 1 + e^{2\pi i \frac{h}{4}\cdot 1} + e^{2\pi i \frac{h}{4}\cdot 4} + e^{2\pi i \frac{h}{4}\cdot 9}\\
&= 2 + 2e^{\pi i \frac{h}{2}}\\
&= 2\left(1 + i^h\right)\\
G(h,8) &= 1 + 1 + 2e^{\pi i h} + 4e^{2\pi i \frac{h}{8}}\\
&\qquad \text{(since } 1^2, 3^2, 5^2, 7^2 \text{ are all } \equiv 1 \text{ modulo 8)}\\
&= 4e^{\pi i \frac{h}{4}} = 4\left(\frac{1+i}{\sqrt{2}}\right)^2
\end{aligned}$$

Before we return to $G(h, p)$, $p > 2$, we shall a digression an connect to the whole thing with the Legendre-Jacobi symbols 302

$$\begin{aligned}
G(h,p) &= \sum_{\ell=0}^{p-1} e^{2\pi i \frac{h}{p}\ell^2}\\
&= 1 + 2\sum_a e^{2\pi i \frac{h}{p}a},
\end{aligned}$$

the summation over all quadratic residues a modulo p, since along with ℓ, $p-\ell$ is also a quadratic residue. We can write this in a compact form, so arranging it that the non-residues get cancelled and the residues appear twice:

$$\begin{aligned}
G(h,p) &= \sum_{r \bmod p} \left\{1 + \left(\frac{r}{p}\right)\right\} e^{2\pi i \frac{h}{p}r}\\
&= \sum_{r \bmod p} \left(\frac{r}{p}\right) e^{2\pi i \frac{h}{p}r}
\end{aligned}$$

This would appear in a completely new aspect if we utilised the fact that hr runs through a full system of residues modulo p. Then

$$\begin{aligned}
G(h,p) &= \sum_{k \bmod p} \left(\frac{h}{p}\right)\left(\frac{hr}{p}\right) e^{2\pi i \frac{h}{p}r}\\
&= \left(\frac{h}{p}\right) \sum_{r \bmod p} \left(\frac{r}{p}\right) e^{2\pi i \frac{r}{p}}
\end{aligned}$$

$$= \left(\frac{h}{p}\right) G(h, p).$$

This is very useful if we new go to the Jacobi symbol. For prime p, the Legendre symbol has the multiplicative property: **303**

$$\left(\frac{r_1}{p}\right)\left(\frac{r_2}{p}\right) = \left(\frac{r_1 r_2}{p}\right)$$

Jacobi has the following generalisation.

Define $\left(\frac{r}{pq}\right)$ by

$$\left(\frac{r}{pq}\right) = \left(\frac{r}{p}\right)\left(\frac{r}{q}\right).$$

Si it is ± 1; if it is $+1$ it does not necessarily mean that r is a quadratic residue modulo pq. The Jacobi symbol no longer discriminates between residues and non residues. From the definition then

$$\left(\frac{a}{p^\alpha q^\beta \cdots}\right) = \left(\frac{a}{p}\right)^\alpha \left(\frac{a}{q}\right)^\beta \cdots .$$

The Jacobi symbol has the properties of a character, as can be verified by using the Chinese remainder theorem.

We can now write

$$G(h, p^\alpha) = \left(\frac{h}{p}\right)^\alpha G(1, p^\alpha)$$

under all circumstances. How does this come about? Separate the cases: α even, α odd.

$$\begin{aligned} G(g, p^\alpha) &= G(1, p^\alpha), && \alpha \text{ even;} \\ &= p^{\frac{\alpha-1}{2}} G(h, p), && \alpha \text{ odd,} \\ &= \left(\frac{h}{p}\right) p^{\frac{\alpha-1}{2}} \quad G(1, p) = \left(\frac{h}{p}\right) G(1, p^\alpha) \end{aligned}$$

We can write both in one sweep as **304**

$$\begin{aligned} G(h, p^\alpha) &= \left(\frac{h}{p}\right)^\alpha G(1, p^\alpha) \\ &= \left(\frac{h}{p^\alpha}\right) G(1, p^\alpha) \end{aligned}$$

Now use the multiplicative law. If p, q are odd primes, then

$$\begin{aligned} G(h, p^\alpha q^\beta) &= G(hp^\alpha, q^\beta) G(hq^\beta, p^\alpha) \\ &= \left(\frac{hp^\alpha}{q^\beta}\right) G(1, q^\beta) \left(\frac{hq^\beta}{p^\alpha}\right) G(1, p^\alpha) \end{aligned}$$

Since the Jacobi symbol is separately multiplicative in numerator and denominator, but not both, this is equal to

$$\left(\frac{h}{q^\beta}\right)\left(\frac{p^\alpha}{q^\beta}\right) G(1, q^\beta) \left(\frac{h}{p^\alpha}\right)\left(\frac{q^\beta}{p^\alpha}\right) G(1, p^\alpha) \;=\; \left(\frac{h}{q^\beta}\right)\left(\frac{h}{p^\alpha}\right) G(p^\alpha, q^\beta) G(q^\beta, p^\alpha),$$

taking the second and third factors together, and also the last two. And this is

$$\left(\frac{h}{p^\alpha q^\beta}\right) G(1, p^\alpha q^\beta)$$

according to the multiplication law.

Suppose that we have

$$G(h_1 k_1) = \left(\frac{h}{k_1}\right) G(1, k_1); G(h, k_2) = \left(\frac{h}{k_2}\right) G(1, k_2).$$

We go through the above worker; literally and get **305**

$$G(h, k_1 k_2) = \left(\frac{h}{1, h_2}\right) G(1, k_1, k_2).$$

So we have proved in general that for odd k,

$$G(h, k) = \left(\frac{h}{k}\right) G(1, k)$$

We can now return to $G(h, p)$.

Lecture 36

We were discussing Gaussian sums and it remained to evaluate 306

$$G(h,p) = \left(\frac{h}{p}\right) G(1,p)$$

We shall do a little more than that; we shall study them in a more flexible form. Define

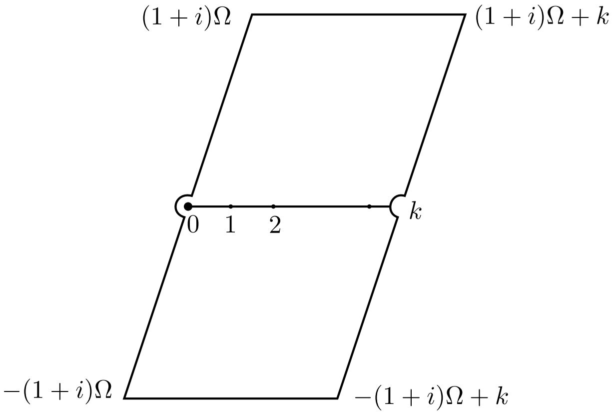
$$S(h,k) = \sum_{\ell=0}^{k-1} e^{\pi i \frac{hg}{k}\ell^2},$$

$h, k > 0$ but not necessarily coprime. We cannot now take the summation over ℓ modulo k. For if k is odd, $(\ell+k)^2 = \ell^2 + 2\ell k + k^2$ and k^2 may give rise to an odd multiple of πi in the exponent and hence introduce a change of sign, We should therefore insist on this particular range of summation. $S(h,k)$ are connected with the Gaussian sums; indeed

$$G(h,k) = S(2h,k)$$

We shall now produce $S(h,k)$ as a sum of residues. To get the integers as poles we should clearly take $e^{2\pi i\mathfrak{z}} - 1$ in the denominator; so we integrate $e^{\pi i \frac{h}{k}\mathfrak{z}^2}/(e^{2\pi i\mathfrak{z}}-1)$ over such a contour as has in its interior the desired poles $\mathfrak{z} = 0, 1, 2, \ldots, k-1$. Indeed

$$S(h,k) = \int_C \frac{e^{\pi i \frac{h}{k}\mathfrak{z}^2}}{e^{2\pi i\mathfrak{z}} - 1} d\mathfrak{z}$$



Where C is the parallelogram with vertices at $\pm(1+i)\Omega$, $\pm(1+i)\Omega+k$, **307**
with the slant sides inclined at 45° (infact this may be anything less than 90°) to the real axis, and making a detour round 0 and k. When we push Ω to ∞, the integrals along the horizontal sides will tend to zero. For instance on the upper side, $\mathfrak{z}=(1+i)\Omega+x$, $0\le x\le k$, and the integrand is therefore

$$\frac{e^{\pi i\frac{h}{k}((1+i)\Omega+x)^2}}{e^{2\pi i((1+i)\Omega+x)}-1}=\frac{e^{\pi i\frac{h}{k}(2i\Omega^2+2(1+i)\Omega x+x^2)}}{e^{2\pi i(\Omega+x)-2\pi\Omega}-1}$$
$$=\frac{e^{-\pi\frac{h}{k}(2\Omega^2+2\Omega x)+\pi i\frac{h}{k}(2\Omega x+x^2)}}{e^{-2\pi\Omega+2\pi i(\Omega+x)}-1}$$

$\to 0$ uniformly as $\Omega\to\infty$ since $\frac{h}{k}>0$. Hence the integral can be written as **308**

$$\int\limits_{-(1+i)\infty+k}^{(1+i)\infty+k}-\int\limits_{-(1+i)\infty}^{(1+i)\infty}\frac{e^{\pi i\frac{h}{k}(\mathfrak{z})^2}}{e^{2\pi\mathfrak{z}}-1}d\mathfrak{z}$$

where, of course, we have to make a small detour round 0 and k. Replacing $\mathfrak{z}$ by $\mathfrak{z}+k$ in the first integral, this becomes

$$\int\limits_{-(1+i)\infty}^{(1+i)\infty}\frac{e^{\pi i\frac{h}{k}(\mathfrak{z}+k)^2}-e^{\pi i\frac{h}{k}\mathfrak{z}^2}}{e^{2\pi i\mathfrak{z}}-1}d\mathfrak{z}=\int\limits_{-(1+i)\infty}^{(1+i)\infty}\frac{e^{\pi i\frac{h}{k}\mathfrak{z}^2}\left(e^{\pi i\frac{h}{k}(2\mathfrak{z}k+k^2)}-1\right)}{e^{2\pi i\mathfrak{z}}-1}d\mathfrak{z}$$
$$=\int\limits_{-(1+i)\infty}^{(1+i)\infty}\frac{e^{\pi\frac{h}{k}\mathfrak{z}^2}\left(e^{2\pi ih\mathfrak{z}+\pi ihk}-1\right)}{e^{2\pi i\mathfrak{z}}-1}d\mathfrak{z}$$

Let us assume from now on that hk is even. Then we can actually divide out and the integral becomes 309

$$\int_{-(1+i)\infty}^{(1+i)\infty} \left(e^{\pi i \frac{h}{k} \mathfrak{z}^2} \sum_{\lambda=0}^{h-1} e^{2\pi i \lambda \mathfrak{z}} \right) d\mathfrak{z}$$

The denominator has now disappeared. There is a further advantage that the integral can now be stretched along the whole line and the detour can be avoided. We then have

$$\sum_{\lambda=0}^{h-1} e^{-\pi i \lambda^2 \frac{h}{k}} \int_{-(1+i)\infty}^{(1+i)\infty} e^{\pi i \frac{h}{k} \left(\mathfrak{z} + \frac{\lambda k}{h}\right)^2} d\mathfrak{z}$$

Write $\mathfrak{z} + \lambda k/h = \omega$; and shift the integral back to the line from $-(1+i)\infty$ to $(1+i)\infty$ - this we can do since the integrand tends to zero along a horizontal segment. This gives

$$\sum_{\lambda=0}^{h-1} e^{-\pi i \frac{h}{k} \lambda^2} \int_{-(1+i)\infty}^{(1+i)\infty} e^{\pi i \frac{h}{k} \omega^2} d\omega,$$

or writing $t = \omega \sqrt{\frac{h}{k}}$, $\sqrt{\frac{h}{k}} > 0$, 310

$$\sqrt{\frac{k}{h}} \sum_{\lambda=0}^{h-1} e^{-\pi i \frac{h}{k} \lambda^2} \int_{-(1+i)\infty}^{(1+i)\infty} e^{\pi i t^2} dt = A \sqrt{\frac{k}{h}} \sum_{\lambda=0}^{h-1} e^{-\pi i \frac{k}{h} \lambda^2}$$

where A is the specific constant:

$$A = \int_{-(1+i)\infty}^{(1+i)\infty} e^{\pi i t^2} dt$$

Hence

$$S(h,k) = A \sqrt{\frac{k}{h}} S(-k,h).$$

In order to evaluate A, take a simple case: $h = 1$, $k = 2$

$$S(1,2) = A\sqrt{2} S(-2,1)$$

i.e.,

$$1 + e^{\frac{\pi i}{2}} = A\sqrt{2},$$

So $A = (1+i)/\sqrt{2}$, an eighth root or unity.
So our reciprocity formula becomes complete:

$$S(h,k) = \frac{1+i}{\sqrt{2}}\sqrt{\frac{k}{h}}S(-k,h).$$

Let us develop some corollaries.
1) $h = 2$, k arbitrary: **311**

$$S(2,k) = G(1,k), \text{ so}$$
$$\begin{aligned} G(1,k) = S(2,k) &= \frac{1+i}{\sqrt{2}}\sqrt{\frac{k}{2}}S(-k,2) \\ &= \frac{1+i}{2}\sqrt{k}(1+e^{-\pi i\frac{k}{2}}) \\ &= \frac{1+i}{2}\sqrt{k}(1+(-i)^k) \end{aligned}$$

We then have explicitly the value of $G(1,k)$

$$G(1,k) = \frac{(1+i)(1+(-i)^k)}{2}\sqrt{k}.$$

We mention the four cases separately:

$$G(1,k) = \begin{cases} \sqrt{k} & \text{if } k \equiv 1 \pmod 4 \\ 0 & \text{if } k \equiv 2 \pmod 4 \\ i\sqrt{k} & \text{if } k \equiv 3 \pmod 4 \\ (1+i)\sqrt{k} & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

Hence the absolute value of $G(1,k)$ can be $0, k$ or $\sqrt{2k}$.

So far k was only positive. The case k odd deserves some special mention. $k-1$ is even and

$$G(1,k) = \begin{cases} \sqrt{k} & \text{if } \frac{k-1}{2} \text{ is even} \\ i\sqrt{k} & \text{if } \frac{k-1}{2} \text{ is odd.} \end{cases}$$

$\left(\frac{k-1}{2}\right)^2 \equiv 0, 1 \pmod 4$ according as $\frac{k-1}{2}$ is even or odd; so we can write this as **312**

$$G(1,k) = i^{\left(\frac{k-1}{2}\right)^2}\sqrt{k}.$$

This we have obtained by a purely function-theoretical argument. From our arithmetical augment, we had, for odd k,

$$G(h,k) = \left(\frac{h}{k}\right) i^{\left(\frac{k-1}{2}\right)^2} \sqrt{k}$$

where $\left(\frac{h}{k}\right)$ is the Jacobi symbol. We can get a little more out of it.

$$G(-1,k) = \left(\frac{-1}{k}\right) i^{\left(\frac{k-1}{2}\right)^2} \sqrt{k}.$$

Multiplying this and the equation for $G(1,k)$ together,

$$\begin{aligned} G(1,k)G(-1,k) &= \left(\frac{-1}{k}\right)(-)^{\left(\frac{k-1}{2}\right)^2} k \\ &= \left(\frac{-1}{k}\right)(-)^{\frac{k-1}{2}} k \end{aligned}$$

But the left side is only $G(l,k)\overline{G(1,k)}$, and this is always > 0. So

$$\left(\frac{-1}{k}\right)(-)^{\frac{k-1}{2}} k > 0,$$

and since $k > 0$ by nature,

$$\left(\frac{-1}{k}\right) = (-)^{\frac{k-1}{2}}$$

which is Euler's criterion for the Jacobi symbol.

2) $h = 2$, k odd.

$$\begin{aligned} G(2,k) = S(4,k) &= \frac{1+1}{\sqrt{2}} b \sqrt{\frac{k}{4}} S(-k,4) \\ &= \frac{1+i}{2\sqrt{2}} \sqrt{k} \left\{1 + e^{-\frac{\pi ik}{4}} + e^{-\pi ik} + e^{-\frac{\pi ik}{4}}\right\} \\ &= \frac{1+i}{\sqrt{2}} \frac{1}{i} \sqrt{2}\sqrt{k} e^{-\pi i \frac{k}{4}} \\ &= e^{-\frac{\pi i}{4}(k-1)} \sqrt{k} \\ &= e^{-\frac{\pi i}{2}\frac{k-1}{2}} \sqrt{2} \\ &= i^{-\frac{k-1}{2}} \sqrt{k} \end{aligned}$$

On the other hand 313

$$G(2,k) = \left(\frac{2}{k}\right) i^{\left(\frac{k-1}{2}\right)^2} \sqrt{k}$$

Hence

$$\begin{aligned}
\left(\frac{2}{k}\right) &= i^{-\frac{k-1}{2} - \left(\frac{k-1}{2}\right)^2} \\
&= i^{-\frac{k-1}{2}\left(1+\frac{k-1}{2}\right)} \\
&= i^{-\frac{k^2-1}{4}} \\
&= i^{-2\frac{k^2-1}{8}} \\
&= (-)^{\frac{k^2-1}{8}}
\end{aligned}$$

3) $(h,k) = 1$; h, k both odd:

$$\begin{aligned}
G(h,k) = S(2h,k) &= \frac{1+i}{\sqrt{2}} \sqrt{\frac{k}{2h}} S(-k, 2h) \\
&= \frac{1+i}{\sqrt{2}} \sqrt{\frac{k}{2h}} \sum_{\lambda=0}^{2h-1} e^{\pi i \frac{k}{2h} \lambda^2} \\
&= \frac{1+i}{\sqrt{2}} \sqrt{\frac{k}{2h}} \sum_{\lambda \bmod 2h} e^{-\pi i \frac{k}{2h} \lambda^2}
\end{aligned}$$

Here it is no longer necessary to insist on the special range of summation, for changing λ by $\lambda + 2h$ would introduce only an even multiple of πi in the exponent. Separating the odd and even $\lambda' s$, this becomes 314

$$\begin{aligned}
\frac{1+i}{\sqrt{2}} \sqrt{\frac{k}{2h}} &\left\{ \sum_{\ell \bmod h} e^{-\pi i \frac{k}{2h}(2\ell)^2} + \sum_{\ell \bmod h} e^{-\pi i \frac{k}{2h}(2\ell+h)^2} \right\} \\
&= \frac{1+i}{\sqrt{2}} \sqrt{\frac{k}{2h}} \left(1 + e^{-\pi i \frac{hk}{2}}\right) \sum_{\ell \bmod h} e^{-2\pi i \frac{k}{h} \ell^2} \\
&= \frac{1+i}{\sqrt{2}} \left(1 + (-i)^{hk}\right) \sqrt{\frac{k}{2h}} G(-k, h) \\
&= i^{\left(\frac{hk-1}{2}\right)^2} \sqrt{\frac{k}{h}} G(-k,h) \\
&= i^{\left(\frac{hk-1}{2}\right)^2} \sqrt{\frac{k}{h}} \overline{G(k,h)}
\end{aligned}$$

Then we have 315

$$\left(\frac{h}{k}\right) i^{\left(\frac{k-1}{2}\right)^2}\sqrt{k} = i^{\left(\frac{hk-1}{2}\right)^2}\sqrt{\frac{k}{h}}\left(\frac{h}{k}\right) i^{-\left(\frac{h-1}{2}\right)^2}\sqrt{h}$$

i.e.,
$$\left(\frac{h}{k}\right)\left(\frac{k}{h}\right) = i^{\left(\frac{hk-1}{2}\right)^2 - \left(\frac{h-1}{2}\right)^2 - \left(\frac{k-1}{2}\right)^2}$$

$$i^b$$

where
$$\begin{aligned} b &= \frac{1}{4}\left(h^2k^2 - h^2 - k^2 + 1 - 2(hk - h - k - 1)\right) \\ &= \frac{1}{4}(h-1)(k-1)\{(h+1)(k+1) - 2\} \\ &= \frac{1}{2}\left[(h-1)(k-1)\right]\left[\frac{(h+1)(k+1)}{2} - 1\right] \end{aligned}$$

So

$$\begin{aligned} i^b &= i^{2\frac{(h-1)(k-1)}{4}\text{ an odd number}} \\ &= (-)^{\frac{(h-1)(k-1)}{4}\text{ (odd number)}} = (-)^{\frac{(h-1)(k-1)}{4}} \end{aligned}$$

$$\therefore \qquad \left(\frac{h}{k}\right)\left(\frac{k}{h}\right) = (-)^{\frac{(h-1)(k-1)}{4}}.$$

which is Jacobi's law of reciprocity.

We shall use all this in the singular series. It may be worth while to do what Gauss himself did and evaluate $G(1, k)$ by an arithmetical method. To distinguish between the different primitive roots of unity is, however, algebraically impossible; in the analytical method we can use the exponential function to uniformise the roots of unity. 316

Lecture 37

We have finished to some extent Gaussian sums; we treated then only in view of their occurrence in the singular series defined as 317

$$S^{(r)}_{(n)} = \sum_{k=1}^{\infty} V^{(r)}_k(n)$$

with
$$V^{(r)}_k(n) = V_k(n) = \sum_{\substack{h \mod k\\(h,k)=1}} \left(\frac{G(h,k)}{k}\right)^r e^{-2\pi i \frac{h}{k} n},$$

which appeared as the principal term in the expression for the number of representation of n as the sum of r squares:

$$A_r(n) = \frac{\pi^{r/2}}{\Gamma\left(\frac{r}{2}\right)} n^{\frac{r}{2}-1} S^{(r)}_{(n)} + O\left(n^{r/4}\right),$$

$r \geq 5$. We did not bother to do this for lower r, although we could for $r = 4$, in which case we know an exact formula; but this is another question. We consider first a fundamental property of the singular series, viz. its expression as an infinite product.

Fundamental Lemma.

$$S^{(r)}_{(n)} = \prod_p \left\{1 + V_p(n) + V_{p^2}(n) + V_{p^3}(n) + \cdots\right\},$$

p prime.

We first prove the multiplicative property of $V_k(n)$: for $(k_1, k_2) = 1$,

$$V_{k_1}(n) V_{k_2}(n) = V_{k_1 k_2}(n)$$

We had a similar situation in connection with $A_k(n)$ for the partition function; but there the multiplication was more complicated. Here we have 318

$$V_{k_1k_2}(n) = \frac{1}{(k_1,k_2)^r} \sum_{\substack{h \mod k_1k_2 \\ (h,k_1k_2)=1}} G(h,k_1k_2)^r e^{-2\pi i \frac{hn}{k_1k_2}}.$$

Writing $h = k_2h_1 + k_1h_2$ with the conditions $(h_1,k_1) = 1 = (h_2,k_2)$, h, running modulo h_1 and h_2 modulo k_2, this becomes

$$\begin{aligned} &\frac{1}{(k_1k_2)^r} \sum_{h_1} \sum_{h_2} G(k_2h_1 + k_1h_2, k_1k_2) e^{-2\pi i \frac{h}{k_1k_2} n} \\ &\qquad = \frac{1}{(k_1k_2)^r} \sum_{h_1} \sum_{h_2} G\left((k_2h_1 + k_1h_2)k_1, k_2\right)^r \\ &\qquad\qquad G((k_2h_1 + k_1h_2)k_2k_1)^r e^{-2\pi i (k_2h_1+k_1h_2) \frac{n}{k_1k_2}} \end{aligned}$$

on using the multiplicativity of the Gaussian sums; and suppressing multiples of k_1, k_2, as we may, this gives

$$\frac{1}{k_1^r k_2^r} \sum_{h_1 \mod k_1} \sum_{h_2 \mod k_2} G(h_2k_1^2, k_2)^r G(k_2^2h_1, k_1)^r e^{-2\pi i \frac{h_1}{k_1} n - 2\pi i \frac{h_2}{k_2} n}$$

Now 319

$$G(ha^2, h) = \sum_{\ell \mod k} e^{2\pi i \frac{h}{k} a^2\ell^2}$$

If $(a,k) = 1$, al also runs modulo k when ℓ does, so that the right side is

$$\sum_{n \mod k} e^{2\pi i \frac{h}{k} m^2} = G(h,k)$$

In our case $(k_1,k_2) = 1$. So we have

$$\begin{aligned} \frac{1}{k_1^r} \sum_{h_1 \mod k_1} G(h_1,k_1)^r e^{-2\pi i \frac{h_1}{k_1} n} \frac{1}{k_2^r} \sum_{h_2 \mod k_2} G(h_2,k_2)^r e^{-2\pi i \frac{h_2}{k_2} n} \\ = V_{k_1}(n) V_{k_2}(n) \end{aligned}$$

We can then break each summand in $S_n^{(r)}$ into factors corresponding to prime powers and multiply them again together, and the rearrangement does not count because of absolute convergence; so

$$S_{(n)}^{(r)} = \prod_p \left\{1 + V_p(n) + V_{p^2}(n) + V_{p^3}(n) + \cdots\right\}$$

$$= \prod_{p} \gamma_p(n),$$

say; this is an absolutely convergent product. This simplifies matters considerably. We have to investigate V only for those $G's$ is which prime powers appear. 320

We first take $p = 2$. then

$$\gamma_2(n) = 1 + V_2(n) + V_{2^2}(n) + \cdots$$
$$V_{2^\lambda}(n) = \frac{1}{2^{\lambda r}} \sum_{\substack{h \mod 2^\lambda \\ 2 \nmid h}} G(h, 2^\lambda)^r e^{-2\pi i h \frac{n}{2^\lambda}}$$

(i) $\lambda = 1$ Since $G(h, 2) = 0$ for odd h,

$$V_2(n) = 0$$

(ii) λ even. For $\lambda \geq 4$,

$$G(h, 2^\lambda) = 2^{\frac{\lambda}{2}-1} 2(1 + i^h) = 2^{\frac{\lambda}{2}}(1 + i^h)$$
$$V_{2^\lambda}(n) = \frac{1}{2^{\lambda r}} 2^{\lambda r/2} \sum_{\substack{h \mod 2\lambda \\ 2 \nmid h}} (1 + i^h)^r e^{-2\pi i \frac{h}{2^\lambda} n}$$
$$= \frac{1}{2^{\lambda r/2}} \left\{ \sum_{\substack{h \equiv 1 \pmod 4 \\ h \mod 2^\lambda}} (1 + i)^r e^{-2\pi i \frac{h}{2^\lambda} n} + \sum_{\substack{h \equiv - \pmod 4 \\ h \mod 2^\lambda}} (1 - i)^r e^{-2\pi i \frac{h}{2^\lambda} n} \right\}$$
$$= \frac{2^{r/2}}{2^{\lambda r/2}} \left\{ \sum_{h \equiv 1 \pmod 4} e^{\pi i \frac{r}{4}} e^{-2\pi i \frac{h}{2^\lambda} n} + \sum_{h \equiv -1 \pmod 4} e^{-\pi i \frac{r}{4}} e^{-2\pi i \frac{h}{2^\lambda} n} \right\}$$
$$= \frac{1}{2^{\frac{\lambda-1}{2} r}} \left\{ e^{\pi i \frac{r}{4} - 2\pi i \frac{r}{2^\lambda}} \sum_{s \mod 2^{\lambda-2}} e^{-2\pi i \frac{s}{2^{\lambda-2}} n} + \right.$$
$$\left. + e^{-\pi i \frac{r}{4} + 2\pi i \frac{r}{2^\lambda}} \sum_{s \mod 2^{\lambda-2}} e^{-2\pi i \frac{s}{2^{\lambda-2}} n} \right\}$$
$$= 0, \text{ if } 2^{\lambda-2} + n;$$
$$\frac{2^{\lambda-2}}{2^{\frac{\lambda-1}{2} r}} \cos\left(\pi \frac{r}{4} - 2\pi \frac{\nu}{4}\right), \text{ if } 2^{\lambda-2}/n, n = 2^{\lambda-2}.\nu$$
$$\text{i.e.,} \frac{1}{2^{(\lambda-1)(\frac{r}{4}-1)}} \cos \frac{\pi}{4}(2\nu - r)$$

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Hence, for λ even, $\lambda \geq 4$,

$$V_{2^\lambda}(n) = \begin{cases} 0, & \text{if } 2^{\lambda-2} + n; \\ \dfrac{\cos\frac{\pi}{4}(2\nu - r)}{2^{(\lambda-1)(\frac{r}{2}-1)}}, & \text{if } 2^{\lambda-2}.\nu = n. \end{cases} \qquad (*)$$

(iii) λ *odd*, $\lambda \geq 3$.

$$G(h, 2^\lambda) = 2G(h, 2^{\lambda-2}) = 2^{\frac{\lambda-3}{2}} G(h, 2^3)$$
$$= 2^{\frac{\lambda-3}{2}} 4 e^{\pi ih/4} = 2^{\frac{\lambda+1}{2}} e^{\pi ih/4}$$
$$V_{2^\lambda}(n) = \frac{1}{2^{\lambda r}} 2^{\frac{\lambda+1}{2}} \sum_{\substack{h \mod 2^\lambda \\ 2\nmid h}} e^{\pi ih\frac{r}{4}} e^{-2\pi i\frac{h}{2^\lambda}n},$$

or, writing $h = 8s + t$, $t = 1, 3, 5, 7$, **322**

$$= \frac{1}{2^{\frac{\lambda-1}{2}r}} \sum_t \sum_{s=1}^{2^{\lambda-3}} e^{\pi itr/4} e^{-2\pi i(8s+t)\frac{n}{2^\lambda}}$$
$$= \frac{1}{2^{\frac{\lambda-1}{2}r}} \sum_t e^{\pi itr/4 - 2\pi itn/2\lambda} \sum_{s=1}^{2^{\lambda-3}} e^{-2\pi isn/2^{\lambda-3}}$$
$$= 0, \text{ if } 2^{\lambda-3} \nmid n.$$

If, however, $2^{\lambda-3}|n$, $n = 2^{\lambda-3}.\nu$, this is

$$\frac{2^{\lambda-3}}{2^{\frac{\lambda-1}{2}r}} \sum_t e^{\pi it\nmid 4(r-\nu)} = o, \text{ if } 4/(r-\nu);$$
$$\frac{2^{\lambda-1}}{2^{\frac{\lambda-1}{2}r}} e^{\pi i(r-\nu)|4}, \text{ if } 4/(r-\nu)$$
$$\text{i.e.,} \quad \frac{1}{2^{(\lambda-1)(\frac{r}{2}-1)}} \cdot (-)^{\frac{\nu-r}{4}}.$$

Hence for λ odd, $\lambda \geq 3$, **323**

$$V_{2^\lambda}(n) = \begin{cases} 0, & \text{if } 2^{\lambda-3} \nmid n; \\ 0, & \text{if } 2^{\lambda-3}. \mid n, n = 2^{\lambda-3}\nu, 4 \nmid (\nu - r); \\ \dfrac{(-)^{\frac{\nu-r}{4}}}{2^{(\lambda-1)(\frac{r}{2}-1)}}, & \text{if } 2^{\lambda-3} \mid n, 4 \mid (\nu - r) \end{cases} \qquad (**)$$

Now, given n, only a finite number of powers of 2 can divide it. So the situation $2^{\lambda-3}/n$ will occur sometime or the other, so that $\gamma_2(n)$ is always a finite sum.

$$\begin{aligned}|\gamma_2(n)-1| &\leq \sum_{\lambda=2}^{\infty}\frac{1}{2^{(\lambda-1)(\frac{r}{2}-1)}}\\ &= \frac{1}{2^{\frac{r}{2}-1}}\cdot\frac{1}{1-1/2\frac{r}{2}-1}\\ &= \frac{1}{2^{r/2-1}-1};\end{aligned}$$

and this is valid for $r \geq 3$. so the singular series behaves much better than we expected.

Lecture 38

It would be of interest to study $\gamma_2(n)$ also for $r = 3, 4$. **324**

$$\gamma_2(n) = 1 + V_2(n) + V_{2^2}(n) + \cdots$$

First consider the case $r = 3, 2/n$. Then

$$V_2(n) = 0.$$

For $V_{2^\lambda}(n)$, $\lambda > 1$, we have to make a distinction between λ even and λ odd.
λ even.

$$V_{2^\lambda}(n) = \begin{cases} 0, & \text{if } 2^{\lambda-2} \nmid n; \\ \frac{\cos\frac{\pi}{4}(2\nu - r)}{2^{(\lambda-1)(\frac{r}{2}-1)}}, & \text{if } 2^{\lambda-2} \nmid n, n = 2^{\lambda-2}.\nu. \end{cases}$$

λ odd.

$$V_{2^\lambda}(n) = \begin{cases} 0, & \text{if } 2^{\lambda-3} \nmid n; \\ 0, & \text{if } 2^{\lambda-3} \mid n, n = 2^{\lambda-3}\nu, \nu - r \not\equiv 0 \pmod 4 \\ \frac{(-)^{\frac{\nu-r}{4}}}{2^{(\lambda-1)(\frac{r}{2}-1)}}, & \text{if } 2^{\lambda-3} \mid n, n = 2^{\lambda-3}\nu, \nu - r \equiv 0 \pmod 4 \end{cases}$$

So for $r = 3$,

$$\begin{aligned} \gamma_2(n) &= 1 + V_4(n) + V_8(n) \\ &= 1 + \frac{\cos\frac{\pi}{4}(2n-3)}{\sqrt{2}} + \frac{(-)^{\frac{n-3}{4}}}{2}, \end{aligned}$$

where the last summand has to be replaced by 0 if $(n-3)/4$ is not an integer. 325
Since $2n-3$ is odd, we have

$$|\cos\frac{\pi}{4}(2n-3)| = \frac{1}{\sqrt{2}},$$

and thus clearly,

$$|\gamma_2(n)| \leq 1 + \frac{1}{2} + \frac{1}{2} = 2$$

Moreover, $\gamma_2(n)$ can vanish. This would require

$$(-)^{\frac{n-3}{4}} = 1$$

and
$$\cos\frac{\pi}{4}(2n-3) = -\frac{1}{\sqrt{2}}$$

simultaneously. But this is the case for

$$n \equiv 7 \pmod 8,$$

as is easily seen. This corresponds to the fact that a number n, $n \equiv 7 \pmod 8$ cannot be represented as the sum of three squares.

Next take $r = 4$. We distinguish between the cases $2 \nmid n$ and $2 \mid n$.

1. $2 \nmid n$. Then from relations (*) and (**) proved in lecture 37, we have

$$\begin{aligned}\gamma_2(n) &= 1 + V_4(n) + V_8(n)\\ &= 1 + \frac{\cos\frac{\pi}{4}(2n-4)}{2} = 1 - \frac{1}{2}\cos\frac{\pi n}{2}\\ &= 1\end{aligned}$$

2. $2 \mid n$ Let $n = 2^\alpha n'$, $2 \mid n'$. Then (*) and (**) show that $V_{2^\lambda}(n) = 0$ for $\lambda > \alpha + 3$. But actually $V_{2^\lambda}(n) = 0$ also for $\lambda = \alpha + 3$. Indeed, for α odd, $\lambda = \alpha + 1$ is the last even, $\lambda = \alpha + 2$ the last odd index for non-vanishing $V_{2^\lambda}(n)$. For α even, $\lambda = \alpha + 2$ is the last even index: $\lambda = \alpha + 3$ is odd and since $4 \nmid (n' - 4)$, we have also $V_{2^\lambda}(n) = 0$ for $\lambda = \alpha + 3$. 326

$$\therefore \qquad \gamma_2(2^\alpha n') = 1 + \sum_{\lambda=2}^{\alpha+2} V_{2^\lambda}(n)$$

Now, in $V_{2^\lambda}(n)$, for λ even,

$$\cos\frac{\pi}{4}(2\nu - r) = -\cos\frac{\pi}{2}n' 2^{\alpha-\lambda+2}$$

$$= -\cos \pi n' 2^{\alpha-\lambda+1}$$

$$= \begin{cases} -1, & \text{for } \lambda \le \alpha, \\ 1, & \text{for } \lambda = \alpha+1, \\ 0, & \text{for } \lambda = \alpha+2. \end{cases}$$

Similarly in $V_{2^\lambda}(n)$, for λ odd,

$$(-)^{\frac{\nu-4}{4}} = -(-)^{n^1.2^{\alpha-\lambda+1}}$$

$$= \begin{cases} -1, & \text{for } \lambda \le \alpha; \\ 1, & \text{for } \lambda = \alpha+1, \end{cases}$$

and $V_{2^\lambda}(n) = 0$ for $\lambda = \alpha + 2$ since then $4 \nmid 2^{\alpha-\lambda+1}$. The numerators of the non-vanishing $V_{2^\lambda}(n)$ are -1 upto the last one, which is 1. And thus 327

$$\gamma_2(2^\alpha n') = 1 - \frac{1}{2} - \frac{1}{2^2} - \cdots - \frac{1}{2^{\alpha-1}} + \frac{1}{2^\alpha}$$
$$= \frac{1}{2^{\alpha-1}} + \frac{1}{2^\alpha} = \frac{3}{2^\alpha}$$

Although here $\gamma_2(2^\alpha n') > 0$, we see that for α sufficiently large $\gamma_2(n)$ can come arbitrarily close to 0.

We now consider $\gamma_p(n)$ for $p \ge 3$.

$$\gamma_p(n) = 1 + V_p(n) + V_{p^2}(n) + \cdots,$$

where
$$V_{p^\lambda}(n) = \frac{1}{p^{\lambda r}} \sum_{\substack{h \bmod p^\lambda \\ p \nmid h}} G(h, p^\lambda)^r e^{-2\pi i \frac{h}{p^\lambda} n}$$

Now

$$G(h, p^\lambda) = \left(\frac{h}{p^\lambda}\right) G(1, p^\lambda)$$
$$= \left(\frac{h}{p}\right)^\lambda i^{\left(\frac{p^\lambda-1}{2}\right)^2} p^{\frac{\lambda}{2}}$$

$$\therefore \quad V_{p^\lambda}(n) = \frac{i^{r\left(\frac{p^\lambda-1}{2}\right)^2}}{p^{\lambda r/2}} \sum_{\substack{h \bmod p^\lambda \\ p \nmid h}} \left(\frac{h}{p}\right) e^{-2\pi i \frac{h}{p^\lambda} n}$$

We have to distinguish between λr odd; and λr even **328**
1) λr *even*. If $p^{\lambda} \equiv 1 \pmod 4$, then

$$(-)^{\frac{r}{2}\left(\frac{p^{\lambda}-1}{2}\right)^2} = (-)^{\frac{r}{2}\frac{p^{\lambda}-1}{2}}$$

So

$$V_{p^{\lambda}}(n) = \frac{i^{r\left(\frac{p^{\lambda}-1}{2}\right)^2}}{p^{\lambda r/2}} \sum_{\substack{h \mod p^{\lambda} \\ p \nmid h}} e^{-2\pi i \frac{h}{p^{\lambda}} n}$$

2) λr *odd*. In this case

$$V_{p^{\lambda}}(n) = \frac{i^{r\left(\frac{p^{\lambda}-1}{2}\right)^2}}{p^{\lambda r/2}} \sum_{\substack{h \mod p^{\lambda} \\ p \nmid h}} \left(\frac{h}{p}\right) e^{-2\pi i \frac{h}{p^{\lambda}} n}$$

The inner sum here is a special case of the so-called Ramanujan sums:

$$C_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{2\pi i \frac{h}{k} n}$$

There sums can be evaluated. Look at the simpler sums

$$S_k(n) = \sum_{\lambda \mod k} e^{2\pi i \frac{\lambda}{k} n}$$
$$= \begin{cases} k, & \text{if } k \mid n; \\ 0, & \text{if } k \nmid n. \end{cases}$$

Classify the $\lambda' s$ in $S_k(n)$ according to their common divisor with k. Then **329**

$$S_k(n) = \sum_{d|k} \sum_{\substack{\lambda \mod k \\ (\lambda,k)=d}} e^{2\pi i \frac{\lambda}{k} n}$$
$$= \sum_{d|k} \sum_{\substack{\lambda \mod k \\ \left(\frac{\lambda}{k}, \frac{k}{d}\right)=1}} e^{2\pi i \frac{\lambda}{d} \cdot \frac{n}{k/d}}$$
$$= \sum_{d|k} \sum_{\substack{\mu \mod \frac{k}{d} \\ \left(\mu, \frac{k}{d}\right)=1}} e^{2\pi i \frac{\mu n}{k/d}}$$

$$= \sum_{d|k} C_{\frac{k}{d}}(n)$$
$$= \sum_{d|k} C_d(n).$$

Now by Möbious inversion formula,

$$C_k(n) = \sum_{d|k} \mu\left(\frac{k}{d}\right) S_d(n),$$

and $S_d(n)$ is completely known- it is either 0 or d; hence

$$C_k(n) = \sum_{d|k, d|n} d\mu\left(\frac{k}{d}\right)$$
$$= \sum_{d|(n,k)} d\mu\left(\frac{k}{d}\right).$$

So these are integers. 330

The Möbious function which appears here arises as a coefficient in a certain Dirichlet series; in fact

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

It is possible to build up a complete formal theory of Dirichlet series as we had in the case of power series. Formal Dirichlet series form a ring without null-divisors. The multiplication law is given by

$$\sum \frac{a_n}{n^2} \sum \frac{b_n}{n^2} = \sum \frac{c_n}{n^s}$$

where $$c_n = \sum_{kj=n} a_k b_j$$

The relation

$$\sum \frac{\mu(n)}{n^s} \sum \frac{1}{n^s} = 1$$

then implies that $$0 = \sum_{jk=n} \mu(j) \cdot 1 = \sum_{d|n} \mu(d), n > 1.$$

Lecture 39

For $p \geq 3$ we had 331

$$\gamma_p(n) = 1 + V_p(n) + V_{p^2}(n) + \cdots$$

where

$$V_{p^\lambda}(n) = \frac{1}{p^{\lambda r}} \sum_{\substack{h \mod p^\lambda \\ (h,p)=1}} G(h, p^\lambda)^r e^{-2\pi \frac{h}{p^\lambda} n}$$

$$= \frac{i^{\left(\frac{p^\lambda - 1}{2}\right)^{2r}}}{p^{\lambda r/2}} \begin{cases} \sum\limits_{\substack{h \mod p^\lambda \\ p \nmid h}} e^{-2\pi i \frac{h}{p^\lambda} n}, & \lambda r \text{ even;} \\ \sum\limits_{\substack{h \mod p^\lambda \\ p \nmid h}} \left(\frac{h}{p}\right) e^{-2\pi i \frac{h}{p^\lambda} n}, & \lambda r \text{ odd.} \end{cases}$$

For λr odd we have to evaluate this directly. If λr is even it is simpler; it is a special case of the Ramanujan sums:

$$C_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{2\pi i \frac{h}{k} n}$$

which could be evaluated by means of the Möbious inversion formula:

$$C_k(n) = \sum_{d|(k,n)} d\mu\left(\frac{k}{d}\right)$$

So if k is a prime power, $k = p^\lambda$, 332

$$\sum_{\substack{h \mod p^\lambda \\ p \nmid h}} e^{-2\pi i \frac{h}{p^\lambda} n} = \sum_{d|(p^\lambda, n)} d\mu\left(\frac{p^\lambda}{d}\right)$$

$$= \begin{cases} 0, & \text{if } \alpha < \lambda - 1, n = p^{\alpha} n', p \nmid n'; \\ -1 \times p^{\lambda-1} = -- p^{\alpha}. & \text{if } \alpha = \lambda - 1; \\ -1 \times p^{\lambda-1} + p^{\lambda} & \\ = p^{\lambda}(1 - \frac{1}{p}), & \text{if } \alpha \geq \lambda. \end{cases}$$

For obtaining these values we observe that in the summation on the right side we have to take into account only such divisors d that $\frac{p^{\lambda}}{\alpha}$ is at most p. This leads in the first case $\alpha < \lambda - 1$ to a vacuous sum. In the second case the only admissible divisor is $p^{\lambda-1}$; in the last we have two divisors $p^{\lambda-1}$ and p^{λ}. Thus

$$V_{p^{\lambda}}(n) = 0$$

for $\lambda > \alpha + 1$; we get again a finite sum for $\gamma_p(n)$

We now take λr odd. We want

$$\sum_{\substack{h \mod p^{\lambda} \\ p \nmid h}} \left(\frac{h}{p}\right) e^{-2\pi i \frac{h}{p^{\lambda}} n}$$

h modulo p is periodic, and we emphasize this by writing

$$h = rp + s; s = 1, 2, \ldots, p - 1; r = 1, \ldots, p^{\lambda-1}$$

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So the above sum becomes

$$\sum_{r=1}^{p^{\lambda}-1} \sum_{s=1}^{p-1} \left(\frac{s}{p}\right) e^{-2\pi i \frac{(rp+s)}{p^{\lambda}}} = \sum_{s=1}^{p-1} \left(\frac{s}{p}\right) e^{-2\pi i \frac{s}{p^{\lambda}}} \sum_{r=1}^{p^{\lambda}-1} e^{-2\pi i \frac{r}{p^{\lambda-1}} n}$$

This is zero when $p^{\lambda-1} \nmid n$ (because the inner sum vanishes). Otherwise, let $n = p^{\lambda-1}\nu$ and $p \nmid \nu$; then it is again zero because we have only a sum of quadratic residue symbols (since the character is not the principal character). If $p \mid \nu$, the sum becomes

$$p^{\lambda-1}\overline{G(\nu, p)} = p^{\lambda-1} \left(\frac{\nu}{p}\right) i^{\left(\frac{p-1}{2}\right)^2} \sqrt{p}$$

So if $n = p^{\alpha} \cdot n'$ where $p \nmid n'$, then

$$V_{p^{\lambda}}(n) = \begin{cases} 0, & \text{if } \lambda - 1 > \alpha; \\ p^{\alpha} \left(\frac{n'}{p}\right) i^{\left(\frac{p-1}{2}\right)^2} \sqrt{p}, & \text{if } \lambda - 1 = \alpha; \\ 0, & \text{if } 0 \leq \lambda - 1 < \alpha. \end{cases}$$

So the only non vanishing term in the case $\alpha + 1$ odd is $V_{p^{\alpha+1}}(n)$.

Let us put things together now. Let r be even. If $p \nmid n$, then

$$\gamma_p = 1 + V_p = 1 - \frac{i^{\left(\frac{p-1}{2}\right)^2 r}}{p^{r/2}}$$
$$= 1 - \frac{(-)^{\frac{r}{2}\frac{p-1}{2}}}{p^{r/2}}$$

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If $p \mid n$, $n = p^\alpha \cdot n'$, then

$$\gamma_p = 1 + V_p + V_{p^2} + \cdots + V_{p^\alpha} + V_{p^\alpha+1}$$
$$= 1 + \frac{\epsilon_p}{p^{r/2}}(p-1) + \frac{\epsilon_p^2}{p^2 r/2} p(p-1) + \cdots$$
$$+ \frac{\epsilon_p^\alpha}{p^\alpha r/2} p^{\alpha-1}(p-1) - \frac{\epsilon_p^{\alpha+1}}{p^{(\alpha+1)r/2}} p^\alpha,$$

where $\epsilon_p = (-)^{r(p-1)/4}$ for $r \neq 4$

$$= \left(1 - \frac{\epsilon_p}{p^{r/2}}\right) + \frac{\epsilon_p}{p^{r/2} - 1}\left(1 - \frac{\epsilon_p}{p^{r/2}}\right)$$
$$+ \frac{\epsilon_p^2}{p^2\left(\frac{r}{2} - 1\right)}\left(1 - \frac{\epsilon_p}{p^{r/2}}\right) + \cdots + \frac{\epsilon_p^\alpha}{p^{\alpha\left(\frac{r}{2}-1\right)}}\left(1 - \frac{\epsilon_p}{p^{r/2}}\right)$$
$$= \left(1 - \frac{\epsilon_p}{p^{r/2}}\right)\left(1 - \frac{\epsilon_p^{\alpha+1}}{p^{(\alpha+1)\left(\frac{r}{2}-1\right)}}\right)\left(1 - \frac{\epsilon_p}{p^{r/2-1}}\right)^{-1}$$

For $r = 4$, the thing becomes critical: Let us look at it more specifically. 335
$\dfrac{r(p-1)}{4}$ is even now and so $\epsilon_p = 1$. Hence

$$\gamma_p = \left(1 - \frac{1}{p^2}\right)\frac{1 - \frac{1}{p^{\alpha+1}}}{1 - \frac{1}{p}}$$

We go to the full singular series.

$$S_4(n) = \prod_p \gamma_p = \gamma_2 \prod_{p\geq 3} \gamma_p$$
$$= \gamma_2 \prod_{p\geq 3}\left(1 - \frac{1}{p^2}\right)\prod_{p\geq 3}\frac{1 - \frac{1}{p^{\alpha+1}}}{1 - \frac{1}{p}}$$

The product is convergent since $\sum \frac{1}{p^2} < \infty$. So

$$|S_4(n)| \geq \gamma_2 \prod_p \left(1 - \frac{1}{p^2}\right) \prod_{p|n} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p}}$$

$$\leq \gamma_2 \prod_p \left(1 - \frac{1}{p^2}\right)^2 \prod_{p|n} \frac{1}{1 - \frac{1}{p}}$$

$\prod \left(1 - \frac{1}{p}\right)$ diverges to zero in the infinite product senses. So $S_4(n)$ is not bounded. $S_4(n)$ could become very small if we keep the odd factors fixed and introduce more even factors.

$S_4(n)$ is unbounded in both senses; it can be as large as we please or as small as zero.

For $r \geq 5$ we are again on the safe side. In this case the first term comes from V_{p^λ}. We have

$$S_5(n) \sim \left(1 \pm \frac{V_p}{p^{5/2}}\right)$$

or
$$C_2 \prod \left(1 + \frac{1}{p^2}\right) < S_5(n) < C_1 \prod \left(1 - \frac{1}{p^2}\right)$$

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For $r = 7$ the situation is similar. For $r = 6$ the series again converges. So for $r \geq 5$.

$$0 < C_1 < S_r(n) < C_2$$

This is of importance in the application to our problem.

We had

$$A_r(n) = \frac{\pi^{r/2}}{\Gamma(r/2)} n^{\frac{r}{2}-1} S_r(n) + O(n^{r/4})$$

If $r \geq 5$, $\frac{r}{2} - 1 > \frac{r}{4}$, and since $S_{r(n)}$ being bounded does not raise the order in the term,

$$A_r(n) \sim \frac{\pi^{r/2}}{\Gamma(r/2)} n^{\frac{r}{2}-1} S_r(n)$$

If, however, if $r = 4$, the sharpness of the analysis is lost. Both the first factor and the error term are $O(r)$ and $S_r(n)$ may contribute to a decrease in the first term. If there are many odd factors for n, the main term is still good. But if there are many powers of 2, it would be completely submerged.

For $r = 4$ the exact formula was given by Jacobi.

We shall consider also representation of n in the form $an_1^2 + bn_2^2 + cn_3^2 + dn_4^2$ in which connection the Kloosherman sums appear. We shall also cast a glance at the meaning of the singular series in the sense of Siegel'e p-edic density.

Lecture 40

Let us look at $S_r(n)$ a little more explicitly. 337

$$S_r(n) = \gamma_2(n)\gamma_3(n)$$

$r \equiv 0 \pmod 4$.

In this case we need not bother about the sign of the Gaussian sums; the fourth power of the coefficient becomes 1.

$$\gamma_2(n) = 1 + V_2(n) + V_{2^2}(n) + \cdots$$

which is a finite sum. If $2 \nmid n$, then $\gamma_2(n) = 1$. If $2 \mid n$, $n = 2^\alpha n'$, $2 \nmid n'$, then

$$V_2(n) = 0$$

$$V_{2^\lambda}(n) = \begin{cases} \frac{(-)^{r/4}}{2^{(\lambda-1)\left(\frac{r}{2}-1\right)}}. & \text{if } \lambda < \alpha + 1; \\ -\frac{(-)^{r/4}}{2^{(\lambda-1)\left(\frac{r}{2}-1\right)}}, & \text{if } \lambda = \alpha + 1; \\ 0, & \text{if } \lambda > \alpha + 1 \end{cases}$$

So

$$\begin{aligned} \gamma_2(n) &= 1 + (-)^{r/4} \left\{ \frac{1}{2^{\frac{r}{2}-1}} + \frac{1}{2^{2\left(\frac{r}{2}-1\right)}} + \cdots + \frac{1}{2^{(\alpha-1)\left(\frac{r}{2}-1\right)}} - \frac{1}{2^\alpha\left(\frac{r}{2}-1\right)} \right\} \\ &= 1 + (-)^{\frac{r}{4}} \sum_{\mu=1}^{\alpha} \frac{(-)^{\frac{n}{2\mu}}}{2^\mu\left(\frac{r}{2}-1\right)}, \end{aligned}$$

if $2^\alpha \| n$ (2^α is the highest power of 2 dividing n). If $2 \nmid n$, $\gamma_2(n) = 1$. **338**

$$\gamma_p(n) = \left(1 - \frac{1}{p^{r/2}}\right)\left(1 - \frac{1}{p^{r/2-1}} + \cdots \frac{1}{p^{\alpha\left(\frac{r}{2}-1\right)}}\right), p^\alpha \| n$$

$$S_r(n) = \gamma_2(n) \prod_{p \geq 3} \left(1 - \frac{1}{p^{r/2}}\right) \prod_{p|n, p \text{ odd}} \left(1 - \frac{1}{p^{r/2-1}} + \cdots + \frac{1}{p^{\alpha\left(\frac{r}{2}-1\right)}}\right)$$

$$= \gamma_2(n) P_1 \cdot P_2(n),$$

where P_1 is a fixed factor and

$$P_2(n) = \prod_{p|n, p \text{ odd}} \left(1 - \frac{1}{p^{\frac{r}{2}-1}} + \cdots + \frac{1}{p^{\alpha\left(\frac{r}{2}-1\right)}}\right)$$

$$= \sum_{d|n, d \text{ odd}} \frac{1}{d^{\frac{r}{2}-1}}.$$

$$P_1 = \left(1 - \frac{1}{2^{r/2}}\right)^{-1} \prod_{p \geq 2} \left(1 - \frac{1}{p^{r/2}}\right)$$

$$= \frac{2^{r/2}}{2^{r/2-1}} \times \frac{1}{\zeta\left(\frac{r}{2}\right)}.$$

It is known (vide: Whittaker & Watson) that

$$\zeta(2k) = (-)^{k-1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}, k \geq 1,$$

where B_{2k} are the Bernoulli numbers. **339**

$$\left(B_1 = -\frac{1}{2}, B_3 = B_5 = B_7 = \cdots = 0; B_{2k} \neq 0; sgn B_{2k} = (-)^{k-1}\right)$$

$$P_1 = \frac{2^{r/2}}{2^{r/2} - 1} \times \frac{2\left(\frac{r}{2}\right)!}{(2\pi)^{r/2} |B_{r/2}|}$$

So for $r > 4$, the principal term

$$A_r(n) \sim \frac{\pi^{r/2}}{\Gamma\left(\frac{r}{2}\right)} n^{\frac{r}{2}-1} S_n(n)$$

$$= C_r(n),$$

say, where

$$C_r(n) = \frac{\pi^{r/2}}{\Gamma(r/2)} \frac{2^{r/2}}{2^{r/2}-1} \frac{2\left(\frac{r}{2}\right)!}{(2\pi)^{r/2}|B_{r/2}|} n^{\frac{r}{2}-1}\gamma_2(n) \sum_{d|n, d\text{odd}} \frac{1}{d^{\frac{r}{2}-1}}$$

(a divisor sum! which is interesting, but not surprising, because the Jacobi formula contains it). 340

$$C_r(n) = \frac{r}{2^{\frac{r}{2}-1}|B_{r/2}|} n^{\frac{r}{2}-1} \sum_{d|n, d \text{ odd}} \frac{1}{d^{\frac{r}{2}-1}}$$

$r \equiv 0 \pmod 8$

$$\begin{aligned}
n^{\frac{r}{2}-1}\gamma_2(n)\cdot \sum_{d|n, d \text{ odd}} \frac{1}{d^{r/2-1}} \\
&= n^{\frac{r}{2}-1}\left(1 + \frac{1}{2^{\frac{r}{2}-1}} + \cdots + \frac{1}{2^{(\alpha-1)(\frac{r}{2}-1)}} - \frac{1}{2^{\alpha\left(\frac{r}{2}-1\right)}} \sum_{d|n, d \text{ odd}} \frac{1}{d^{\frac{r}{2}-1}}\right). \\
&= n^{\frac{r}{2}-1} \sum_{\delta|n} \frac{(-)^{\frac{n}{\delta}}}{\delta^{\frac{r}{2}-1}}, \text{ if } n \text{ is even;} \\
& n^{\frac{r}{2}-1} \sum_{\delta|n} \frac{1}{\delta^{\frac{r}{2}-1}}, \text{ if } n \text{ is odd.}
\end{aligned}$$

So for any n,

$$\begin{aligned}
n^{\frac{r}{2}-1}\gamma_2(n) \sum_{d|n, d\text{odd}} \frac{1}{d^{\frac{r}{2}-1}} &= n^{\frac{r}{2}-1}(-)^n \sum_{\delta|n} \frac{(-)^{\frac{n}{\delta}}}{\delta^{\frac{r}{2}-1}} \\
&= (-)^n \sum_{\delta|n} (-)^{n/\delta}\left(\frac{n}{\delta}\right)^{\frac{r}{2}-1} \\
&= (-)^n \sum_{t|n} (-)^t t^{\frac{r}{2}-1}
\end{aligned}$$

So 341

$$C_r(n) = Q_r(-)^n \sum_{t|n} (-)^t t^{\frac{r}{2}-1},$$

here $$Q_r = \frac{r}{2^{\frac{r}{2}-1}|B_{\frac{r}{2}}|}$$

This is exactly what appears for $r = 4$ in the Jacobi formula.

$r \equiv 4 \pmod 8$

$$\begin{aligned}
n^{\frac{r}{2}-1}\gamma_2(n) \sum_{d|n, d\text{odd}} \frac{1}{d^{\frac{r}{2}-1}} \\
&= n^{\frac{r}{2}-1}\left(1 - \frac{1}{2^{\frac{r}{2}-1}} - \cdots - \frac{1}{2^{(\alpha-1)(\frac{r}{2}-1)}} + \frac{1}{2^{\alpha\left(\frac{r}{2}-1\right)}}\right)\sum_{d|n}, d \text{ odd } \frac{1}{d^{\frac{r}{2}-1}} \\
&= n^{\frac{r}{2}-1} \sum_{\delta|n} \frac{(-)^{\delta+\frac{n}{\delta}+1}}{\delta^{\frac{r}{2}-1}} \\
&= \sum_{\delta|n} (-)^{\delta+\frac{n}{\delta}+1} \left(\frac{n}{\delta}\right)^{\frac{r}{2}-1} \\
&= \sum_{t|n} (-)^{\frac{n}{t}+t+1} t^{\frac{r}{2}-1}, \text{ if } n \text{ is even;} \\
&\quad \sum_{t|n} t^{\frac{r}{2}-1}, \text{ if } n \text{ is odd;}
\end{aligned}$$

or in either case **342**

$$(-)^n \sum_{t|n} (-)^{t+\frac{n}{t}+1} t^{\frac{r}{2}-1}$$

So $$C_r(n) = (-)^n Q_r \sum_{t|n} (-)^{\frac{n}{t}+t-1} t^{\frac{r}{2}-1}$$

$$A_r(n) \sim Q_n (-)^n \sum_{t|n} (-)^{t+\frac{r}{4}\left(\frac{n}{t}+1\right)} t^{\frac{r}{2}-1};$$

where $$A_r = \frac{r}{2^{\frac{r}{2}-1}|B_{r/2}|}$$

The Bernoulli numbers are all rational numbers and we can show that $2(2^{r/2} - 1)B_{r/2}$ is an odd integral i.e., $2(2^{2k} - 1)B_{2k}$(k integral) is an all integer. Suppose q is an odd prime; then, by Fermat's theorem, **343**

$$2^{q-1} \equiv 1(\mod q)$$

Let $(q - 1) \mid 2k$. Then

$$2^{2k} \equiv 1 \pmod q$$

$$2^{2k} - 1 \equiv 0 \pmod q$$

We now appeal to the non-Steadt-Clausen theorem, which is a beautiful theorem describing fully the denominators of the Bernoulli numbers:

$$B_{2k} = G_{2k} - \sum_{(p-1)|2k} \frac{1}{p}$$

where G_{2k} is an integer.

$$\therefore \quad (2^{2k}-1)B_{2k} = (2^{2k}-1)G_{2k} - (2^{2k}-1)\sum_{(p-1)|2k} \frac{1}{p}$$
$$= \text{integer} + \frac{1}{2}\text{ integer}$$

So $2(2^{2k}-1)B_{2k}$ is an odd integer.

Let us obtain some specimens of

$$Q_r = \frac{2r}{(2(2^{r/2}-1)|B_{r/2}|)}$$
$$A_4 = 8, \quad Q_8 = 16, \quad Q_{12} = 8, \quad Q_{16} = \frac{32}{17},$$
$$Q_{20} = \frac{8}{31}, \quad Q_{24} = \frac{16}{691}, \quad Q_{28} = \frac{8}{5461}, \quad q_{32} = \frac{64}{929569}$$

The conspicuous prime 691 appears in connection with the representation **344**
as the sum 24 squares; it has to do with η^{24}.

Can $A_r(n)$ be exactly equal to the asymptotic expression? (as for $r = 4$). $A_4(n) = C_4(n), A_8(n) = C_8(n)$. From Q_{16} on wards, $A_{16}(n) \neq C_{16}(n)$. This is because Q_{16} has an odd prime factor in the denominator. Suppose p divides the denominator. Then the fraction produced by Q_{16} cannot be destroyed by the other factor and $C_r(n)$ is not always an integer. If $p \mid n$. the numerator of $C_r(p^\alpha)$ is congruent to $\pm 1 \mod p$.

Lecture 41

It might be of interest to take $C_r(n)$, the main term in the formula for $A_r(n)$ and make some remarks about it. 345

$$C_r(n) = Q_r \sum_{d|n} (-)^{n+d+\frac{r}{4}\left(\frac{n}{d}+1\right)} d^{\frac{r}{2}-1}$$

Let us form the generating function

$$H_r(x) = 1 + \sum_{n=1}^{\infty} C_r(n) x^n;$$

this will give a sort of partial fraction decomposition. In the case where $r \equiv 0 \pmod 8$, it is simpler:

$$\begin{aligned}
H_r(x) &= 1 + Q_r \sum_{n=1}^{\infty} x^n \sum_{d|n} (-)^{n+d} d^{\frac{r}{2}-1} \\
&= 1 + Q_r \sum_{n=1}^{\infty} (-x)^n \sum_{d|n} (-)^d d^{\frac{r}{2}-1} \\
&= 1 + Q_r \sum_{d=1}^{\infty} (-)^d d^{\frac{r}{2}-1} \sum_{q=1}^{\infty} (-x)^{qd} \\
&= 1 + Q_n \sum_{d=1}^{\infty} (-)^d d^{\frac{r}{2}-1} \frac{(-x)^d}{1-(-x)^d} \\
&= 1 + Q_r \sum_{d=1}^{\infty} d^{\frac{r}{2}-1} \frac{x^d}{1-(-x)^d} \\
&= 1 + Q_r \left\{ \frac{1.x}{1+x} + 2^{\frac{r}{2}-1} \frac{x^2}{1-x^2} + 3^{\frac{r}{2}-1} \frac{x^3}{1+x^3} + \cdots \right\}
\end{aligned}$$

This is a Lambert Series. Replacing x by $e^{\pi i\tau}$, it becomes

$$1 + Q_r\left\{\frac{e^{\pi i\tau}}{1+e^{\pi i\tau}} + 2^{\frac{r}{2}-1}\frac{e^{2\pi i\tau}}{1-e^{2\pi i\tau}} + \cdots\right\}$$

The series above can be transformed into an Eisen stein series. If r is taken to be 8, it is actually the 8th power of the $\mathscr{V}$−function 346

Next, take $r \equiv 4 \pmod 8$

$$\begin{aligned}
G_r(x) &= 1 + Q_r\sum_{n=1}^{\infty} x^n \sum_{d|n}(-)^{n+d+\frac{n}{d}+1} d^{\frac{r}{2}-1}\\
&= 1 - Q_r\sum_{d=1}^{\infty}(-)^d d^{\frac{r}{2}-1}\sum_{d|n}(-x)^n(-)^{n/d}\\
&= 1 - Q_r\sum_{d=1}^{\infty}(-)^d d^{\frac{r}{2}-1}\sum_{q=1}^{\infty}(-x)^{qd}(-)^q\\
&= 1 + Q_r\sum_{d=1}^{\infty}(-)^d d^{\frac{r}{2}-1}\frac{(-x)^d}{1+(-x)^d}\\
&= 1 + Q_r\sum_{d=1}^{\infty} d^{\frac{r}{2}-1}\frac{x^d}{1-(-x)^d}\\
&= 1 + Q_r\left\{\frac{1\cdot x}{1-x} + 2^{\frac{r}{2}-1}\frac{x^2}{1+x^2} + 3^{\frac{r}{2}-1}\frac{x^3}{1-x^3} + \cdots\right\}
\end{aligned}$$

This is again a Lambert Series. This shows that a $\mathscr{V}$−power has to do with Lambert series which appears as an evaluation of certain Eisenstein series not that they are identical.

We now go to something quite different. We had for $r \geq 5$,

$$A_r(n) \sim \frac{\pi^{r/2}}{\Gamma\left(\frac{r}{2}\right)} n^{\frac{r}{2}-1} S_r(n) \qquad (*)$$

This comes out as a nice formula. Now could we not make some sense out of this formula? What is its inner meaning? We shall show that the first factor $\left(\pi^{r/2}/\Gamma(r/2)\right) n^{\frac{r}{2}-1}$ gives the average value of the number of representations of n as the sum of r squares; the second factor also is an average, in the p-adic measurement. We shall show that 347

$$\sum_{n\leq x} A_r(n) \sim \frac{\pi^{r/2}}{\Gamma(r/2)}\sum_{n\leq x} n^{\frac{r}{2}-1}$$

So for each individual n, $S_r(n)$ gives the deviation of $A_r(n)$ from $\left(\pi^{r/2}/\Gamma(r/2)\right) n^{\frac{r}{2}-1}$; but on the average there is no deviation.

Let us first look at $\sum_{n\leq x} A_r(n)$.

$$\sum_{n\leq x} a_r(n) = \sum_{n\leq x} \sum_{m_1^2+\cdots m_r^2=n} 1$$
$$= \sum_{m_1^2+\cdots+m_r^2\leq x} 1 ,$$

which is the number of lattice-points in the r-sphere with centre at the origin and radius $\sqrt{x}$, and so is proportional asymptotically to a certain volume (because the point lattice has cells or volume 1 and to each points belongs a cell). So this is roughly the volume of the sphere of radius $\sqrt{x}$ which is

$$\underset{x_1^2+\cdots+x_n^2\leq x}{\int\cdots\int} dx_1\cdots dx_r$$
$$= \frac{\pi^{r/2}}{\Gamma(r/2)} x^{r/2}$$

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The difference will not be zero but of the order of magnitude of the surface of the sphere, i.e., $O\left(x^{r/2}-1\right)$

Now consider the other side.

$$\frac{\pi^{r/2}}{\Gamma(r/2)} \sum_{n\leq x} n^{\frac{r}{2}-1} \sim \frac{\pi^{r/2}}{\Gamma(r/2)} \int_0^x \mathscr{V}^{\frac{r}{2}-1} d\mathscr{V}$$
$$= \frac{\pi^{r/2} x^{r/2}}{\Gamma\left(\frac{r}{2}+1\right)}$$

So the first factor on the right of (*) gives the average. $S_r(n)$ has to be adjusted. $S_r(n)$ is also, surprisingly, an average. It was defined as

$$S_r(n) = \gamma_2(n)\gamma_3(n)\gamma_5(n)\cdots\gamma_p(n)\cdots,$$

and $\gamma_p(n)$ in turn was given by **349**

$$\gamma_p(n) = 1 + \sum_{\lambda=1}^{\infty} V_{p^\lambda}(n)$$
$$= 1 + \sum_{\lambda=1}^{\infty} \frac{1}{p^{\lambda r}} \sum_{\substack{h \bmod p^\lambda \\ p\nmid h}} G(h,p^\lambda)^r e^{-2\pi i \frac{h}{p^\lambda} n} \frac{1}{p^{\lambda r}} \sum_{\substack{h \bmod p^\lambda \\ p\nmid h}} G(h,p^\lambda)^r e^{-2\pi i \frac{h}{p^\lambda} n}$$

$$= \frac{1}{p^{\lambda r}} \sum_{\substack{h \mod p^\lambda \\ p\nmid h}} \left\{ \sum_{\ell_1 \mod p^\lambda} e^{2\pi i \frac{h}{p^\lambda}\ell_1} \sum_{\ell_2 \mod p^\lambda} e^{2\pi i \frac{h}{p^\lambda}\ell_2} \right.$$

$$\left. \sum_{\ell_r \mod p^\lambda} e^{2\pi i \frac{h}{p^\lambda}\ell r^2} \right\} e^{-2\pi i \frac{h}{p^\lambda} n}$$

$$= \frac{1}{p^{\lambda r}} \sum_{\ell_1,\ldots,\ell_4 \mod p^\lambda} \sum_{h \mod p^\lambda} e^{2\pi i \frac{h}{p^\lambda}} (\ell_1^2 + \cdots + \ell_r^2 - n)$$

$$= \frac{1}{p^{\lambda r}} \sum_{\ell_1,\ldots,\ell_r \mod p^\lambda}$$

$$\left\{ \sum_{s \mod p^\lambda} e^{2\pi i \frac{s}{p^\lambda}(\ell_1^2+\cdots+\ell_r^2-n)} - \sum_{t \mod p^{\lambda-1}} e^{2\pi i \frac{t}{p^{\lambda-1}}(\ell_1^2+\cdots+\ell_r^2-n)} \right\}$$

$$= \frac{1}{p^{\lambda r}} \sum_{\ell_1,\ldots\ell_n \mod p^\lambda} \sum_{s \mod p^\lambda} e^{2\pi i \frac{s}{p^\lambda}(\ell_1^2+\cdots+\ell_r^2-n)}$$

$$- \sum_{\ell_1,\ldots,\ell_r \mod p^{\lambda-1}} \sum_{t \mod p^{\lambda-1}} e^{2\pi i \frac{t}{p^{\lambda-1}}(\ell_1^2+\cdots+\ell_2^2-n)}$$

$$= W_{p^\lambda}(n) - W_{p^{\lambda-1}}(n), \text{ say,}$$

$$\therefore \qquad 1 + V_p(n) + V_{p^2}(n) + \cdots + V_{p^\lambda}(n) = W_{p^\lambda}(n) \to \gamma_p(n)$$

So for λ large enough $W_{p^\lambda}(n) = V_{p^\lambda}(n)$: the partial sums get identical. The **350**
value of λ for which this occurs depends on the structure of n, on how many primes that specific n contains. Now

$$\sum e^{2\pi i \frac{s}{p^\lambda}(\ell_1^2+\cdots+\ell_r^2-n)} = 0 \text{ or } p^\lambda$$

$$\therefore \qquad W_{p^\lambda}(n) = \frac{p^\lambda}{p^{\lambda r}} \sum_{\substack{\ell_1,\ldots,\ell_r \mod p^\lambda \\ \ell_1^2+\cdots+\ell_r^2 \equiv n \pmod{p^\lambda}}}$$

The sum on the right gives the number of times the congruence $\ell_1^2 + \cdots + \ell_r^2 \equiv n \pmod{p^\lambda}$ can be solved, $N_{p^\lambda}(n)$, say. Then

$$W_{p^\lambda}(n) = \frac{1}{p^{\lambda(r-1)}} N_{p^\lambda}(n)$$

We have therefore divided the number of solutions of the congruence by $p^{\lambda(r-1)}$. Now how many $\ell_1, \ldots, \ell_r \mod p^\lambda$ do we have? There are $p^{\lambda r}$ possibilities discarding n. n is one of the numbers modulo p^λ. So dividing by p^r, the average number of possibilities is $p^{\lambda(r-1)}$. Hence $\dfrac{N_{p^\lambda}(n)}{p^{\lambda(r-1)}}$ is the average density modulo p^λ of the solution of the congruence. And since the $W_{p^\lambda}(n)$ eventually becomes $\gamma_p(n)$, each factor $\gamma_p(n)$ acquires a density interpretation, viz. the p-adic density of the lattice points modulo p^λ.

Lecture 42

The error term in the formula for the number of representations of n as the sum of r squares, $r \geq 5$, was $O(n^{r/4})$. For $r = 4$ this did not suffice. We shall therefore study the problem by Kloosterman's method and find out what happens when we want to decompose n in the form $n = n_1^2 + n_2^2 + n_3^2 + n_4^2$. We shall see that we can diminish the order in the error term by nearly $\frac{1}{18}$. When Kloosterman did this for the first time (Act a Mathematic as 1927) he took a slightly more general problem, that of representing n in the form $n = an_1^2 + b_2^2 + cn_3^2 + dn_4^2, a, b, c, d$ integers. This works nicely; we get the singular series and an error term which is smaller than before. The difficult not will be about the arithmetical interpretation. The singular series will now be a difficult phenomenon; we shall have multiplicativity, but the interpretation of the factors γ_p becomes complicated. We shall content ourselves with the analytical power of the discussion. The generating function which will have to be discussed is quite clear: 351

$$F(x) = \Theta(x^a) \quad \Theta(x^b) \quad \Theta(x^c) \quad \Theta(x^d)$$

where
$$\Theta(x) = \sum_{n=-\infty}^{\infty} x^{n^2}$$

And we will have

$$A_4(n) = \frac{1}{2\pi i} \int_C \frac{\Theta(x^a)\Theta(x^b)\Theta(x^c)\Theta(x^d)}{x^{n+1}} dx$$

and the analysis goes on as before with Farey series.

We are here representing n by a positive definite quadratic form which is a diagonal form. Let us make the problem more general.

Let us represent n by a positive definite form with integral coefficients. 352

(We could very well unsedes also the 'semi-integral' case). Let S be a positive definite integral symmetric matrix and $\underline{x}$ a column vector with elements $x_1, x_2, \ldots x_r$ in r-space. $\underline{x'}$ is the transposed row-vector. $\underline{x'}S\,\underline{x}$ is a quadratic form in r variables. The question is how often can we express an integer n by integer vectors with respect to this quadratic form in r variables.

The generating function to be studied this time is

$$F_r(x) = \sum_{\underline{n}} x^{\underline{n'}S\,\underline{n}}, |x| < 1,$$

the summation over all integral vectors $\underline{n}$. Convergence is easily assured by positive definiteness. Indeed

$$\underline{x'}S\,\underline{x} \geq C(x_1^2 + \cdots + x_r^2), C > 0$$

For $\underline{x'}S\,\underline{x}$ has a minimum $C > 0$ on $|x| = 1$ by positive definiteness; the inequality follows from the homogeneity of the quadratic form. And $\sum x^{c(n_1^2+\cdots+n_r^2)}$ is trivially a product of convergent series.

In a later paper (Hamburger Abhandlungen, 1927) Kloosterman, on the advice of Hecke, took up a more general problem. This would require a little more preparation on modular forms. The generating function will now be a modular form of dimension $-\frac{r}{2}$ of a certain 'stafe'; so we have to discuss modular forms not only with respect to the full modular group, but also the substitutions

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{mod } N),$$

(N will the 'stafe') which from a subgroup finite index in the modular group. **353**
Kloosterman's work goes through for all modular forms of this sort, but we should want generalisations of $\eta(\tau)$ and $\mathscr{V}(\tau)$. To do this we need a good deal of Heeke's theory about Eisenstein series of higher stafe of the type:

$$\sum_{\substack{m_1 \equiv a \quad (\text{mod } N) \\ m_2 \equiv b \quad (\text{mod } N)}} \frac{1}{(m_1 + m_2\tau)^r}$$

which is a modular form of dimension $-\frac{r}{2}$ and stafe N. These were investigated by Hecke in a famous paper (Hamburger Abhandlungen 1927). Kloosterman could carry out his theory for these also. We shall, however, compromise on the quadratic form.

We had the generating function

$$F_r(x) = \sum_{\underline{n}} x^{\underline{n'}S\,\underline{n}}, |x| < 1,$$

$$= 1 + \sum_{n=1}^{\infty} A_r(n) x^n.$$

$F_r(x)$ is a modular form. This can be seen directly by the transformation formulae. Let us start with Kloosterman's method and see what happens. The problem is to get

$$A_r(n) = \frac{1}{2\pi i} \int_C \frac{F_n(x)}{x^{n+1}} dx$$

At a certain moment later on we shall need a greater knowledge of $F_r(x)$

Let us carry out the Farey dissection:

$$x = e^{2\pi i \frac{h}{k} - 2\pi\mathfrak{z}} = e^{2\pi i \frac{h}{k} - 2\pi(\delta_N - i\varphi)}$$

$$A_r(n) = \sum_{0 \le h < k \le N} e^{-2\pi i \frac{h}{k} n} \int_{-\mathscr{V}'_{hk}}^{\mathscr{V}''_{hk}} F_r(e^{2\pi i \frac{h}{k} - 2\pi\mathfrak{z}}) e^{2\pi n\mathfrak{z} d\varphi}$$

with $(h,k) = 1$, $\mathscr{V}'_{hk} = \frac{1}{k(k_1+k)}$, $\mathscr{V}''_{hk} = \frac{1}{k(k+k_2)}$ where in the Farey situation, **354**
$\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$. The refinement of Kloosterman consists in not merely making the rough remark that

$$\frac{1}{2kN} \le \mathscr{V}'_{hk}, \mathscr{V}''_{hk} \le \frac{1}{k(N+1)},$$

but in a finer following up of the number theoretical determination of the adjacent fractions. We have

$$h_1 k - h k_1 = -1, hk_2 - h_2 k = -1;$$

i.e., $$hk_1 \equiv 1 \mod k, hk_2 \equiv -1 \mod k$$

$\frac{h}{k}$ is given. What we are worried about is, how long is its environment. k_1 and k_2 are given as solutions of certain congruences. We have the habit of calling h' a number such that

$$hh' \equiv -1 \pmod k; \quad \text{so let us write}$$
$$k_1 \equiv -h' \pmod k, k_2 \equiv h' \pmod k$$

So we know in which residue class modulo k k_1 and k_2 have to lie. $k_1 + k$, **355**
being the denominator of a mediant, had to exced N. $N < k_1 + k \le N + k$, or $N - k < k_1 \le N$. So k_1 has a span of size k. This along with $k_1 \equiv -h \mod k$ determines k_1 completely. Similarly, for k_2, $N - k < k_2 \le N$ So there is no

uncertainty at all about $\mathscr{V}'_{hk}$, $\mathscr{V}''_{hk}$; and we could single them out if we insisted on that.

For example, let $\frac{h}{k} = \frac{5}{9}$, $N = 12$; what are the neighbours? $\frac{h_1}{k_1} < 5/9 < \frac{h_2}{k_2}$. First determine h'. $5h' \equiv -1 \pmod 9$ or $h' = 7$. Then $12 - 9 < k_1 \leq 12$ and $k_1 \equiv -7 \pmod 9$, so $k_1 = 11$. Similarly $3 < k_2 \leq 12$, $k_2 \equiv 7 \pmod 9$ so $k_2 = 7$. We need only k_1 and k_2; but for our own enjoyment let us calculate h_1 and h_2.

$$\begin{vmatrix} h, & 5 \\ 11 & 9 \end{vmatrix} = -1, \begin{vmatrix} 5 & h_2 \\ 9 & 7 \end{vmatrix} = -1,$$

or $h_1 = 6$, $h_2 = 4$, so that we have $\frac{6}{11} < \frac{5}{9} < \frac{4}{7}$ as adjacent fractions in the Farey series of order 12. We do not need to display the whole Farey series.

Now utilise this in the following way.

$$A_r(n) = \sum_{o \leq h < k \leq N}{}' e^{2\pi i \frac{h}{k} n} \int\limits_{-\frac{1}{k(k_1+k)}}^{\frac{1}{k(k+k_2)}} F_r\left(e^{2\pi i \frac{h}{k} - 2\pi \mathfrak{z}}\right) e^{2\pi n \mathfrak{z}} d\varphi$$

Kloosterman does the following investigation. In any case we are sure that k_1, k_2 can at most become N. If we take k_1 and k_2 big we have a small interval of integration. Since

$$k_1 + k < k_1 + 1 + k < \cdots < N + k,$$
$$k_2 + k < k_2 + 1 + k < \cdots < N + k,$$
$$\frac{1}{k_1 + k} > \frac{1}{N + k}, \frac{1}{k_2 + k} > \frac{1}{N + k},$$

so that the interval of integration should be at least as big as the interval **356**
$-1/k(k + N)$ to $1/k(k + N)$. This interval is always present whatever be k_1 and k_2. So $A_r(n)$ is equal to the always present kernel

$$\sum_{0 \leq h < k \leq N}{}' e^{2\pi i \frac{h}{k} n} \int\limits_{\frac{-1}{k(k+N)}}^{\frac{1}{k(k+N)}} (\cdots) d\varphi,$$

with the possible additional terms

$$\sum_{0 \leq h < k \leq N}{}' e^{-2\pi i \frac{h}{k} n} \sum_{\ell = k_2}^{N-1} \int\limits_{\frac{1}{k(k+\ell+1)}}^{\frac{1}{k(k+\ell)}} (\cdots) d\varphi \sum_{0 \leq h < k \leq N} e^{-2\pi i \frac{h}{k} n} \sum_{\ell = k_1}^{N-1} \int\limits_{\frac{-1}{k(k+\ell)}}^{\frac{1}{k(k+\ell+1)}} (\cdots) d\varphi$$

There is no doubt about the integrals. The limits are all well-defined. This will help us to appraise certain roots of unity closely-by the Kloosterman sums. **357**

We shall now return to the integrand; that is a $\mathscr{V}$-function and requires the usual $\mathscr{V}$ treatment. Consider the r-fold $\mathscr{V}$-series:

$$\Theta(t) = \sum_{\underline{n}} e^{-\pi t \underline{n}' S \underline{n}}, \mathscr{R}_e t > 0.$$

Modify this slightly by introducing a vector α of real numbers; $\underline{\alpha'} = (\alpha_1, \ldots, \alpha_r)$. Let

$$\Theta(t; \alpha_1, \ldots, \alpha_r) = \sum_{\underline{n}} e^{-\pi t(\underline{n}' + \underline{\alpha}')S(\underline{n} + \underline{\alpha})}$$

This is periodic in α_j, of period 1, and so permits a Fourier expansion. The convergence is so good that the function is analytic in each α_j and so we are sure that it is equal to the sum

$$\sum_{\underline{m}} C(\underline{m}) e^{2\pi i \underline{m}' \underline{\alpha}}$$

where $C(\underline{m})$ is the Fourier coefficient:

$$\begin{aligned} C(\underline{m}) &= \int_0^1 \cdots \int_0^1 \Theta(t; \beta_1, \ldots, \beta_r) e^{-2\pi i \underline{m}' \underline{\beta}} d\beta_1, \ldots d\beta_r \\ &= \int_0^1 \cdots \int_0^1 \sum_{\underline{n}} e^{-\pi t(\underline{n}' + \underline{\beta}')S(\underline{n} + \underline{\beta})} e^{-2\pi i \underline{m}' \underline{\beta}} d\beta_1, \ldots, d\beta_r \\ &= \int_0^1 \cdots \int_0^1 \sum_{\underline{n}} e^{-\pi t(\underline{n}' + \underline{\beta}')S(\underline{n} + \underline{\beta})} e^{-2\pi i \underline{m}'(\underline{n} + \underline{\beta})} d\beta_1, \ldots, d\beta_r \end{aligned}$$

which is an integral over the unit cube W, and so on translation with respect to the vector $\underline{n}$, becomes **358**

$$\sum_{\underline{n}} \int \cdots \int_{W + \underline{n}} e^{-\pi t(\underline{\mathscr{V}} S \underline{\mathscr{V}})} e^{-2\pi i \underline{m}' \mathscr{V}} d\underline{\mathscr{V}}_1 \cdots d\mathscr{V}_r$$

(the exchange of integration and summation orders being trivial)

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\pi t \mathscr{V}' D \mathscr{V}} e^{-2\pi i \underline{m}' \underline{\mathscr{V}}} d\mathscr{V}_1 \ldots d\mathscr{V}_r.$$

Lecture 43

Let us return to the generalised theta-formula: 359

$$\begin{aligned}\Theta(t;\alpha_1,\ldots\alpha_r) &= \sum_{\underline{n}} e^{-\pi t(\underline{n}'+\underline{\alpha})S(\underline{n}+\underline{\alpha})}\\ &= \sum_{\underline{m}} c(\underline{m})e^{2\pi i\underline{m}'\underline{\alpha}}\end{aligned}$$

where

$$c(\underline{m}) = \int_{-\infty}^{\infty}\cdots\int e^{-\pi t\mathscr{V}'S\mathscr{V}}e^{-2\pi i\underline{m}'\underline{\mathscr{V}}}d\mathscr{V}_1\ldots d\underline{\mathscr{V}_r}$$

To get this into shape, consider the quadratic complement

$$-\frac{\pi}{t}(t\underline{\mathscr{V}'}+i\underline{m}'S^{-1})S(t\underline{\mathscr{V}}+iS^{-1}\underline{m}) = -\pi t\underline{\mathscr{V}'}S\underline{\mathscr{V}}-\pi i\underline{\mathscr{V}'m}-\pi i\underline{m}'\underline{\mathscr{V}}+\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}$$

Since $m'\underline{\mathscr{V}} = \underline{\mathscr{V}'m}$,

$$\begin{aligned}c(m) &= \int_{-\infty}^{\infty}\cdots\int e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}}e^{-\frac{\pi}{t}(t\underline{\mathscr{V}'}+i\underline{m}'S^{-1})S(t\underline{\mathscr{V}}+iS^{-1}\underline{m})}d\mathscr{V}_1\ldots d\mathscr{V}_r\\ &= e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}}\int_{-\infty}^{\infty}\cdots\int e^{-\pi(\sqrt{t}\underline{\mathscr{V}'}+\frac{i}{\sqrt{t}}\underline{m}'S^{-1})S(\sqrt{t}\underline{\mathscr{V}}+\frac{i}{\sqrt{t}}S^{-1}\underline{m})}d\mathscr{V}_1\cdots d\mathscr{V}_r\end{aligned}$$

Put $\sqrt{t}\underline{\mathscr{V}} = \underline{w}$ and $\underline{\mu} = \frac{1}{\sqrt{t}}\underline{m}'S^{-1}$. Then 360

$$c(\underline{m}) = \frac{e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}}}{(\sqrt{t})^r}\int_{-\infty}^{\infty}\cdots\int e^{-\pi(\underline{w}'+i\underline{\mu}')S(\underline{w}+i\underline{\mu})}dw_1\cdots dw_r$$

Since every positive definite quadratic form may be turned into a sum of squares, we can put $S = A'A$, so that the exponent in the integrand become $-\pi(\underline{w}'A' + i\underline{\mu}A')(A\underline{w} + iA\underline{\mu})$; and writing $A\underline{w} = \underline{\mathfrak{z}}$, we have

$$c(\underline{m}) = \frac{e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}}}{(\sqrt{t})^r} \int_{-\infty}^{\infty} \cdots \int e^{-\pi(\underline{\mathfrak{z}}'+i\underline{\mathscr{V}}')(\underline{\mathfrak{z}}+i\underline{\mathscr{V}})} \frac{d_{\mathfrak{z}_1} \ldots d_{\mathfrak{z}_r}}{|A|}$$

where $\underline{\mathscr{V}} = \underline{\mu}A$, and $|A|$ = determinant of A. Let $D = |A|^2 = |S|$, $\underline{\mathfrak{z}'} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_r)$. Then

$$\begin{aligned} c(\underline{m}) &= \frac{e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}}}{D^{1/2}t^{r/2}} \prod_{j=1}^{r} \int_{-\infty}^{\infty} e^{-\pi(\mathfrak{z}_j+i\mathscr{V}_j)^2} d\mathfrak{z}_j \\ &= \frac{e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}}}{D^{1/2}t^{r/2}} \left(\int_{-\infty}^{\infty} e^{-\pi\mathfrak{z}^2} d\mathfrak{z}\right)^r \\ &= \frac{e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}}}{D^{1/2}t^{r/2}}, \end{aligned}$$

the last factor being unity. So we have ultimately

$$\Theta(t; \alpha_1, \ldots, \alpha_r) = \frac{1}{D^{1/2}t^{r/2}} \sum_{\underline{m}} e^{-\frac{\pi}{t}\underline{m}'S^{-1}\underline{m}} e^{2\pi i\underline{m}'\underline{\alpha}}$$

Let us now we back to our study of $A_r(n)$. We had integrals with now limits which were the special feature of the Kloosterman method. **361**

$$A_r(n) = \sum_{0\le h<k\le N}{}' e^{-2\pi i\frac{h}{k}n} \int_{-\frac{1}{k(k+N)}}^{\frac{1}{k(k+N)}} F_r\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right) e^{2\pi n\mathfrak{z}} d\mathfrak{z} + \sum_{\ell=0}^{N-1} \cdots + \sum_{\ell=0}^{N-1} \cdots$$

Now

$$\begin{aligned} F_r(x) &= \sum_{\underline{n}} x^{\underline{n}'S\underline{n}} = 1 + \sum_{n=1}^{\infty} A_r(n)x^n \\ F_r\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right) &= \sum_{\underline{n}} e^{(2\pi i\frac{h}{k}-2\pi\mathfrak{z})} \underline{n}'S\,\underline{n} \\ &= \sum_{\underline{n}} e^{2\pi i\frac{h}{k}\underline{n}'S\underline{n}} e^{-2\pi\mathfrak{z}\underline{n}'S\underline{n}} \end{aligned}$$

$\underline{n}$ is of interest only modulo k, so put

$$\underline{n} = kq + \ell, \ell = (\ell_1, \ldots, \ell_n), 0 \leq \ell_j < k.$$

So dismissing multiples of k,

$$F_r(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}) = \sum_{\underline{\ell} \bmod k} e^{2\pi i\frac{h}{k}\underline{\ell}'S\ell'} \sum_{q} e^{-2\pi\mathfrak{z}k^2(q'+\frac{\ell'}{k})S\left(q+\frac{\ell}{k}\right)},$$

and applying the transformation formula we derived earlier, with $t = 2\mathfrak{z}k^2$ and $\underline{\alpha} = \frac{1}{k}\underline{\ell}$, this becomes

$$\frac{1}{\sqrt{D}k^r e^{r/2}\mathfrak{z}^{r/2}} \sum_{\underline{\ell}} e^{2\pi i\frac{h}{k}\underline{\ell}'S\underline{\ell}} \sum_{\underline{m}} e^{-\frac{\pi}{2\mathfrak{z}k^2}\underline{m}'S^{-1}\underline{m}} e^{2\pi i\underline{m}'\frac{\underline{\ell}'}{k}}$$
$$= \frac{1}{\sqrt{D}k^r(2\mathfrak{z})^{r/2}} \sum_{\underline{m}} e^{-\frac{\pi}{2\mathfrak{z}k^2}\underline{m}'S^{-1}\underline{m}} T_k(h, m),$$

on exchanging summations, where **362**

$$T_k(h, m) = \sum_{\underline{\ell}} e^{2\frac{\pi i}{k}(h\underline{\ell}'S\underline{\ell}+\underline{m}'\underline{\ell})}$$

$T_k(h, 0)$ will be the most important; the others we only estimate. We require a little more number theory for this. We cannot tolerate the presence of a both a quadratic form and a linear form in the exponent. There will be a common denominator in $\underline{m}'S^{-1}\underline{m}$ and that will have to be discussed.

Lecture 44

We had **363**

$$F_r(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}) = \frac{1}{k^r(2\mathfrak{z})^{r/2}D^{1/2}}\sum_{\underline{m}} e^{-\frac{\pi}{2\mathfrak{z}k^2}\underline{m}'S^{-1}\underline{m}}T_k(h,\underline{m}),$$

and

$$T_k(h,m) = \sum_{\ell \mod k} e^{\frac{2\pi i}{k}(h\underline{\ell}'S\underline{\ell}+\underline{m}'\underline{\ell})}$$

The common denominator in $\underline{m}'S^{-1}\underline{m}$ will be at most D, the determinant; define k^* and D_k by

$$kD = k\cdot(k,D)\cdot D_k = k^*D_k, (D_k,k) = 1,$$

so that D_k is D stripped of all its common divisors with k. Suppose first that k is odd. Let ρ be a solution of the congruence

$$4hD_k\rho \equiv 1 \pmod{k^*}$$

$$\begin{aligned}
T_k(h,\underline{m}) &= \sum_{\underline{\ell} \mod k} e^{2\pi i\frac{h}{k}(\ell'S\underline{\ell+4D_k\rho\underline{m}'\underline{\ell}})}\\
&= \sum_{\underline{\ell} \mod k} e^{2\pi i\frac{h}{k}(\underline{\ell}'+2D_k\rho\underline{m}'S^{-1})S(\underline{\ell}+2D_k\rho S^{-1}\underline{m})}e^{-(4D_k^2\rho^2\underline{m}'S^{-1}\underline{m})2\pi i\frac{h}{k}}.\\
&= e^{-2\pi i\frac{h}{k}\cdot 4D_k^2\rho^2\underline{m}'S^{-1}\underline{m}}\sum_{\underline{\ell} \mod k} e^{2\pi i\frac{h}{k}(\underline{\ell}'+2D_k\rho\underline{m}'S^{-1})S(\underline{\ell}+2D_k\rho S^{-1}\underline{m})}\\
&= e^{-2\pi\frac{D_k\rho}{k}\underline{m}'S^{-1}\underline{m}}\mathscr{U}_k,
\end{aligned}$$

say, (using the definition of ρ), where $\mathscr{U}_k = \mathscr{U}_k(h,\underline{m})$ is periodic in $\underline{m}$ with **364**
period (k,D); it is enough if we take this period to be D itself. So

$$F_r(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}) = \frac{1}{k^r(2\mathfrak{z})^{r/2}D^{1/2}}\sum_{\underline{s} \mod D}\mathscr{U}_k(h,\underline{s})\sum_{\underline{m}\equiv\underline{s} \pmod D} e^{-\left(\frac{\pi}{2\mathfrak{z}k^2}+2\pi i\frac{D_k\rho}{k}\right)\underline{m}'S^{-1}\underline{m}}$$

This is a linear combination of finitely many $\mathscr{V}$-series of the form

$$\sum_{\underline{m}\equiv\underline{s} \pmod D} x^{\underline{m}' S^{-1}\underline{m}}$$

The power series goes in powers of $\frac{x}{D}$ because $\frac{1}{D}$ remains silent inside. This is for k odd.

For k even, define. σ by

$$hD_k\sigma \equiv 1 \pmod{4k^*}$$

$$\begin{aligned} T_k(h,m) &= \sum_{\underline{\ell} \bmod k} e^{2\pi i \frac{h}{4k}(4\underline{\ell}' S\underline{\ell} + 4D_k\sigma \underline{m}'\underline{\ell})} \\ &= e^{-2\pi i \frac{h}{4k} D_k^2\sigma^2 \underline{m}' S^{-1}\underline{m}} \sum_{\underline{\ell} \bmod k} e^{2\pi i \frac{h}{4k}(2\ell' + D_k\sigma\underline{m}' S^{-1}) S (2\underline{\ell} + D_k\sigma S^{-1}\underline{m})} \\ &= e^{-2\pi i \frac{D_k}{4k}\sigma \underline{m}' S^{-1}\underline{m}} \mathscr{U}_k(h,\underline{m}), \end{aligned}$$

where $\mathscr{U}_k$ again has a certain periodicity; we can take the period to be $2D$ and **365**
forget about the refinement. So

$$F_r\left(e^{2\pi i \frac{h}{k} - 2\pi\mathfrak{z}}\right) = \frac{1}{k^r (2\mathfrak{z})^{r/2} D^{1/2}} \sum_{\underline{s} \bmod 2D} \mathscr{U}_k(h,\underline{s}) \sum_{\underline{m}\equiv\underline{s} \pmod{2D}} e^{-\left(\frac{\pi}{2\mathfrak{z}k^2} + \frac{2\pi i}{4k} D_k\sigma\right)\underline{m}' S^{-1}\underline{m}}$$

which is again a linear combination of theta-series with coefficients $\mathscr{U}_k$. Observe that T_k and $\mathscr{U}_k$ differ only by a purely imaginary quantity:

$$|T_k(h,\underline{m})| = |\mathscr{U}_k(h,\underline{m})|,$$

and for $\underline{m} = \underline{0}$, $T_k(h,\underline{0}) = \mathscr{U}_k(h,\underline{0})$.

We shall use as essential only those theta-series which are congruent to zero modulo D or $2D$; and the rest will be thrown into the error term. Only these corresponding to $\underline{o}$ have a constant term. The general shape in both cases is

$$\sum_{\underline{s} \bmod 2D} \mathscr{U}_k(h,\underline{s}) = \sum_{\underline{m}\equiv\underline{s} \pmod{2D}} x^{\underline{m}' S^{-1}\underline{m}}$$

Lecture 45

We have to get a clear picture of that we are aiming at. We are discussing the function under the integral sign. We get it as **366**

$$F_r\left(e^{2\pi i\frac{h}{k}-2\pi\mathfrak{z}}\right) = \frac{1}{k^r(2\mathfrak{z})^{r/2}D^{1/2}} \sum_{\underline{s} \mod 2D} \mathscr{U}_k(h,\underline{s}) \sum_{\underline{m}\equiv\underline{s} \pmod{2D}} e^{-\left(\frac{\pi}{2k^2\mathfrak{z}}+2\pi i\frac{D_k\sigma}{4k}\right)m'S^{-1}m}$$

where $k \cdot D = k^* D_k, (k, D_k) = 1$. k is even; if k is odd the formula looks finitely many different values. This most important fact we formulate as a lemma.

Lemma 1. *For k even, $\mathscr{U}_k(h,\underline{s})$ depends only on h modulo 2D.*

This depends on a theorem on the behaviour of quadratic forms the equivalence of quadratic forms modulo a given number. This is a lemma of Siegel's (Annals of Mathematics, 1935, 527-606).

Let us recall that for k even

$$T_k(h,m) = \sum_{\underline{\ell} \mod k} e^{2\pi\frac{i}{k}(h\underline{\ell}'S\underline{\ell}+\underline{m}'\underline{\ell})}$$
$$= e^{-2\pi i\frac{h}{4k}D_k\sigma\underline{m}'S^{-1}\underline{m}}\mathscr{U}_k(h,\underline{m})$$

Lemma 2.

$$|T_k(h,\underline{m})| \leq Ck^{r/2}$$

We have

$$|T_k(h,\underline{m})|^2 = \sum_{\ell \mod k} e^{2\pi i\frac{h}{k}(\ell S\ell+\sigma\underline{m}'\underline{\ell})} \sum_{\lambda \mod k} e^{-2\pi i\frac{h}{k}(\underline{\lambda}'S\underline{\lambda}+\sigma\underline{m}'\underline{\ell})}$$

$$= \sum_{\underline{\ell},\underline{\lambda}} e^{2\pi i \frac{h}{k}(\underline{\ell}' S\, \underline{\ell} - \underline{\lambda}' S\, \underline{\lambda} + \sigma \underline{m}'(\underline{\ell} - \underline{\lambda}))},$$

and since **367**

$$\begin{aligned}\underline{\ell}' S\, \underline{\ell} - \underline{\lambda}' S\, \underline{\lambda} &= (\underline{\ell}' - \underline{\lambda}')S(\underline{\ell} + \underline{\lambda}) + \underline{\lambda}' S\, \underline{\ell} - \underline{\ell}' S\, \underline{\lambda}\\ &= (\underline{\ell}' - \underline{\lambda}')S(\underline{\ell} + \underline{\lambda}) + \underline{\ell}' S\, \underline{\ell} - \underline{\ell}' S\, \underline{\lambda}\\ &= (\underline{\ell}' - \underline{\lambda}')S(\underline{\ell} + \underline{\lambda}),\end{aligned}$$

this is equal to

$$\begin{aligned}&\sum_{\underline{\ell},\underline{\lambda}} e^{2\pi i \frac{h}{k}(\underline{\ell}' - \underline{\lambda}')(S(\underline{\ell} + \underline{\lambda}) + \sigma \underline{m})}\\ &= \sum_{\underline{\alpha} \bmod k} \; \sum_{\underline{\ell} - \underline{\lambda} \equiv \underline{\alpha} \bmod k} e^{2\pi i \frac{h}{k}\underline{\alpha}'(S(\underline{\ell} + \underline{\lambda}) + \sigma \underline{m})}\\ &= \sum_{\underline{\alpha} \bmod k} \; \sum_{\underline{\ell} - \underline{\lambda} \equiv \bmod k} e^{2\pi i \frac{h}{k}\underline{\alpha}'(S(2\underline{\lambda} + \underline{\alpha}) + \sigma \underline{m})}\\ &= \sum_{\underline{\alpha} \bmod k} e^{2\pi i \frac{h}{k}\underline{\alpha}'(S\underline{\alpha} + \sigma \underline{m})} \sum_{\underline{\lambda} \bmod k} e^{2\pi \frac{h}{k} 2\underline{\alpha}' S\, \underline{\lambda}}\end{aligned}$$

If we write $2\underline{\alpha}' S = \underline{\beta}'$, the inner sum is **368**

$$\sum_{\lambda_1,\ldots\lambda_r \bmod k} e^{2\pi i \frac{h}{k}(\beta_1\lambda_1 + \cdots + \beta_r\lambda_r)} = k^2, \text{ if } k \mid \beta_1, \ldots, k \mid \beta_r;$$

$$0 \text{ otherwise}$$

So $|T_k(h, \underline{m})|^2 = 0$ if at least one β is not divisible by k; otherwise it is equal to

$$k^r \sum_{\underline{\alpha} \bmod k} e^{2\pi i \frac{h}{k}\underline{\alpha}'(S\underline{\alpha} + \sigma \underline{m})}$$

Writing $S = (s_{jk})$, the system of congruences

$$\begin{aligned}2\alpha_1 s_{11} + 2\alpha_2 s_{21} + \cdots + 2\alpha_r s_{r1} &\equiv 0 \pmod{k}\\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot\\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot\\ 2\alpha_1 s_{1r} + 2\alpha_2 s_{2r} + \cdots + 2\alpha_r s_{rr} &\equiv 0 \pmod{k}\end{aligned}$$

has at most $2^r |S|^r$ solutions, and thus

$$|T_k(h, \underline{m})|^2 \leq 2^r |S|^r k^r,$$

i.e., $$|T_k(h,\underline{m})| \leq 2^{r/2}|S|^{r/2}k^{r/2}.$$

We shall now outline the main argument a little more skillfully, putting the thing back on its track. $A_r(n)$ is the sum of integrals over the finer-prepared Farey arcs of Kloosterman: **369**

$$A_r(n) = \frac{1}{2^{r/2}D^{1/2}} \sum_{0\leq h<k\leq N}{}' e^{-2\pi i \frac{h}{k} n} \int\limits_{-\frac{1}{k(k+N)}}^{\frac{1}{k(k+N)}} \frac{e^{2\pi n \mathfrak{z}}}{\mathfrak{z}^{r/2}} \sum_{\underline{s} \mod 2D} \mathscr{U}_k(h,\underline{s})\Theta_s\left(e^{-\frac{\pi}{2k^2\mathfrak{z}}+\frac{2\pi i}{4k}D_k\sigma}\right)d\varphi + \frac{1}{2^{r/2}D^{1/2}} \sum_{h,k} e^{-2\pi i\frac{h}{k}n} \sum_{\ell=k_2}^{N-1} \int\limits_{\frac{1}{k(k+\ell+1)}}^{\frac{1}{k(k+\ell)}} + \frac{1}{2^{r/2}D^{1/2}} \sum_{h,k} e^{-2\pi i \frac{h}{k}n} \sum_{\ell=k_1}^{N-1} \int\limits_{-\frac{1}{k(k+\ell)}}^{-\frac{1}{k(k+\ell+1)}} \cdots,$$

where

$$\begin{aligned}\Theta_s(x) &= \sum_{\underline{m}\equiv\underline{s} \pmod{2D}} x^{\underline{m}'S^{-1}\underline{m}};\\ &= S_0 + S_2 + S_1, \text{ say,}\\ &= \left(S_{00} + \sum_{\underline{m}\neq 0} S_{0m}\right) + \left(S_{20} + \sum_{\underline{m}\neq 0} S_{2\underline{m}}\right) + \left(S_{10} + \sum_{\underline{m}\neq 0} S_{1\underline{m}}\right)\end{aligned}$$

in an obvious notation. Now treat the things separately. By inspection of $\mathscr{U}_k(h,\underline{m})$ we find how it depends on h, it is only modulo 4 k^*. We have to reconcile Lemma 1 with this. This actual period therefore is neither $2D$ nor $4k^*$ but the greatest common divisor

$$\begin{aligned}(2D,4k^*) = 2(D,2k^*) &= 2\left(d, \frac{kD}{D_k}\right)\\ &= \frac{2}{D_k}(DD_k, 2kD) = \frac{2D}{D_k}(D_k,2k)\\ &= \frac{2D}{D_k} \text{ or } \frac{4D}{D_k}\end{aligned}$$

So we have **370**

Corollary of Lemma 1. $\mathscr{U}_k(h,m)$ for k even depends on h only modulo $\frac{2D}{D_k} = \wedge$, say.

$$S_{00} = \frac{1}{2^{r/2}D^{1/2}} \sum_{0\le h<k\le N} e^{-2\pi i \frac{h}{k} n} T_k(h,\underline{o}) \int\limits_{-\frac{1}{k(k+N)}}^{\frac{1}{l(k+N)}} \frac{e^{2\pi n \mathfrak{z}}}{\mathfrak{z}^{r/2}} d\varphi$$

This goes into the principal term. We shall make it a little more explicit later.

$$S_{o\underline{m}} = \frac{1}{2^{r/2}D^{1/2}} \sideset{}{'}\sum_{0\le h<k\le N} e^{-2\pi i \frac{h}{k} n} \mathscr{U}_k(h,\underline{m}) e^{2\pi i \frac{D_k}{4k}\sigma \underline{m}' S^{-1}\underline{m}} \int\limits_{-\frac{1}{k(k+N)}}^{\frac{1}{k(k+N)}} \frac{e^{-\frac{\pi}{2k^2\mathfrak{z}}\underline{m}' S^{-1}\underline{m}}}{\mathfrak{z}^{r/2}} d\varphi$$

$$= \frac{1}{2^{r/2}D^{1/2}} \sum_{k=1}^{N} K_k(n,\underline{m}) \int\limits_{-\frac{1}{k(k+N)}}^{\frac{1}{k(k+N)}} \cdots,$$

where **371**

$$K_k(n,\underline{m}) = \sideset{}{'}\sum_{h \mod k} e^{-2\pi i \frac{h}{k}} \mathscr{U}_k(h,\underline{m}) e^{2\pi \frac{i}{4k} D_k \sigma \underline{m}' S^{-1}\underline{m}}$$
$$= \frac{1}{a} \sum_{\lambda \mod \wedge} \mathscr{U}_k(\lambda,\underline{m}) \sum_{\substack{h\equiv\lambda \pmod{\wedge} \\ h \mod 4k^*}} e^{-\frac{2\pi i a h n.4 + 2\pi i \mathscr{V}\sigma}{4k^*}},$$

where $4k^* = ak$, $a \le 4D$, and $\mathscr{V} = \frac{k^*}{k} D_k \underline{m}^{-1} S^{-1}\underline{m}$

We defined σ by

$$hD_k\sigma \equiv 1 \pmod{4k^*}$$

Let

$$D\bar{D}_k \equiv 1 \pmod{4k^*}, h\bar{H} \equiv 1 \pmod{4k^*}$$

Then

$$K_k(n,\underline{m}) = \frac{1}{a} \sum_{\lambda \mod \wedge} \mathscr{U}_k(\lambda,\underline{m}) \sum_{\substack{h\equiv\lambda \pmod{\wedge} \\ h \mod k^*, (h,k^*)=1}} e^{\frac{-2\pi i a h n + 2\pi i \mathscr{V}\sigma}{4k^*}} D_k \bar{D} k$$
$$= \frac{1}{a} \sum_{\lambda \mod \wedge} \mathscr{U}_k(\lambda,\underline{m}) \sideset{}{'}\sum_{\substack{h\equiv\lambda \pmod{\wedge} \\ h \mod k^*}} e^{\frac{2\pi i}{4k}\left(-4anh + \mathscr{V}\bar{D}_k \bar{h}\right)}$$

The inner sum here is a Kloosterman sum. It has essentially $4k^*$ terms. A **372**
trivial estimate of this would be $O(k)$, and this is what we had tacitly assumed in the older method. The advantage here is, however, that they can be appraised letter. We shall not estimate them here but only quote the result as

Lemma 3.

$$K_k(u,v) = \sum_{\substack{h \equiv \lambda \pmod \wedge \\ h \bmod k}} e^{2\pi \frac{i}{k}(uh + v\bar{h})}, \wedge \mid k, h\bar{h} \equiv 1 \pmod k$$
$$= O\left(k^{1-\alpha+\epsilon}(u,k)^{\alpha}\right)$$

There has been a lot of discussion about the size of the α in this formula. Kloosterman and Esternann proved that $\alpha = \frac{1}{4}$ (Hamb, Ab. 1930), Salie' that $\alpha = 1/3$ and A.Weil that $\alpha = \frac{1}{2}$ (P.N.A.S', 48) Weil's was a very complicated and deep method going into the zeta-functions of Artin type and the Riemann hypothesis for these functions.

We thus save a good deal in the order of magnitude. The further S's will be nearly similar; the complete Kloosterman sums will be replaced by sums with certain conditions.

$$|S_{o\underline{m}}| \leq C \sum_{k=1}^{N} k^{1-\alpha+\epsilon} \frac{(n,k)^{\alpha}}{k^{r/2}} e^{-\frac{\pi}{4}\left(\underline{m}' S^{-1} \underline{m} - \frac{1}{D}\right)} \int_{-\frac{1}{k(k+N)}}^{\frac{1}{k(k+N)}} \frac{e^{-\mathscr{R}\left(\frac{\pi}{2k^2 \mathfrak{z}} \cdot \frac{1}{D}\right)}}{|\mathfrak{z}|^{r/2}} d\varphi$$

Since $\mathscr{R}\frac{1}{k^2}\mathfrak{z} \geq \frac{1}{k}$ on the Farey arc, the integrand is majorised by

$$e^{-\frac{\pi}{2D} \frac{\delta_N}{k^2 |\delta_N^2 + \varphi^2}1} |k^2(\delta_N^2 + \varphi^2)|^{-r/4} = \delta_N^{-r/4} \left(\frac{\delta_N}{k^2(\delta_N^2 + \varphi^2)}\right)^{r/4} e^{-\frac{\pi}{2D} \frac{\delta_N}{k^2\left(\delta_N^2 + \varphi^2\right)}}$$
$$= O(n^{r/4})$$
$$|S_{o\underline{m}}| \leq Cn^{r/4} \sum_{k=1}^{\sqrt{n}} k^{1-\alpha+\epsilon} (n,k)^{\alpha} e^{-\frac{\pi}{4}\underline{m}' S^{-1} \underline{m}} \frac{1}{k\sqrt{n}},$$

since the path of integration has a length of the order $1/k\sqrt{n}$. Now summing **373**
over all $\underline{m} \neq 0$,

$$\left|\sum_{\underline{m} \neq 0} S_{o\underline{m}}\right| \leq Cn^{\frac{r}{4}-\frac{1}{2}} \sum_{k=1}^{\sqrt{n}} k^{-\alpha+\epsilon}(n,k)^{\alpha}$$
$$< Cn^{\frac{r}{4}-\frac{1}{2}} \sum_{d|n} d^{\alpha} \sum_{dt \leq \sqrt{n}} (dt)^{-\alpha+\epsilon}$$
$$= Cn^{\frac{r}{4}-\frac{1}{2}} \sum_{d|n} d^{\epsilon} \sum_{t \leq \frac{\sqrt{n}}{d}} t^{-\alpha+\epsilon}$$

$$< Cn^{\frac{r}{4}-\frac{1}{2}} \sum_{d|n} d^{\epsilon} \left(\frac{\sqrt{n}}{d}\right)^{1-\alpha+\epsilon}$$
$$= Cn^{\frac{r}{4}-\frac{\alpha}{2}+\frac{\epsilon}{2}} \sum_{d|n} d^{\alpha-1},$$

and since the number of divisors of n is $O(n^{\epsilon/2})$. This is **374**

$$= Cn^{\frac{r}{4}-\frac{\alpha}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}}$$
$$= Cn^{\frac{r}{4}-\frac{\alpha}{2}+\epsilon}$$

Improving α has been the feature of many investigations.

Lecture 46

All the other sums that we have to estimate behave some what similarly. We take as specimen S_{20}. **375**

$$S_{20} = \frac{1}{D^{1/2}2^{r/2}} \sideset{}{'}\sum_{o\leq h<k\leq N} e^{-2\pi i \frac{h}{k}n} T_k(h, \underline{o}) \times \frac{1}{k^r} \times \sum_{\ell=k_2}^{N-1} \int\limits_{\frac{1}{k(k+\ell+1)}}^{\frac{1}{k(k+\ell)}} \frac{e^{2\pi n \mathfrak{z}}}{\mathfrak{z}^{r/2}} d\varphi$$

$$= \frac{1}{D^{1/2}2^{r/2}} \sum_{k=1}^{N} \frac{1}{k^r} \sum_{\ell=N-k+1}^{N-1} \int\limits_{\frac{1}{k(k+\ell+1)}}^{\frac{1}{k(k+\ell)}} \frac{e^{2\pi n \mathfrak{z}}}{\mathfrak{z}^{r/2}} \sum_{\substack{h \mod k\\ N-k<k_2\leq\ell}} e^{-2\pi i \frac{h}{k}n} T_k(h, \underline{o}) d\varphi$$

The original interval for k_2 was bigger: $N - k < k_2 \leq N$. Now the full interval is not permissible, i.e., we have admitted not all residues modulo k, but only a part of these, and the N may lie in two adjacent classes of residues.

Here we have a new type of sum of interest. We know how to discuss T_k; h plays a role there. The sums we have now get are

$$\sum_{\lambda \mod \wedge} T_k(\lambda, \underline{o}) \sideset{}{'}\sum_{\substack{h\equiv\lambda \pmod{\wedge}\\ N-k<k_2\leq\ell}} e^{-2\pi i \frac{h}{k}n}$$

The inner sum is an incomplete Ramanujan sum, with restriction on k_2 implying (see lecture 45) actually a restriction on h! The Kloosterman sums are a little more general: **376**

$$\sum_{h\bar{h}\equiv 1 \pmod{k}} e^{2\pi \frac{i}{k}(uh+\mathscr{V}\bar{h})}$$

Our present sums are incomplete Kloosterman sums (with $\mathscr{V} = o$ and $u =$

1), and the interesting fact is that they also permit the same appraisal, viz.

$$O\left(k^{r/2}k^{1-\alpha+\epsilon}(k,n)^{\alpha}\right)$$

From there on things go just as smoothly as before.

$$S_{2\underline{o}} = O\left(\sum_{k=1}^{\sqrt{n}} k^{-r/2}k^{1-\alpha+\epsilon}(k,n)^{\alpha}\int\limits_{\frac{1}{k(k+N)}}^{\frac{1}{k(N+1)}} \frac{d\varphi}{(\delta_N^2+\varphi^2)^{r/4}}\right)$$

and here for convergence of the integral we want $r \geq 3$. This would give again the old order. Similar estimates hold for the other pieces:

$$\sum_{\underline{m}\neq\underline{o}} S_{2\underline{m}} = O\left(n^{r/4} - \frac{\alpha}{2} + \epsilon\right)$$

(The incomplete Kloosterman sums here are actually incomplete Ramanujan sums and so we may got a slightly better estimate; but this is of no consequence as the other terms have a higher order). **377**

We then have

$$A_r(n) = S_{oo} + O\left(n^{r/4-\alpha/2-\epsilon}\right), \alpha = \frac{1}{2}.$$

Let us look at S_{oo}. It is classical, but not quite what we like it to be.

$$S_{oo} = \frac{1}{D^{1/2}2^{r/2}} \sum_{o\leq h<k\leq N}{}' e^{-2\pi i\frac{h}{k}n} \frac{T_k(h,\underline{o})}{k^r} \int\limits_{\frac{1}{k(k+N)}}^{\frac{1}{k(k+N)}} \frac{e^{2\pi n\mathfrak{z}}}{\mathfrak{z}^{r/2}} d\varphi + O\left(n^{r/4-\alpha/2+\epsilon}\right)$$

Replace the integral by an infinite integral:

$$\frac{1}{D^{1/2}2^{r/2}} \sum_{k=1}^{\sqrt{n}} \frac{H_k(n)}{k^r} \int\limits_{-\infty}^{\infty} \frac{2\pi n\mathfrak{z}}{\mathfrak{z}^{r/2}} d\varphi + O\left(n^{r/4-\alpha/2+\epsilon}\right),$$

with
$$H_k(n) = \sum_h e^{-2\pi i\frac{h}{k}nT_k(h,\underline{o})}$$
$$= O\left(k^{r/2}k^{1-\alpha+\epsilon}(k,n)^{\alpha}\right),$$

thereby adding an error term of order

$$O\left(\sum_{k=1}^{\sqrt{n}} k^{-\frac{r}{2}+1-\alpha+\epsilon}(k,n)^{\alpha}\int\limits_{\frac{1}{kN}}^{\infty} \frac{d\varphi}{(\delta_N^2+\varphi^2)^{r/4}}\right)$$

Now

$$\int_{\frac{1}{kN}}^{\infty} \frac{d\varphi}{(\delta_N^2+\varphi^2)^{r/4}} = \int_{\frac{1}{kN}}^{\infty} \frac{d\varphi}{\left(1+\left(\frac{\varphi}{\delta_N}\right)^2\right)^{r/4}} \delta_N^{1-r/2}$$
$$= O\left(n^{\frac{r}{2}-1} \int_{\frac{N}{k}}^{\infty} \frac{d\psi}{(1+\varphi^2)^{r/4}}\right)$$

with $\psi = N^2\varphi$. ψ is never smaller than 1 as $\frac{N}{k} > 1$. So we can drop 1 in the denominator without committing any error in the order of magnitude. So this gives **378**

$$O\left(n^{\frac{r}{2}-1} \int_{\frac{N}{k}}^{\infty} \frac{d\psi}{\psi^{r/2}}\right)$$

and the integral converging for $r \geq 3$, it is equal to

$$O\left(n^{\frac{r}{2}-1} \left(\frac{\sqrt{n}}{k}\right)^{-\frac{r}{2}+1}\right) = O\left(n^{\frac{r}{4}-\frac{1}{2}} k^{\frac{r}{2}-1}\right)$$

Hence our new error term is

$$O\left(\sum_{k=1}^{\sqrt{n}} k^{-\alpha+\epsilon}(n,k)^{\alpha} n^{\frac{r}{4}-\frac{1}{2}}\right) = O\left(n^{r/4-\alpha/2+\epsilon}\right)$$

which is what has already appeared.

We than have on writing $2\pi n\mathfrak{z} = \omega$,

$$A_r(n) = \frac{1}{D^{1/2}2^{r/2}} \sum_{k=1}^{\sqrt{n}} \frac{H_k(n)}{k^r} \frac{1}{i} (2\pi n)^{\frac{r}{2}-1} \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega}}{\omega^{r/2}} d\omega + O\left(n^{r/4-\alpha/2+\epsilon}\right),$$

and the integral being the Hankel integral for the gamma-function, **379**

$$A_r(n) = \frac{(2\pi)^{r/2} n^{r/2}-1}{D^{1/2}2^{r/2}} \sum_{k=1}^{\sqrt{n}} \frac{H_k(n)}{k^r} \frac{1}{\Gamma(r/2)} + O\left(n^{r/4-\alpha/2+\epsilon}\right)$$
$$= \frac{\pi^{r/2}}{\Gamma\left(\frac{r}{2}\right) D^{1/2}} n^{r/2-1} \sum_{k=1}^{\infty} \frac{H_k(n)}{k^r} + O\left(n^{r/4-\alpha/2+\epsilon}\right)$$

$$+ O\left(n^{\frac{r}{2}-1} \sum_{k=\sqrt{n}+1}^{\infty} \frac{k^{\frac{r}{2}+1-\alpha+\epsilon}(k,n)^{\alpha}}{k^r}\right)$$

This new error term is

$$O\left(n^{\frac{r}{2}-1} \sum_{d|n} d^{\alpha} \sum_{q>\frac{\sqrt{n}}{d}} (qd)^{1-\frac{r}{2}-\alpha+\epsilon}\right) = O\left(n^{\frac{r}{2}-1} \sum_{d|n} d^{1+\epsilon-r/2} \sum_{q>\frac{\sqrt{n}}{d}} q^{-r/2}\right)$$

(This is because for the Ramanujan sum we have)

$$\sum_{h \mod k}{}' e^{-2\pi i \frac{h}{k} n} = \sum_{d|(k,n)} d\mu\left(\frac{k}{d}\right)$$
$$= O\left((k,m) \sum_{d|(k,n)} 1\right) = O\left((k,n)^{1+\epsilon}\right);$$

and then we use the old appraisal ($k^{1-\alpha+\epsilon}(k,n)^{\alpha}$ with $\alpha = 1+\epsilon$). So we have **380**

$$O\left(n^{\frac{r}{2}-1} \sum_{d|n} d^{1+\epsilon-r/2} \left(\frac{\sqrt{n}}{d}\right)^{-r/2+1}\right) = O\left(n^{\frac{r}{4}-\frac{1}{2}} \sum_{d|n} d^{\epsilon}\right) = O\left(n^{r/4-1/2+2\epsilon}\right)$$

This is of smaller order than the old error term. So we have our final result:

$$A_r(n) = \frac{\pi^{r/2}}{\Gamma(r/2)D^{1/2}} n^{r/2-1} \sum_{k=1}^{\infty} \frac{H_k(n)}{k^r} + O\left(n^{r/4-\alpha/2+\epsilon}\right);$$

the singular series plus the error term.

What remains to be shown is that the singular series again enjoys the multiplicative property:

$$H_{k_1k_2}(n) = H_{k_1}(n)H_{k_2}(n)$$

We shall then have it as the product

$$\prod_p \gamma_p$$

where
$$\gamma_p = 1 + \frac{H_k(n)}{p^r} + \frac{H_{p^2}(n)}{p^{2r}} + \cdots$$

The arithmetical interpretation now becomes difficult, because all the properties that the quadratic form may have will have to show up. One or other of the factors γ_p may be zero in which case we have no representation. **381**

We should like to throw some light on the Kloosterman sums. We take for granted the estimate

$$\sum_{\substack{h \bmod k \\ h\bar{h}\equiv 1 \pmod k}}' e^{2\pi\frac{i}{k}(uh+\mathscr{V}\bar{h})} = O\left(k^{1-\alpha+\epsilon}\cdot(k,u)^{\alpha}\right)$$

Kloosterman and Esterman (Hamburger Abhandlungen Vol.7) proved $\alpha = \frac{1}{4}$; Salie' (Math. Zeit., vol. 36) proved $\alpha = \frac{1}{3}$. Using the multiplicativity, in a certain sense, of the sums, Salie' could prove that if $k = p^{\beta}$, p prime and $\beta \geq 2$, then $\alpha = \frac{1}{2}$ but he could not prove this in the other cases. The difficult case was that of

$$\sum_{h \bmod p}' e^{2\pi i/p(uh+\mathscr{V}\bar{h})}.$$

For this nothing better than $O\left(p^{2/3+\epsilon}(p,u)^{1/3}\right)$ could be obtained; and it defied all efforts until A.Weil proved $\alpha = 1/2$ in all cases by using deep methods (Proc. Nat. Acad. Sc.1948). Further application of the Kloosterman sums offer no difficulty.

The (generalised) Kloosterman sums are symmetrical in u and $\mathscr{V}$, for

$$\sum_{\substack{h\equiv\lambda(\wedge)\\ h \bmod k}}' e^{\frac{2\pi i}{k}(uh+\mathscr{V}\bar{h})} = \sum_{\substack{\bar{h}\equiv\bar{\lambda}(\wedge)\\ h \bmod k}}' e^{\frac{2\pi i}{k}(u\bar{h}+\mathscr{V}h)}$$

since $(\lambda,\wedge) = 1$, $h \equiv \lambda \pmod{\wedge}$ and $h\bar{h} \equiv 1 \pmod{\wedge}$ imply $\bar{h} \equiv \bar{\lambda} \pmod{\wedge}$ **382**
and $\lambda\bar{\lambda} \equiv 1 \pmod{\wedge}$. The last we can write as

$$\sum_{h \bmod k}' g(\bar{h})e^{\frac{2\pi i}{k}(uh+\mathscr{V}\bar{h})},$$

where $g(m)$ is the periodic function defined as

$$g(m) = \begin{cases} 1 & \text{if } m \equiv \bar{\lambda} \pmod{\wedge} \\ 0 & \text{otherwise.} \end{cases}$$

$g(m)$ has therefore the finite Fourier expansion

$$g(m) = \sum_{j=1}^{\wedge} C_j e^{2\pi i j \frac{m}{\wedge}}$$

The coefficients c_j can be calculated in the usual way:

$$c_q = \frac{1}{\wedge} e^{\frac{-2\pi i q\bar{\lambda}}{\wedge}} mq = 1, 2, \ldots, \wedge$$

Substituting for C_q, the sum becomes

$$\sum_{j \mod \wedge} C_j \sum'_{h \mod k} e^{2\pi i j \frac{\bar{h}}{\wedge}} e^{2\pi \frac{i}{k}(uh+\mathscr{V}\bar{h})} = \frac{1}{\wedge} \sum_{j \mod \wedge} e^{-2\pi i j \frac{\bar{\lambda}}{\wedge}} \sum_{h \mod k} e^{2\pi \frac{i}{k}\left(uh+(\mathscr{V}+\frac{jk}{\wedge})\bar{h}\right)}$$

so that the generalised sum becomes a finite combination of undisturbed Kloosterman sums and so has the estimate $O\left(k^{1-\alpha+\epsilon}(k,u)^{\alpha}\right)$

This works just as well in the other case when there is an inequality on $\bar{h}$. 383

$$\sum_{\substack{h\equiv\lambda \pmod{\wedge}, h \mod k \\ a\le\bar{h}\le b}} e^{\frac{2\pi i}{k}(uh+\mathscr{V}\bar{h})} = \sum_{\substack{h\equiv\lambda \pmod{\wedge} \\ h \mod k}} f(\bar{h}) e^{\frac{2\pi i}{k}(uh+\mathscr{V}\bar{h})}$$

where
$$f(m) = \begin{cases} 1, & 0 < m \le a, \\ 0, & a < m \le k, \end{cases}$$

and $f(m)$ is periodic modulo k.

Then
$$f(m) = \sum_{j=1}^{k} c_j e^{2\pi i j \frac{m}{k}}$$

where
$$c_j = \frac{1}{k} \frac{e^{-\frac{2\pi i j}{k}} - e^{-2\pi i j(a+1)/k}}{1 - e^{-2\pi i j/k}}, \; j \ne k,$$
$$c_k = \frac{a}{k}$$
$$|c_j| \le \frac{2}{k \sin \pi j/k}$$

The sum becomes 384

$$\sum_{j=1}^{k-1} c_j \sum'_{\substack{h \mod k \\ h\equiv\lambda \pmod{\wedge}}} e^{\frac{2\pi i}{k}(uh+(\mathscr{V}+j)\bar{h})} + c_k \sum'_{\substack{h \mod k \\ h\equiv\lambda \pmod{\wedge}}} e^{\frac{2\pi i}{k}(uh+\mathscr{V}\bar{h})}$$
$$= O\left(k^{1-\alpha+\epsilon}(h,k)^{\alpha}\right)\left\{1 + \frac{1}{k}\sum_{j=1}^{k-1} \frac{1}{|\sin \frac{\pi j}{k}|}\right\}$$

Since $\sin\alpha \ge \frac{2}{\pi}$,

$$2\sum_{j=1}^{\frac{k-1}{j}} \frac{1}{\sin\frac{\pi j}{k}} \le 2\frac{\pi}{2} \sum \frac{1}{\frac{\pi j}{k}}$$

$$= k \sum_{j \le \frac{k-1}{2}} \frac{1}{j} = O(k \log k)$$

so that again the sum becomes

$$O\left(k^{1-\alpha+\epsilon}(k,u)^{\alpha}\right)$$

Kloosterman first discussed his method for a diagonal quadratic form. Later on he applied it to modular forms and for this he could derive on the investigations by Hecke comparing modular forms with Eisenstein series. In this case the theory becomes simpler: we can subtract suitable Eisenstein series and the principle term then becomes zero. The r-fold theta-series that we had are in fact modular forms.