REAL ANALYSIS NOTES
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Contents

Chapter 1

Logic and Methods of Proof

1.1 Logic

In this course you will be expected to read, understand and construct proofs. The purpose of these notes is to teach you the language of Mathematics. Once you have understood the language of Mathematics, you will be able to communicate your ideas in a clear, coherent and comprehensible manner.

1.1.1 Definition
A **proposition** (or **statement**) is a sentence that is either true or false (not both).

1.1.2 Examples

[1] South Africa was beaten by New Zealand in the 2003 cricket world cup.

[2] February 17, 2003 was on a Tuesday.

[3] $3 + 6 = 11$.

[4] $\sqrt{2}$ is irrational.

1.1.3 Examples
(Examples of non-propositions).

[1] Jonty is handsome.

[2] What is the date?

[3] This statement is true.

There are two types of propositions: **atomic** and **compound** propositions.

- An atomic proposition is a proposition that cannot be divided into smaller propositions.
- A compound proposition is a proposition that has parts that are propositions. Compound propositions are built by using **connectives**.

1.1.4 Examples
(Examples of atomic propositions).

[1] John's leg is broken.

[2] Our universe is infinite.

[3] 2 is a prime number.

[4] There are infinitely many primes.

1.1.5 Examples
(Examples of compound propositions).

[1] Jim and Anne went to the movies.

[2] $3 \leq 7$.

[3] n^2 is odd whenever n is an odd integer.

[4] If a function is differentiable, then it is continuous.

[5] If $f' > 0$, then f is increasing.

[6] If f is increasing and f' exists, then $f' > 0$.

Let us look at some of the most commonly used connectives:

Name	English name	Symbol
Conjunction	and	$\wedge$
Disjunction	or	$\vee$
Implication	If ... then	$\Rightarrow$
Biconditional	if and only if	$\Leftrightarrow$
Negation	not	$\neg$

One has to be careful when using everyday English words in Mathematics as they may not carry the same meaning in Mathematics as they do in everyday non-mathematical usage. One such word is *or*. In everyday parlance, the word *or* means that you have a choice of one thing or the other but **not both** - *exclusive* disjunction. In Mathematics, on the other hand, the word *or* stands for an *inclusive* disjunction, i.e., you have a choice of one thing or the other or both.

We shall use the capital letters $P,\ Q,\ R,\ \ldots$ to denote atomic propositions.

1.1.6 Examples
(Using symbols to represent compound statements).

[1] If Lucille has credit for MAT 1E1 and MAT1E2, then she cannot get credit for MAT101.

Let P stand for the statement "Lucille has credit for MAT 1E1", Q stand for the statement "Lucille has credit for MAT 1E2", and R stand for the statement "Lucille can get credit for MAT 101." Then the above statement can represented symbolically as $(P \wedge Q) \Rightarrow \neg R$.

[2] If Lucille has credit for MAM100W or has credit for MAM105H and MAM106H, then she do MAM200W.

Let P stand for the statement "Lucille has credit for MAM100W", Q stand for the statement "Lucille has credit for MAM105H", R stand for the statement "Lucille has credit for MAM106H", and S stand for the statement "Lucille can do MAM200W." Then the above statement can represented symbolically as $[P \vee (Q \wedge R)] \Rightarrow S$.

[3] Either you pay your rent or I will kick you out of the apartment.

Let P stand for the statement “You pay your rent”, and Q stand for the statement “I will kick you out of the apartment.” Then the above statement can represented symbolically as $P \vee Q$.

[4] Joe will leave home and not come back again.

Let P stand for the statement “Joe will leave home”, and Q stand for the statement “Joe will come back again.” Then the above statement can represented symbolically as $P \wedge \neg Q$.

[5] The lights are on if and only if either John or Mary is at home.

Let P stand for the statement “The lights are on”, Q stand for the statement “John is at home”, and S stand for the statement “Mary is at home.” Then the above statement can represented symbolically as $P \Leftrightarrow (Q \vee S)$.

A **truth table** is a convenient device to specify all of the possible truth values of a given atomic or compound proposition. We use truth tables to determine the truth or falsity of a compound proposition based on the truth or falsity of its constituent atomic propositions.

When we evaluate the truth or falsity of a statement, we assign to it one of the labels T for “true” and F for “false”. We also use 1 for “true” and 0 for “false”.

Let us construct truth tables for the above connectives.

[1] **Conjunction**: Let P and Q be two propositions. The proposition $P \wedge Q$ is called the conjunction of P and Q. The proposition $P \wedge Q$ is true if and only if both atomic propositions P **and** Q are true. In other words, if either or both atomic propositions P and Q are false, then the conjunction $P \wedge Q$ is false.

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

1.1.7 Examples

P: Cape Town is in the Western Cape and $\sqrt{3}$ is irrational.
Q: $\sqrt{5} < 3$ and $f(x) = |x|$ is differentiable at $x = 0$.
R: Harare is the capital of Botswana and $f(x) = \cos x$ is continuous on $\mathbb{R}$.
S: $-2 < -10$ and 8 is an odd number.

Only P is true; all the others are false.

[2] **Disjunction**: Let P and Q be two propositions. The proposition $P \vee Q$ is called the disjunction of P and Q. The proposition $P \vee Q$ is true if and only if at least one of the atomic propositions P or Q is true.

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

It is clear from this truth table that the proposition $P \vee Q$ will be false only when **both** P and Q are false.

1.1.8 Examples

(a) $\pi > 2$ or π is an irrational number.

(b) $\pi > 2$ or π is a rational number.

(c) $\pi < 2$ or π is an irrational number.

(d) $\pi < 2$ or π is a rational number.

All these propositions, except (d), are true.

[3] **Implication**: Let P and Q be two propositions. The proposition $P \Rightarrow Q$ is referred to as a *conditional* proposition. It simply means that P implies Q. In the statement $P \Rightarrow Q$, P is called the *hypothesis* (or *antecedent or condition*) and Q is called the *conclusion* (or *consequent*).

There are various ways of stating that P implies Q:

- If P, then Q.
- Q if P.
- P is sufficient for Q.
- Q is necessary for P.
- P only if Q.
- Q whenever P.

P	Q	$P \Rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

It is clear from this truth table that the proposition $P \Rightarrow Q$ will be false only when P is true and Q is false.

In order to have some appreciation of why the above truth table is reasonable, consider the following: If you pass MAM200W exam, I will buy you a cell-phone.

Let P: You pass MAM200W exam.

Let Q: I will buy you a cell-phone.

At the end of MAM200W exam, there are various scenarios that may arise.

(a) You have passed MAM200W exam and then I buy you a cell-phone. You will be happy and feel that I was telling the truth . Therefore $P \Rightarrow Q$ is true.

(b) You have passed MAM200W exam but I refuse to buy you a cell-phone. You will feel cheated and lied to. Therefore $P \Rightarrow Q$ is false.

(c) You have failed MAM200W, but I still buy you a cell-phone. You are unlikely to question that, are you? We did not cover this contingency in my conditional statement.

(d) You have failed MAM200W and, consequently I do not buy you a cell-phone. You will not feel that I have been unfair to you and that I have not kept my promise.

1.1.9 Examples

(a) If $\pi > 2$, then π is an irrational number.

(b) If $\pi > 2$, then π is a rational number.

(c) If $\pi < 2$, then π is an irrational number.

(d) If $\pi < 2$, then π is a rational number.

All these propositions, except (b), are true.

1.1.10 Definition
Let P and Q be propositions. The **converse** of the proposition $P \Rightarrow Q$ is the proposition $Q \Rightarrow P$.

1.1.11 Examples
(Examples of converse statements).

(a) If it is cold, then the lake is frozen.
Converse: If the lake is frozen, then it is cold.

(b) Johny is happy if he is healthy.
Converse: If Johny happy, then he is healthy.

(c) If it rains, Zinzi does not take a walk.
Converse: If Zinzi does not take a walk, then it rains.

The truth table of a proposition and its converse:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
1	1	1	1
1	0	0	1
0	1	1	0
0	0	1	1

Note that the truth tables of $P \Rightarrow Q$ and $Q \Rightarrow P$ are not the same.

Consider the following conditional proposition and its converse:
Proposition: If $\pi > 2$, then $\sqrt{3}$ is rational.
Converse: If $\sqrt{3}$ is rational, then $\pi > 2$.
In this example the conditional statement is false whereas its converse is true. Hence this conditional proposition and its converse are not equivalent.

Consider the following conditional proposition and its converse:
Proposition: If $\pi > 2$, then $\sqrt{3}$ is irrational.
Converse: If $\sqrt{3}$ is irrational, then $\pi > 2$.
Here both that conditional proposition and its converse are true. If, in this example, we let P stand for the proposition "$\pi > 2$" and Q for "$\sqrt{3}$ is irrational", then we have that both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true.

[4] **Biconditional Proposition**: Let P and Q be propositions. The proposition $P \Leftrightarrow Q$ is referred to as a *biconditional* proposition. It simply means that $P \Rightarrow Q$ and $Q \Rightarrow P$. It is called a "biconditional proposition" because it represents two conditional propositions.

There are various ways of stating the proposition $P \Leftrightarrow Q$:

- P if and only if Q (also written as P iff Q).

- P implies Q and Q implies P.
- P is necessary and sufficient for Q.
- Q is necessary and sufficient for P.
- P is equivalent to Q.

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Note that the statement $P \Leftrightarrow Q$ is true precisely in the cases where P and Q are both true or P and Q are both false.

[5] **Negation**: Let P be a proposition. The proposition $\neg P$, meaning "not P", is used to denote the negation of P. If P is true, then $\neg P$ is false and vice versa.

P	$\neg P$
1	0
0	1

Let us construct a few more truth tables.

1.1.12 Examples

[1] Let P and Q be propositions. Construct a truth table for the proposition $(P \wedge Q) \Rightarrow (P \vee Q)$.

Solution:

P	Q	$P \wedge Q$	$P \vee Q$	$P \wedge Q \Rightarrow P \vee Q$
1	1	1	1	1
1	0	0	1	1
0	1	0	1	1
0	0	0	0	1

[2] Let P, Q and R be propositions. Construct the truth table for the proposition $\neg(P \wedge Q) \vee R$.

Solution:

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg(P \wedge Q) \vee R$
1	1	1	1	0	1
1	1	0	1	0	0
1	0	1	0	1	1
0	1	1	0	1	1
1	0	0	0	1	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	1	1

1.2 Tautologies, Contradictions and Equivalences

Some compound propositions are always true while others are always false.

1.2.1 Definition
A compound proposition is a **tautology** if it is always true regardless of the truth values of its atomic propositions. If, on the other hand, a compound proposition is always false regardless of its atomic propositions, we say that such a proposition is a **contradiction**.

1.2.2 Example
The statement $P \vee \neg P$ is always true while the statement $P \wedge \neg P$ is always false.

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
1	0	1	0
0	1	1	0

1.2.3 Remark
In a truth table, if a proposition is a tautology, then every line in its column will have 1 as its entry; if a proposition is a contradiction, every line in its column will have 0 as its entry.

1.2.4 Definition
Let P and Q be propositions. The **contrapositive** of the proposition $P \Rightarrow Q$ is the proposition $\neg Q \Rightarrow \neg P$.

1.2.5 Examples
(Examples of contrapositive statements).

[1] If it is cold, then the lake is frozen.

Contrapositive: If the lake is not frozen, then it is not cold.

[2] If Johny is healthy, then he is happy.

Contrapositive: If Johny not happy, then he is not healthy.

[3] If it rains, Zinzi does not take a walk.

Contrapositive: If Zinzi takes a walk, then it does not rain.

DO NOT CONFUSE THE CONTRAPOSITIVE AND THE CONVERSE. Here is the difference:

Converse: The hypothesis of a converse statement is the conclusion of the conditional statement and the conclusion of the converse statement is the hypothesis of the conditional statement.

Contrapositive: The hypothesis of a contrapositive statement is the *negation* of conclusion of the conditional statement and the conclusion of the contrapositive statement is the *negation* of hypothesis of the conditional statement.

1.2.6 Examples

[1] If Bronwyne lives in Cape Town, then she lives of South Africa.

Converse: If Bronwyne lives in South Africa, then she lives in Cape Town.

Contrapositive: If Bronwyne does not live in South Africa, then she does not live Cape Town.

[2] If it is morning, then the sun is in the east.

Converse: If the sun is in the east, then it is morning.

Contrapositive: If the sun is not in the east, then it is not morning.

1.2.7 Definition
Two propositions P and Q are said to be **logically equivalent**, written as $P \equiv Q$, if $P \Leftrightarrow Q$ is a tautology. Logically equivalent statements have the same truth values.

1.2.8 Remark
When we write "$P \equiv Q$", we basically say that proposition P *means the same* as proposition Q.

Here is an important example: $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$. That is, the conditional and its contrapositive say the same thing.

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
1	1	1	0	0	1	1
1	0	0	0	1	0	1
0	1	1	1	0	1	1
0	0	1	1	1	1	1

1.2.9 Theorem
Let P, Q and R be propositions. Then

(a) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

(b) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

(c) $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

(d) $P \Rightarrow Q \equiv \neg P \vee Q$

(e) $\neg(\neg P) \equiv P$

(f) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

(g) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(h) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

(i) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

Proof. (a) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$
1	1	1	0	0	0	0	1
1	0	0	1	0	1	1	1
0	1	0	1	1	0	1	1
0	0	0	1	1	1	1	1

(c) $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$	$\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$
1	1	1	0	0	0	1
1	0	0	1	1	1	1
0	1	1	0	0	0	1
0	0	1	0	1	0	1

Try to convince yourself that all the other statements are valid.

Let us analyze the following argument: If girls are blonde, they are popular with boys. Ugly girls are unpopular with boys. Intellectual girls are ugly. Therefore blonde girls are not intellectual.

Is this argument valid?

Solution: Let us use letters and connectives to represent the above statement.

P: Girls are blonde.

Q: Girls are popular with boys.

R: Girls are ugly.

S: Girls are intellectual.

We can represent the above argument as follows:

$$P \Rightarrow Q, \quad R \Rightarrow \neg Q, \quad S \Rightarrow R.$$

Since $S \Rightarrow R$ and $R \Rightarrow \neg Q$, we can conclude that $S \Rightarrow \neg Q$.

Since $P \Rightarrow Q$, we have, by contrapositive, that $\neg Q \Rightarrow \neg P$. Hence, $S \Rightarrow \neg P$.

Again, by contrapositive, $P \Rightarrow \neg S$, which says that "Blonde girls are not intellectual." Therefore the argument is valid.

1.3 Open Sentences and Quantifiers

In mathematics, one frequently comes across sentences that involve a variable. For example, $x^2+2x-3=0$ is one such. The truth or falsity of this statement depends on the value you assign for the variable x. For example, if $x=1$, then this sentence is true, whereas if $x=-1$, this sentence is false.

1.3.1 Definition

An **open sentence**(also called a **predicate**) is a sentence that contains variables and whose truth or falsity depends on the values assigned for the variables. We represent an open sentence by a capital letter followed by the variable(s) in parenthesis, e.g., $P(x)$, $Q(x, y)$ etc.

1.3.2 Examples

(Open statements).

[1] $x+4=-9$

[2] $x<y$.

[3] She is the queen of jazz.

[4] It has four legs.

1.3.3 Definition

The collection of all allowable values for the variable in an open sentence is called the **universe of discourse**.

Let $P(x)$ be an open sentence containing a free variable x. We want to *quantify* the number of x for which $P(x)$ is true. In particular, we want to say that $P(x)$ is true for at least one x or for all x in the universe of discourse.

Universal Quantifier ($\forall$): To say that $P(x)$ is true for all x in the universe of discourse, we write $(\forall x)P(x)$. Think of the symbol $\forall$ as an inverted A (representing *all*). $\forall$ is called the **universal quantifier**.

$$\forall \text{ means } \begin{cases} \text{all} \\ \text{for all} \\ \text{for every} \\ \text{for each} \end{cases}$$

Existential Quantifier ($\exists$): To say that there is (at least one) x in the universe of discourse for which $P(x)$ is true, we write $(\exists x)P(x)$. Think of the symbol $\exists$ as the backwards capital E (representing *exists*). $\exists$ is called the **existential quantifier**.

$$\exists \text{ means } \begin{cases} \text{there is} \\ \text{there exists} \\ \text{for some} \end{cases}$$

Symbolic Statement	*Translation*
$(\forall x)P(x)$	For all x, $P(x)$ is true
$(\forall x)(\neg P(x))$	For all x, $P(x)$ is false (There is no x for which $P(x)$ is true)
$(\exists x)P(x)$	There exists an x for which $P(x)$ is true
$(\exists x)(\neg P(x))$	There is an x for which $P(x)$ is false
$(\forall x)(\forall y)P(x, y)$	$P(x, y)$ is true for all pairs (x, y)
$(\exists x)(\exists y)P(x, y)$	There is a pair (x, y) for which $P(x, y)$ is true
$(\forall x)(\exists y)P(x, y)$	For each x, there is a y for which $P(x, y)$ is true
$(\exists x)(\forall y)P(x, y)$	There is an x for which $P(x, y)$ is true for every y

1.3.4 Remark

Quantifying an open sentence makes it a proposition.

1.3.5 Examples

Write the following statements using quantifiers.

(a) For each real number $x > 0$, $x^2 + x - 6 = 0$.

Solution: $(\forall x > 0)(x^2 + x - 6 = 0)$.

(b) There is a real number $x > 0$ such that $x^2 + x - 6 = 0$.

Solution: $(\exists x > 0)(x^2 + x - 6 = 0)$.

(c) The square of any real number is nonnegative.

Solution: $(\forall x \in \mathbb{R})(x^2 \geq 0)$.

(d) For each integer x there is an integer y such that $x + y = -1$.

Solution: $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x + y = -1)$.

(e) There is an integer x such that for each integer y, $x + y = -1$.

Solution: $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x + y = -1)$.

Do examples (d) and (e) convey the same message?
The answer is NO. Statement (d) is true: given any integer x, there is an integer, namely, $y = -1 - x$, such that $x + y = -1$. Statement (e) is false.
ORDER DOES MATTERS AFTER ALL!

1.3.6 Remark
In the statement $(\forall x)(\exists y)P(x, y)$, the choice of y is allowed to depend on x - the y that works for one x need not work for another x. On the other hand, in the statement $(\exists y)(\forall x)P(x, y)$, the y must work for all x, i.e., y is *independent* of x.

1.3.7 Examples
Translate the following into English.

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x = y^2)$.

Solution: Every real number is a perfect square.

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 0)$.

Solution: Every real number has an additive inverse.

Negation of Quantifiers

Symbolic Statement	*Translation*
$\neg[(\forall x)P(x)] \equiv (\exists x)(\neg P(x))$	There is an x for which $P(x)$ is false
$\neg[(\exists x)P(x)] \equiv (\forall x)(\neg P(x))$	$P(x)$ is false for every x
$\neg[(\forall x)(\exists y)P(x, y)] \equiv (\exists x)(\forall y)(\neg P(x, y))$	There is an x for which $P(x, y)$ is false for every y
$\neg[(\exists y)(\forall x)P(x, y)] \equiv (\forall y)(\exists x)(\neg P(x, y))$	For each y there is an x for which $P(x, y)$ is false
$\neg[(\forall x)(\forall y)P(x, y)] \equiv (\exists x)(\exists y)(\neg P(x, y))$	There is a pair (x, y) for which $P(x, y)$ is false
$\neg[(\exists x)(\exists y)P(x, y)] \equiv (\forall x)(\forall y)(\neg P(x, y))$	$P(x, y)$ is false for every pair (x, y)

1.3.8 Remark
To negate a statement that involves the quantifiers $\forall$ and $\exists$, change each $\forall$ to $\exists$, change each $\exists$ to $\forall$, and negate the open sentence (predicate).

1.3.9 Examples

[1] All birds can fly.

Negation: There is (at least one) bird that cannot fly.

1.3.10 Exercise
Write the following statements using quantifiers.

(a) A function f has *limit* L at a point a, denoted by $\lim_{x\to a} f(x) = L$, if and only if given any $\epsilon > 0$, there is a $\delta > 0$ such that for each x in the domain of f, we have that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Ans. $\left(\lim_{x\to a} f(x) = L\right) \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \text{dom}(f))[0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon]$.

(b) Write down the negation of (a).

Ans. $\left(\lim_{x\to a} f(x) \neq L\right) \Leftrightarrow (\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \text{dom}(f))[(0 < |x-a| < \delta) \wedge (|f(x)-L| \geq \epsilon)]$.

(c) A function f is *continuous* at $x = a$ if and only if given any $\epsilon > 0$, there is a $\delta > 0$ such that for each x in the domain of f, we have that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Ans. (f is continuous at $x = a$) $\Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \text{dom}(f))[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon]$.

(d) Write down the negation of (c).

(f is discontinuous at $x = a$) $\Leftrightarrow (\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \text{dom}(f))[(|x - a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon)]$.

Overgeneralization and Counterexample

Overgeneralization occurs when a pattern searcher discovers a pattern among finitely many cases and then claim that the pattern holds in general (when in fact it doesn't).

To disprove a general (universally quantified) statement such as $(\forall x)P(x)$, we must exhibit one x for which $P(x)$ is false. That is, $(\exists x)\neg P(x)$. This particular x is called a **counterexample** to the statement that $(\forall x)P(x)$ is true.

1.3.11 Examples

[1] *Statement*: $(\forall x \in \mathbb{R})(x < x^2)$.

The above statement is false. $x = \frac{1}{2}$ is a counterexample since $\frac{1}{2} \in \mathbb{R}$ but $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$.

[2] For all real numbers x and y, $|x + y| = |x| + |y|$.

This is false. Counterexample: take $x = 1$ and $y = -1$. Then $0 = |0| = |-1 + 1| \neq |-1| + |1| = 2$.

The statement $(\forall x)[P(x) \Rightarrow Q(x)]$ occurs frequently in Mathematics. Recall that

$$\neg[(\forall x)(P(x) \Rightarrow Q(x))] \equiv (\exists x)[P(x) \wedge \neg Q(x)].$$

Therefore, to show that the implication $P(x) \Rightarrow Q(x)$ is false, all that you have to do is produce ONE x for which $P(x)$ is true but $Q(x)$ is false.

1.3.12 Examples

[1] If a function f is continuous, then it is differentiable.

This statement is false since $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

[2] For all real numbers a, b, and c, if $ac = bc$, then $a = b$.

This statement is false. Take $a = 1$, $b = 7$, and $c = 0$. Then $0 = ac = bc = 0$, but $1 = a \neq b = 7$.

[3] For all prime numbers p, $2p + 1$ is prime.

While this statement is true for $p = 2, 3, 5$, it is false for $p = 7$ since $2 \times 7 + 1 = 15$ which is not prime. So $p = 7$ is a counterexample to the given statement.

1.4 Methods of Proof in Mathematics

In Mathematics we make assertions about systems, e.g. number system. The process of establishing the truth of an assertion is called a **proof**. That is, a **proof** in Mathematics is a sequence of logically sound arguments which establish the truth of a statement in question.

Theorem statements are normally in conditional form ($P \Rightarrow Q$) or biconditional form ($P \Leftrightarrow Q$).

Suppose that we wish to establish the truth of the assertion $P \Rightarrow Q$.

1.4.1 Direct Method

In this method of proof, we assume that P is true and proceed through a sequence of logical steps to arrive at the conclusion that Q is also true.

1.4.1 Examples

(a) Show that if m is an even integer and n is an odd integer, then $n + m$ is an odd integer.

Solution: Assume that m is an even integer and m is an odd integer. Then $m = 2k$ and $n = 2\ell + 1$ for some integers k and ℓ. Therefore

$$m + n = 2k + 2\ell + 1 = 2(k + \ell) + 1.$$

Since $k + \ell$ is an integer whenever k and ℓ are integers, we conclude that $m + n$ is an odd integer.

(b) Show that if n is an even integer, then n^2 is also an even integer.

Solution: Assume that n is an even integer. Then $n = 2k$ for some integer k. Now,

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since $2k^2$ is an integer, it follows that n^2 is an even integer.

1.4.2 Contrapositive Method

Associated with the implication $P \Rightarrow Q$ is the logically equivalent statement $\neg Q \Rightarrow \neg P$, the contrapositive of the conditional $P \Rightarrow Q$. Therefore one way of proving the conditional $P \Rightarrow Q$ is to give a direct proof of its contrapositive $\neg Q \Rightarrow \neg P$. The first step in the proof is to write down the negation of the conclusion. Then you show by a series of logical steps that this leads to the negation of the hypothesis of the original conditional statement.

1.4.2 Examples

(a) Show that if n^2 is an even integer, then n is an even integer.

Solution: We will show the contrapositive -if n is an odd integer, then n^2 is an odd integer. To that end, assume that n is an odd integer. Then, $n = 2\ell + 1$ for some integer ℓ. Now,

$$n^2 = (2\ell + 1)^2 = 4\ell^2 + 4\ell + 1 = 2(2\ell^2 + 2\ell) + 1.$$

Since $2\ell^2 + 2\ell$ is an integer, we conclude that n^2 is an odd integer.

(b) Show that if $3n$ is an odd integer, then n is an odd integer.

Solution: We will show the contrapositive - if n is an even integer, then $3n$ is an even integer. To that end, assume that n is an even integer. Then $n = 2k$ for some integer k. Therefore $3n = 3(2k) = 2(3k)$. It follows that $3n$ is an even integer.

1.4.3 Contradiction Method

Proof by contradiction, also called *reductio ad absurdum*, is one of the most powerful methods of proof in Mathematics. It also tends to be harder to understand than the direct or contrapositive methods. Here is how it works: assume that the P is true and Q is false, i.e. assume that the statement $P \wedge \neg Q$ is true. Then show, in a series of logical steps, that this leads to a contradiction, impossibility or absurdity e.g., $R \wedge \neg R$. This will then mean that the assumption that $P \wedge \neg Q$ must have been fallacious, and therefore its negation $\neg(P \wedge \neg Q)$ must be true. Since $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$, it follows that $(P \Rightarrow Q) \equiv \neg(P \wedge \neg Q)$, and hence $P \Rightarrow Q$ must be true.

Before giving some examples, let us define what it means for a number to be rational.

1.4.3 Definition
A real number r is said to be **rational** if there are integers m and n ($n \neq 0$) such that $r = m/n$. We denote the set of all rational numbers by the letter $\mathbb{Q}$. A real number that is not rational is said to be **irrational**.

1.4.4 Examples
(Proof by Contradiction).

(a) Show that $\sqrt{2}$ is irrational. That is, there do not exist integers p and q such that $\frac{p}{q} = \sqrt{2}$.

Solution: Proceeding by contradiction, assume that there are integers p and q such that $\frac{p}{q} = \sqrt{2}$. By cancelling any common factors, we may suppose that p and q have no common factors. Then squaring both sides, we have that

$$\frac{p^2}{q^2} = 2 \;\Leftrightarrow\; p^2 = 2q^2.$$

Hence p^2 is even. By Example 1.4.2(a), we have that p is even. Hence we can express p as $p = 2k$ for some integer k. So,

$$2q^2 = p^2 = (2k)^2 = 4k^2 \text{ and, consequently, } q^2 = 2k^2.$$

This means that q^2 is even and so, again by Example 1.4.2(a), we have that q is even. Hence p and q are both even, contradicting the assumption that p and q have no factors in common. Therefore $\sqrt{2}$ is not of the form $\frac{p}{q}$ for some integers p and q. That is, $\sqrt{2}$ is irrational.

(b) Show that if $3n$ is an odd integer, then n is an odd integer.

Solution: We will use contradiction: Assume that $3n$ is an odd integer and n is an even integer. Then $3n = 2k + 1$ and $n = 2\ell$ for some integers k and ℓ. Thus

$$2k + 1 = 3n = 3(2\ell) = 2(3\ell).$$

This shows that $3n$ is both odd and even,which is absurd. Hence n is an odd integer.

Chapter 2

Sets and Functions

2.1 Introduction

The concept of a *set* permeates every aspect of Mathematics. Set theory underlies the language and concepts of modern Mathematics. The term *set* refers to a well-defined collection of objects that share a certain property or certain properties. The term "well-defined" here means that the set is described in such a way that one can decide whether or not a given object belongs in the set. If A is a set, then the objects of the collection A are called the *elements* or *members* of the set A. If x is an element of the set A, we write $x \in A$. If x is NOT an element of the set A, we write $x \notin A$.

As a convention, we use capital letters to denote the names of sets and lowercase letters for elements of a set.

There are several ways of describing sets, but two are common:

[1] **The Roster method**:- listing the elements of a set, separated by commas and enclosed in braces; e.g., $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. There are two important facts to bear in mind: (1) the order in which the elements are listed is irrelevant, (2) each element should be listed only once in the roster.

[2] **The rule or description method**:- we describe a set in terms of one or more properties that the objects in the set must satisfy. We use set-builder notation to write such a set, e.g., $A = \{x \mid x \text{ satisfies some property or properties}\}$. The vertical bar " | " is read as "such that". Other people use " : " instead of the bar " | ".

If a set A consists of a large (or infinite) number of elements, it is general practice to list a few of its elements followed with ellipsis (...). This method requires recognition of the pattern in the list of elements of A. This practice tends to introduce some ambiguity as the list may be continued in many different ways. It is safer practice to define such a set by spelling out the pattern that determines membership of the set.

2.1.1 Examples

(a) The set $\mathbb{E} = \{2, 4, 6, 8, \ldots\}$ is best described as

$$\mathbb{E} = \{n \in \mathbb{N} \ : \ n = 2k \ \text{ for some } \ k \in \mathbb{N}\}.$$

(b) The set $B = \{1, 4, 9, 16, \ldots\}$ is best described as

$$B = \{n \in \mathbb{N} \ : \ n = k^2 \ \text{ for some } \ k \in \mathbb{N}\}.$$

Some sets come up often in Mathematics and they have special names assigned to them.

$\mathbb{N}_0 = \{0,\ 1,\ 2,\ 3, \ldots\}$	Natural numbers
$\mathbb{N} = \{1,\ 2,\ 3, \ldots\}$	Positive natural numbers
$\mathbb{Z} = \{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2, \ldots\}$	Integers
$\mathbb{Q} = \{p/q\ :\ p,\ q \in \mathbb{Z}, q \neq 0\}$	Rational numbers
$\mathbb{Q}^+$	Positive rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{C} = \{z = a + bi\ :\ a, b \in \mathbb{R} \text{ and } i^2 = -1\}$	Complex numbers.

2.1.2 Definition

Let A and B be sets. We say that

(a) *B is a* **subset** *of A (or is* **contained** *in A), denoted by $B \subseteq A$, if every element of B is an element of A, i.e., $(\forall x)(x \in B \Rightarrow x \in A)$.*

(b) *$A = B$ if $(A \subseteq B) \wedge (B \subseteq A)$, i.e., $(\forall x)(x \in A \Leftrightarrow x \in B)$.*

(c) *If B is a subset of A and $A \neq B$, then B is a* **proper** *subset of A. In this case we write $B \subsetneq A$. It is clear that*

$$B \subsetneq A \Leftrightarrow [(\forall x)(x \in B \Rightarrow x \in A) \wedge (B \neq A)].$$

2.1.3 Example

Let

$$\begin{aligned} A &= \{-1,\ 0,\ 1,\ 2,\ 3,\ 4\} \\ B &= \{1,\ 2,\ 3\} \\ C &= \{x \in \mathbb{R}\ :\ x^3 - 6x^2 + 11x - 6 = 0\} \\ D &= \{-1,\ 0,\ 1,\ 8\} \\ E &= \{x \in \mathbb{Z}\ :\ -2 < x < 5\}. \end{aligned}$$

Then

$$3 \in A, \qquad -2 \notin B, \qquad B \subseteq A, \qquad D \nsubseteq A,$$

$$B = C, \qquad A = E, \qquad \{8\} \subseteq D, \qquad \{2, 3\} \subseteq B.$$

We say that a set is **empty** if it has no elements. For example,

$$\{x \in \mathbb{R}\ :\ x^2 + 1 = 0\}$$

is an empty set since the equation $x^2 + 1 = 0$ has no solution in $\mathbb{R}$.

2.1.4 Proposition

(a) *If B is an empty set, then $B \subseteq A$ for any set A.*

(b) *All empty sets are equal, i.e., if B and C are empty sets, then $B = C$.*

Proof. (a) We must show that every element of B is an element of A; i.e., $(\forall x)(x \in B \Rightarrow x \in A)$. Using the contrapositive method, it suffices to show that $(\forall x)(x \notin A \Rightarrow x \notin B)$. This is vacuously true since if $x \notin A$, then $x \notin B$ (since B contains no elements!)

(b) From (a) we have that $(B \subseteq C) \wedge (C \subseteq B)$. Hence $B = C$. ■

It follows from Proposition 2.1.4(b) that there is a unique empty set.

Axiom of the empty set: *There is a set that contains no elements. This is called the* **empty set**. *It is denoted by* $\emptyset$ *or* $\{\ \}$.

2.1.5 Proposition

(a) For any set A, $A \subseteq A$.

(b) Let $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$.

Russell's Paradox

We have been very casual and informal in our definition of a set. One has to be careful though if one is to avoid some unpleasant surprises. Russell's Paradox is a salutary reminder that one has to exercise care when defining sets.

Consider the set $A = \{1, 2, 3, 4\}$. Then, $3 \in A$, $A \subseteq A$, but $A \notin A$. The set A does not contain itself as an element.

Let us now consider the set S of all sets; i.e., $S = \{B : B \text{ is a set}\}$. Notice that not only is $S \subseteq S$, $S \in S$, since S is a set.

There are therefore sets that contain themselves as elements (e.g., S), and there are sets that do not contain themselves as elements (e.g. A).

Let R be the set of all those sets that do not contain themselves, i.e.,

$$R = \{X \mid (X \text{ is a set}) \wedge (X \notin X)\}.$$

The question is "Does R contain itself an element?"

Well, let's assume $R \notin R$, i.e., R does not contain itself as an element. So by definition of R, R is a member of R. So our assumption that R is not an element of R logically leads to the statement that R is a member of R. This is a contradiction, so our assumption must be wrong.

Let's assuming that R is an element of R, i.e., $R \in R$. But R is the set that has only members that do not contain themselves, so R cannot be a member of R. So our assumption that R is a member of R logically leads to the statement that R is not a member of R. This is a contradiction, so our assumption must be wrong.

In short, we have the situation that $R \in R \Leftrightarrow R \notin R$. □

The main point of Russell's Paradox is that there are properties that do not define sets, i.e., all objects with those properties cannot be collected into one set.

As Russell's Paradox indicates, there are logical difficulties that arise in the foundations of Set Theory if one is not careful. We can avoid such difficulties by assuming that each discussion in which a number of sets are involved is taking place within a context of a fixed set. This set is called the **universal set**.

Some notation...

We use special notation to designate intervals of various kinds on the real line. Let $a, b \in \mathbb{R}$ with

$a \le b$.

$$\begin{aligned}
[a,b] &= \{x \in \mathbb{R} \ : \ a \le x \le b\} \\
[a,b) &= \{x \in \mathbb{R} \ : \ a \le x < b\} \\
(a,b] &= \{x \in \mathbb{R} \ : \ a < x \le b\} \\
(a,b) &= \{x \in \mathbb{R} \ : \ a < x < b\} \\
[a,\infty) &= \{x \in \mathbb{R} \ : \ x \ge a\} \\
(a,\infty) &= \{x \in \mathbb{R} \ : \ x > a\} \\
(-\infty,b] &= \{x \in \mathbb{R} \ : \ x \le b\} \\
(-\infty,b) &= \{x \in \mathbb{R} \ : \ x < b\} \\
(-\infty,\infty) &= \mathbb{R}.
\end{aligned}$$

2.2 Operations on Sets

2.2.1 Definition
Let A be a set. The **power set** *of A, denoted by $\mathcal{P}(A)$, is the set whose elements are all subsets of A. That is,*

$$\mathcal{P}(A) = \{B \ : \ B \subseteq A\}.$$

2.2.2 Example
Let $A = \{x,\ y,\ z\}$. Then

$$\begin{aligned}
\mathcal{P}(\emptyset) &= \{\emptyset\} \\
\mathcal{P}(\{x\}) &= \{\emptyset,\ \{x\}\} \\
\mathcal{P}(\{x,\ y\}) &= \{\emptyset,\ \{x\},\ \{y\},\ \{x,\ y\}\} \\
\mathcal{P}(A) &= \{\emptyset,\ \{x\},\ \{y\},\ \{z\},\ \{x,\ y\},\ \{x,\ z\},\ \{y,\ z\},\ \{x,\ y,\ z\}\}.
\end{aligned}$$

$\bigotimes$: Note that $\emptyset$ is not the same as $\{\emptyset\}$.

2.2.3 Definition
Let A and B be subsets of a universal set U.

(a) *The* **union** *of A and B, denoted by $A \cup B$, is the set of all elements in U that are either in A or in B (or in both sets). That is,*

$$A \cup B = \{x \in U \ : \ (x \in A) \ \vee \ (x \in B)\}.$$

(b) *The* **intersection** *of A and B, denoted by $A \cap B$, is the set of all elements in U that are in A and B. That is,*

$$A \cap B = \{x \in U \ : \ (x \in A) \ \wedge \ (x \in B)\}.$$

Sets A and B are said to be **disjoint** *if $A \cap B = \emptyset$.*

(c) *The* **complement of** *A* **in** *(or* **relative to***) B, denoted by $B \setminus A$ or $B - A$ and read "B minus A", is the set of all elements of B that are not in A, i.e.,*

$$B - A = \{x \in U \ : \ (x \in B) \ \wedge \ (x \notin A)\}.$$

(d) *The* **complement of** *A, denoted by A', is the set of all elements in U that are not in A, i.e.,*

$$A' = \{x \in U \ : \ x \notin A\}.$$

(e) *The* **symmetric difference of** A **and** B *denoted by* $A \triangle B$ *is the set*

$$A \triangle B = (B - A) \cup (A - B).$$

2.2.4 Example

Let $U = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$, $A = \{2,\ 4,\ 6,\ 8,\ 9,\ 10\}$, and $B = \{3,\ 5,\ 7,\ 9\}$. Then

$A \cup B = \{\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$.

$A \cap B = \{9\}$.

$B - A = \{3,\ 5,\ 7\}$.

$A - B = \{2,\ 4,\ 6,\ 8,\ 10\}$.

$A \triangle B = \{\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 10\}$.

$A' = \{1,\ 3,\ 5,\ 7\}$.

2.2.5 Proposition

Let A, B, *and* C *be subsets of a universal set* U.

(a) $A \cup A = A$ *(idempotent law for union)*

(b) $A \cap A = A$ *(idempotent law for intersection)*

(c) $A \cup \emptyset = A$

(d) $A \cap \emptyset = \emptyset$

(e) $A \cup U = U$

(f) $A \cap U = A$

(g) $A \cup B = B \cup A$ *(commutative law for union)*

(h) $A \cap B = B \cap A$ *(commutative law for intersection)*

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ *(associative law for union)*

(j) $(A \cap B) \cap C = A \cap (B \cap C)$ *(associative law for intersection)*

(k) $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$

(l) $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$

(m) $A'' = A$

(n) $A \cup A' = U$

(o) $A \cap A' = \emptyset$

(p) $\emptyset' = U$

(q) $U' = \emptyset$

(r) $(A \cup B)' = A' \cap B'$ *(De Morgan's law)*

(s) $(A \cap B)' = A' \cup B'$ *(De Morgan's law)*

(t) $A \subseteq B$ *if and only if* $B' \subseteq A'$

(u) $A - B = A \cap B'$

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *(intersection distributes over union)*

(w) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *(union distributes over intersection)*

(x) $A \triangle B = (A \cup B) - (A \cap B)$.

Proof. Be sure that you can prove these properties.

(r) In order to show that $(A \cup B)' = A' \cap B'$, we must show that $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$.

	$(A \cup B)' \subseteq A' \cap B'$		$A' \cap B' \subseteq (A \cup B)'$
	Let $x \in (A \cup B)'$		Let $x \in A' \cap B'$
	Then $x \notin A \cup B$		Then $x \in A'$ and $x \in B'$
$\therefore$	$\neg[x \in A \cup B]$	$\therefore$	$(x \notin A) \wedge (x \notin B)$
$\therefore$	$\neg[(x \in A) \vee (x \in B)]$	$\therefore$	$\neg[(x \in A) \vee (x \in B)]$
$\therefore$	$(x \notin A) \wedge (x \notin B)$	$\therefore$	$\neg[x \in A \cup B]$
$\therefore$	$(x \in A') \wedge (x \in B')$	$\therefore$	$x \in (A \cup B)'$
$\therefore$	$x \in A' \cap B'$	$\therefore$	$A' \cap B' \subseteq (A \cup B)'$
$\therefore$	$(A \cup B)' \subseteq A' \cap B'$		

Proof of (v): Here we use the fact that if P, Q, and R are propositions, then $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.

For each x,

$$
\begin{aligned}
& x \in A \cap (B \cup C) \\
\Leftrightarrow\ & x \in A \text{ and } x \in B \cup C \\
\Leftrightarrow\ & x \in A \text{ and } (x \in B \text{ or } x \in C) \\
\Leftrightarrow\ & (x \in A) \wedge [(x \in B) \vee (x \in C)] \\
\Leftrightarrow\ & [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)] \\
\Leftrightarrow\ & x \in A \cap B \text{ or } x \in A \cap C \\
\Leftrightarrow\ & x \in (A \cap B) \cup (A \cap C).
\end{aligned}
$$

2.3 Indexed families of sets

In Mathematics we often work with large collections of sets. Instead of naming each of those sets using the twenty-six letters of the alphabet, we usually index the sets using some convenient indexing set.

Suppose that I is a set and that to each $i \in I$, there corresponds one and only one subset A_i of a universal set U. Then the collection $\{A_i \ : \ i \in I\}$ is called an **indexed family of sets** (or an **indexed collection of sets**). The set I is called an **indexing set** for the collection $\{A_i \ : \ i \in I\}$. If $I = \{1, 2, 3, \ldots, n\}$, then the indexed collection of sets $\{A_i \ : \ i \in I\}$ is called a finite sequence of sets. If $I = \mathbb{N}^+$, the set of positive natural numbers, then the indexed collection $\{A_i \ : \ i \in \mathbb{N}^+\}$ is called an infinite sequence of sets.

We can extend the definition of *union* and *intersection* discussed earlier to cover an indexed family of sets.

2.3.1 Definition
Let $\{A_i \ : \ i \in I\}$ be an indexed family of subsets of the universal set U.

(a) The union of the family $\{A_i \ : \ i \in I\}$, denoted by $\bigcup_{i \in I} A_i$, is the set of all those elements of U which belong to at least one of the A_i. That is,

$$\bigcup_{i \in I} A_i = \{x \in U \ : \ x \in A_i \text{ for some } i \in I\} = \{x \in U \ : \ (\exists i \in I)(x \in A_i)\}.$$

(b) The intersection of the family $\{A_i \ : \ i \in I\}$, denoted by $\bigcap_{i \in I} A_i$, is the set of all those elements of U which belong to all the A_i. That is,

$$\bigcap_{i \in I} A_i = \{x \in U \ : \ x \in A_i \text{ for each } i \in I\} = \{x \in U \ : \ (\forall i \in I)(x \in A_i)\}.$$

2.3.2 Proposition
Let $\{A_i \ : \ i \in I\}$ be an indexed family of subsets of the universal set U and let B be a subset of U. Then

(a) $A_k \subseteq \bigcup_{i \in I} A_i$ *for each* $k \in I$.

(b) $\bigcap_{i \in I} A_i \subseteq A_k$ *for each* $k \in I$.

(c) $B \cap \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} (B \cap A_i)$.

(d) $B \cup \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} (B \cup A_i)$.

(e) $B - \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (B - A_i)$.

(f) $B - \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (B - A_i)$.

(g) $\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i'$ *(de Morgan's law).*

(h) $\left(\bigcap_{i\in I} A_i\right)' = \bigcup_{i\in I} A_i'$ *(de Morgan's law).*

Proof. Be sure that you can prove these statements. We shall prove (c), (e), and (h).

Proof of (c): We should show that $B \cap \left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I}(B\cap A_i)$ and $\bigcup_{i\in I}(B\cap A_i) \subseteq B \cap \left(\bigcup_{i\in I} A_i\right)$. We shall do this in one fell swoop.

$$
\begin{aligned}
x \in B \cap \left(\bigcup_{i\in I} A_i\right) \quad &\Leftrightarrow \quad x \in B \text{ and } x \in \bigcup_{i\in I} A_i \\
&\Leftrightarrow \quad x \in B \text{ and } x \in A_i \text{ for some } i \in I \\
&\Leftrightarrow \quad (\exists i \in I)[(x \in B) \wedge (x \in A_i)] \\
&\Leftrightarrow \quad x \in \bigcup_{i\in I}(B \cap A_i)\,.
\end{aligned}
$$

Proof of (e): We use the same technique as applied in (c).

$$
\begin{aligned}
x \in B - \left(\bigcup_{i\in I} A_i\right) \quad &\Leftrightarrow \quad x \in B \text{ and } x \notin \bigcup_{i\in I} A_i \\
&\Leftrightarrow \quad x \in B \text{ and } \neg[x \in \bigcup_{i\in I} A_i] \\
&\Leftrightarrow \quad x \in B \text{ and } \neg[(\exists i \in I)(x \in A_i)] \\
&\Leftrightarrow \quad (x \in B) \;\wedge\; [(\forall i \in I)(x \notin A_i)] \\
&\Leftrightarrow \quad (\forall i \in I)[(x \in B) \wedge (x \notin A_i)] \\
&\Leftrightarrow \quad (\forall i \in I)(x \in B - A_i) \\
&\Leftrightarrow \quad x \in \bigcap_{i\in I}(B - A_i)\,.
\end{aligned}
$$

Proof of (h):

$$\begin{aligned} x \in \left(\bigcap_{i\in I} A_i\right)' &\Leftrightarrow x \notin \bigcap_{i\in I} A_i \\ &\Leftrightarrow \neg[x \in \bigcap_{i\in I} A_i] \\ &\Leftrightarrow \neg[(\forall i \in I)(x \in A_i)] \\ &\Leftrightarrow (\exists i \in I)(x \notin A_i) \\ &\Leftrightarrow (\exists i \in I)(x \in A_i') \\ &\Leftrightarrow x \in \bigcup_{i\in I} A_i'. \end{aligned}$$

2.4 Functions

2.4.1 Definition
Let X and Y be sets. A **function** *f from X to Y, denoted by $f : X \to Y$, is a rule that assigns to each $x \in X$ a unique element $y \in Y$. We write $y = f(x)$ to denote that f assigns the element $x \in X$ to the element $y \in Y$.*

2.4.2 Definition
Let X and Y be sets. A function $f : X \to Y$ is said to be

(i) **injective** *(or* **one-to-one***) if for for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Symbolically,*

$$(\forall x_1, x_2 \in X)[(f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)].$$

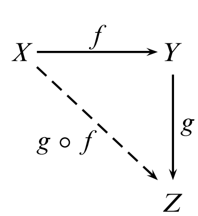
(ii) **surjective** *(or* **onto***) if for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$. Symbolically,we write*

$$(\forall y \in Y)(\exists x \in X)(f(x) = y).$$

(iii) **bijective** *if f is both injective and surjective.*

2.4.3 Definition
Let X, Y, and Z be sets, $f : X \to Y$ and $g : Y \to Z$ be sets. The composition of f and g, denoted by $g \circ f$, is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.

A diagrammatic view of the composition is



2.4.4 Theorem
Let $f : X \to Y$ and $g : Y \to Z$ such that ran$(f) \subseteq$ dom(g). Then

(a) If f and g are onto, then so is the composite function $g \circ f$;

(b) *If f and g are one-to-one, then so is the composite function $g \circ f$;*

(c) *If f and g are bijective, then so is the composite function $g \circ f$;*

(d) *If $g \circ f$ is one-to-one, then so is f;*

(e) *If $g \circ f$ is onto, then so is g;*

(f) *If $g \circ f$ is a bijection, then f is one-to-one and g is onto.*

Proof.

(a) Let $z \in Z$. Since g is onto, there is a $y \in Y$ such that $g(y) = z$. Since f is onto, there is an $x \in X$ such that $f(x) = y$. Therefore $(g \circ f)(x) = g(f(x)) = g(y) = z$. Hence, $g \circ f$ is onto.

(b) Let x_1 and x_2 be in X such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then

$$\begin{aligned} (g \circ f)(x_1) &= (g \circ f)(x_2) \\ \iff g(f(x_1)) &= g(f(x_2)) \\ \iff f(x_1) &= f(x_2) \quad \text{since } g \text{ is one-to-one} \\ \iff x_1 &= x_2 \quad \text{since } f \text{ is one-to-one.} \end{aligned}$$

(c) This follows from (a) and (b).

(d) Let x_1 and x_2 be elements of X such that $f(x_1) = f(x_2)$. Then $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = g(f(x_2))$. Since $g \circ f$ is one-to-one, it follows that $x_1 = x_2$. Thus, f is one-to-one.

(e) Let $z \in Z$. We must produce a $y \in Y$ such that $g(y) = z$. Since $g \circ f$ is onto, there is an $x \in X$ such that $(g \circ f)(x) = g(f(x)) = z$. Let $y = f(x)$ $(\in Y)$. Then $g(y) = z$, which proves that g is onto.

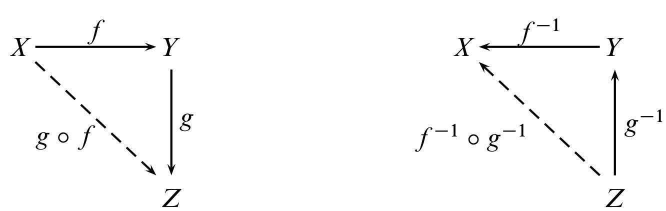
(f) This follows from (d) and (e). ■

2.4.5 Theorem
Let $f : X \to Y$ be a bijection. Then $f^{-1} : Y \to X$ is a bijection.

Proof. Exercise. ■

2.4.6 Theorem
Let $f : X \to Y$ and $g : Y \to Z$ be bijections. Then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$



Proof. Exercise. ■

Let X and Y be sets, $f : X \to Y$, and $A \subset X$. We denote by $f(A)$ the **image of** A **in** Y. It is defined by

$$f(A) = \{f(x) \mid x \in A\}.$$

If $B \subset Y$, we denote by $f^{-1}(B)$ the **pre-image** (or **inverse image**) of B in X. It is defined by

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

2.4.7 Theorem
Let X and Y be sets, $f : X \to Y$, and $\{A_i : i \in I\}$ an indexed family of subsets of X. Then

(a) $f(\emptyset) = \emptyset$;

(b) $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$;

(c) $f\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} f(A_i)$;

(d) *If f is injective, then* $f\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$.

$\prec$Exercise. ■

2.4.8 Theorem
Let X and Y be sets, $f : X \to Y$, $\{B_i : i \in I\}$ an indexed family of subsets of Y and $D \subset Y$. Then

(a) $f^{-1}(\emptyset) = \emptyset$;

(b) $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$;

(c) $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$;

(d) $f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$.

$\prec$Exercise. ■

2.4.9 Theorem
Let X and Y be sets and $f : X \to Y$. If $A \subset X$ and $B \subset Y$, then

(a) $A \subseteq f^{-1}(f(A))$;

(b) *If f is injective, then* $A = f^{-1}(f(A))$;

(c) $f\left(f^{-1}(B)\right) \subseteq B$;

(d) *If f is surjective, then* $f\left(f^{-1}(B)\right) = B$;

(e) $f(A \cap f^{-1}(B)) = f(A) \cap B$.

$\prec$Exercise. ■

2.5 Cardinality: the size of a set

2.5.1 Definition
Two sets A and B are said to **have the same cardinality**, *denoted by $|A| = |B|$, if there is a one-to-one function from A onto B. Sets that have the same cardinality are also said to be* **equipotent** *or* **equinumerous**.

2.5.2 Examples

[1] $\mathbb{N}_0$ has the same cardinality as $\mathbb{N}$.

Proof. Define $f : \mathbb{N}_0 \to \mathbb{N}$ by $f(n) = n + 1$ for each $n \in \mathbb{N}_0$.

Claim: f is one-to-one. Let $n,\ m \in \mathbb{N}_0$ such that $f(n) = f(m)$ Then $n + 1 = m + 1$, and consequently $n = m$.

Claim: f is onto. Let $m \in \mathbb{N}$. Then $m - 1 \in \mathbb{N}_0$ and $f(m-1) = m - 1 + 1 = m$.

[2] Let $\mathbb{E} = \{2n \ : \ n \in \mathbb{N}\} -$ the set of even natural numbers. Then $\mathbb{N}$ and $\mathbb{E}$ have the same cardinality.

Proof. Define $f : \mathbb{N} \to \mathbb{E}$ by $f(n) = 2n$ for each $n \in \mathbb{N}$.

Claim: f is one-to-one. Let n_1 and n_2 be elements of $\mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $2n_1 = 2n_2$ and consequently $n_1 = n_2$.

Claim: f is onto. Let $m \in \mathbb{E}$. Then $m = 2k$ for some $k \in \mathbb{N}$. Hence, $f(k) = 2k = m$.

[3] $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

Proof. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} -\dfrac{n}{2} & \text{if } \quad n \text{ is even} \\[2ex] \dfrac{n-1}{2} & \text{if } \quad n \text{ is odd.} \end{cases}$$

In tabular form

n	1	2	3	4	5	6	7	$\cdots$
$f(n)$	0	-1	1	-2	2	-3	3	$\cdots$

It is clear that $\text{dom}(f) = \mathbb{N}$ and $\text{ran}(f) \subseteq \mathbb{Z}$.

Claim: f is one-to-one. For each $n \in \mathbb{N}$, $f(n) < 0$ if n is even and $f(n) \geq 0$ if n is odd. Let $m_1,\ m_2 \in \mathbb{N}$ such that $f(m_1) = f(m_2)$. We must show that $m_1 = m_2$. If m_1 and m_2 are both even, then

$$\begin{aligned} f(m_1) = f(m_2) \quad &\Longleftrightarrow \quad -\frac{m_1}{2} = -\frac{m_2}{2} \\ &\Longleftrightarrow \quad m_1 = m_2. \end{aligned}$$

If m_1 and m_2 are both odd, then

$$f(m_1) = f(m_2) \iff \frac{m_1 - 1}{2} = \frac{m_2 - 1}{2} \iff m_1 = m_2.$$

Hence, f is one-to-one.

Claim: f is onto. If $m \in \mathbb{Z}$ and is negative, then $-2m$ is in $\mathbb{N}$ and is even. Therefore

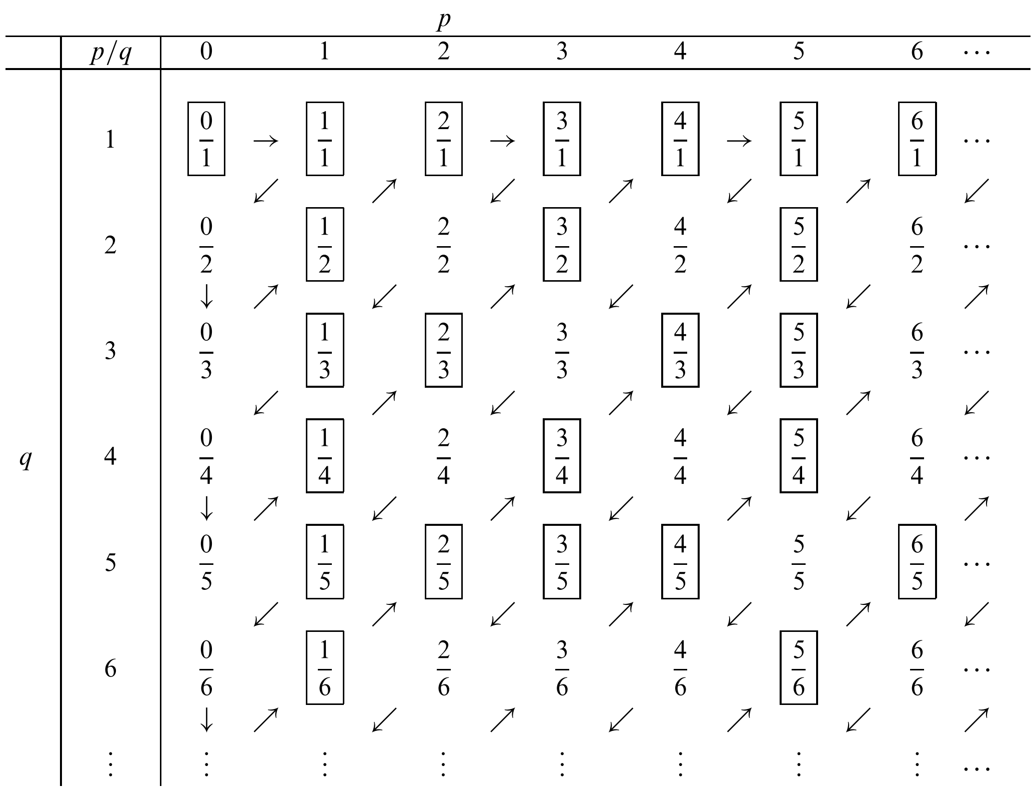
$$f(-2m) = (-1)\frac{(-2m)}{2} = m.$$

If $m \in \mathbb{Z}$ and $m \geq 0$, then $2m + 1$ is in $\mathbb{N}$ and is odd. Therefore

$$f(2m+1) = \frac{(2m+1) - 1}{2} = m.$$

[4] $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality.

Proof. We start by listing nonnegative rational numbers in an infinite matrix as follows:



Starting with $\frac{0}{1}$ at the top left corner, we follow the arrows, putting a box around a rational number that occurs for the first time. This assigns a unique natural number to each nonnegative rational number. That is, this defines a function g from $\mathbb{N}_0$ to the set of nonnegative rational numbers $\mathbb{Q}^+ \cup \{0\}$ given by the following table:

n	0	1	2	3	4	5	6	$\cdots$
$g(n)$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{1}{3}$	$\frac{1}{4}$	$\cdots$

Define $f : \mathbb{N}_0 \to \mathbb{Q}$ by

$$f(n) = \begin{cases} -\,g\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even} \\[2ex] g\left(\dfrac{n+1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

In tabular form

n	0	1	2	3	4	5	6	$\cdots$
$f(n)$	0	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	2	-2	$\cdots$

Then f is a bijection between $\mathbb{N}_0$ and $\mathbb{Q}$. Since $\mathbb{N}$ and $\mathbb{N}_0$ have the same cardinality, there is a bijection $h : \mathbb{N} \to \mathbb{N}_0$. Therefore $f \circ h$ is a bijection from $\mathbb{N}$ onto $\mathbb{Q}$.

[5] $\mathbb{R}$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ have the same cardinality.

Proof. Define $f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by $f(x) = \arctan x$.

Claim: f is one-to-one. Let x_1 and x_2 be elements of $\mathbb{R}$ such that $f(x_1) = f(x_2)$ Then

$$\begin{aligned} \arctan x_1 &= \arctan x_2 \\ \Rightarrow \tan(\arctan x_1) &= \tan(\arctan x_2) \\ \Rightarrow \qquad x_1 &= x_2. \end{aligned}$$

Claim: f is onto. Let $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then, $\tan y \in \mathbb{R}$ and, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have that $\arctan(\tan y) = y$. Let $x = \tan y$. Then $f(x) = y$.

[6] The intervals $(0, 1)$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ have the same cardinality.

$\prec$Define $f : (0, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by $f(x) = \pi x - \frac{\pi}{2}$. It is easy to show that f is a well-defined bijection from $(0, 1)$ onto $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We immediately deduce from examples 5 and 6 that $(0, 1)$ and $\mathbb{R}$ have the same cardinality.

2.5.3 Definition
A set S is said to be

(a) **finite** *if $S = \emptyset$ or if there is an $n \in \mathbb{N}$ such that $|S| = |\{1, 2, 3, \ldots, n\}|$.*

(b) **infinite** *if S is not finite.*

(c) **countably infinite** *if $|S| = |\mathbb{N}|$.*

(d) **countable** *if S is finite or is countably infinite.*

(e) **uncountable** *if S is not countable.*

The cardinality of $\mathbb{N}$ is called $\aleph_0$ (aleph nought).
We have shown that the sets $\mathbb{E}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

2.5.4 Theorem
There does not exist a surjection from a set X onto its power set $\mathcal{P}(X)$.

Proof. (By Contradiction). Suppose there were such a surjection $f : X \to \mathcal{P}(X)$. Let A be the subset of X defined by

$$A = \{x \in X \ : \ x \notin f(x)\}.$$

Then $A \in \mathcal{P}(X)$. Since f is assumed to be surjective, there exists an $a \in X$ such that $f(a) = A$. Either $a \in A$ or $a \notin A$. If $a \in A$, then by definition of A, $a \notin f(a) = A$, a contradiction. Therefore, $a \notin A$. But now again by definition of A, it follows that $a \in A$, a contradiction again. We conclude that there is no function from X onto $\mathcal{P}(X)$. ∎

2.5.5 Corollary
$\mathcal{P}(\mathbb{N})$ is uncountable.

2.5.6 Theorem
The set of real numbers in the interval $(0, 1)$ is uncountable.

$\prec$(By contradiction). Assume that $(0, 1)$ is countable. Let $\{x_1, x_2, x_3, \ldots\}$ be the enumeration of elements of $(0, 1)$; that is, there is a bijection $f : \mathbb{N} \to (0, 1)$ given by $f(k) = x_k$. Each $x_n \in (0, 1)$ has a decimal expansion of the form

$$\begin{array}{rcl} x_1 & = & 0.a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\ x_2 & = & 0.a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\ x_3 & = & 0.a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\ & \vdots & \\ x_n & = & 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}\cdots \\ \vdots & & \vdots \end{array}$$

where $a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let b be the real number that has the decimal expansion:

$$b = 0.b_1b_2b_3b_4b_5\cdots$$

where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} \neq 1. \end{cases}$$

Then, clearly, $b \in (0, 1)$ and $b \neq x_k$ for all $k \in \mathbb{N}$ since b and x_k differ at the k-place after the decimal point. Hence, the function $f : k \mapsto x_k$ is **not** surjective. ■

2.5.7 Corollary
The set $\mathbb{R}$ of real numbers is uncountable.

$\prec$This follows immediately from examples 2.5.2 (5 and 6).

In order to establish the next set of important results, we shall need the following result called the Well Ordering Principle or Least Natural Number Principle:

2.5.8 Theorem
Every nonempty subset A of natural numbers has a least member - a number $a_0 \in A$ such that $a_0 \leq a$ for all $a \in A$.

The Least Natural Number Principle is equivalent to the Principle of Mathematical Induction. That is, assuming one principle you can prove the other. Below we prove that the Principle of Mathematical Induction implies the Least Natural Number Principle. We leave the proof of the converse of this statement as an exercise.

2.5.9 Theorem
The Principle of Mathematical Induction implies the Least Natural Number Principle.

Proof. Let T be a subset of $\mathbb{N}$ with no least element. We prove that T is an empty set. Let

$$S = \{n \in \mathbb{N} : \{1, 2, \ldots, n\} \cap T = \emptyset\}.$$

Claim 1: $1 \in S$. If $1 \notin S$, then $\{1\} \cap T \neq \emptyset$. But then $1 \in T$ and 1 would be the least element of T, contradicting the fact that T has no least element. Hence $1 \in S$.

Claim 2: $k \in S \Rightarrow k+1 \in S$. Since $k \in S$ (by the assumption), it follows that $\{1, 2, \ldots, k\} \cap T = \emptyset$. This says that no positive natural number less than or equal to k belongs to T. We must show that $k + 1$ does not belong to T or equivalently, $k + 1 \in S$. If $k + 1 \notin S$, then $\{1, 2, \ldots, k, k + 1\} \cap T \neq \emptyset$. Since $\{1, 2, \ldots, k\} \cap T = \emptyset$, it follows that $k + 1 \in T$. But then $k + 1$ would be the least element of T, contradicting the fact that T has no least element. Hence $k + 1 \in S$.
By the Principle of Mathematical Induction, we have that $S = \mathbb{N}$. This, of course, means that no natural numbers belongs to T, i.e., $T = \emptyset$. ■

We are now ready to establish some important results.

2.5.10 Theorem
A subset of a countable set is countable.

≺Let A be a subset of a countable set B. If A is finite, then it is obviously countable. Assume that A is infinite. Then B is countably infinite. Let $\{b_1,\ b_2,\ b_3,\ \ldots\}$ be an enumeration of elements of B That is, there is a bijection $f : \mathbb{N} \to B$ given by $f(k) = b_k$.

Let $\mathbb{M} = \{n \in \mathbb{N} \mid b_n \in A\}$. Then $\mathbb{M}$ is a nonempty subset of $\mathbb{N}$. By the Least Natural Number Principle, $\mathbb{M}$ has the least element m_1. Similarly, $\mathbb{M} - \{m_1\}$ has the least element m_2. In general, having chosen $m_1,\ m_2,\ \ldots,\ m_k$, let m_{k+1} be the least element of $\mathbb{M} - \{m_1,\ m_2,\ \ldots,\ m_k\}$. Define $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = m_n$. Since A is infinite, g is defined for each $n \in \mathbb{N}$.

Claim: g is injective. Indeed, if $i < j$, then $m_i \neq m_j$ since $m_j \notin \{m_1,\ m_2,\ \ldots,\ m_i\}$. Thus $g(i) \neq g(j)$.

We have the diagram:

$$\mathbb{N} \xrightarrow{g} \mathbb{N} \xrightarrow{f} B.$$

It now follows that $f \circ g$ is injective. Since each element of A appears somewhere in the enumeration of elements of B, we have that $g(\mathbb{N})$ includes all the subscripts of elements of A. Thus, $f \circ g$ is a bijection from $\mathbb{N}$ onto A. Hence, A is countable. ■

Here is another argument that $\mathbb{R}$ is uncountable: Assume that $\mathbb{R}$ is countable. Then, by Theorem 2.5.10, every subset of $\mathbb{R}$ would be countable. In particular, the set or real numbers in the interval $(0, 1)$ would be countable. This contradicts Theorem 2.5.6. Hence, $\mathbb{R}$ is countable.

2.5.11 Corollary
An intersection of any collection of countable sets is countable.

≺Let $\{A_\lambda \mid \lambda \in I\}$ be a collection of sets such that A_λ is countable for each $\lambda \in I$. Choose and fix $\alpha \in I$. Then

$$\bigcap_{\lambda \in I} A_\lambda \subset A_\alpha.$$

Since A_α is countable, it follows from Theorem 2.5.10 that $\bigcap_{\lambda \in I} A_\lambda$ is countable. ■

2.5.12 Theorem
Let A be a nonempty set. The following statements are equivalent:

(a) A is countable;

(b) There is a surjection $f : \mathbb{N} \to A$.

(c) There is an injection $f : A \to \mathbb{N}$.

≺(a) ⇒ (b): Assume that A is countable. If A is finite, then there is nothing to prove. Assume that A is infinite. Then A is countably infinite. Thus, there is a bijection $f : \mathbb{N} \to A$. Therefore, f is a surjection from $\mathbb{N}$ onto A.

(b) ⇒ (c): Assume that there is a surjection $f : \mathbb{N} \to A$. Then the set

$$f^{\leftarrow}(a) := \{n \in \mathbb{N} \mid f(n) = a\} \neq \emptyset$$

for each $a \in A$.

Define $g : A \to \mathbb{N}$ by

$$g(a) = \text{the least element of the set } f^{\leftarrow}(a)$$

for each $a \in A$. By the Least Natural Number Principle, we have that g is well-defined. We show that g is injective. Note first that since $g(a) \in f^{\leftarrow}(a)$, it follows that $f(g(a)) = a$.

Let $a, b \in A$ such that $g(a) = g(b)$. Then

$$a = f(g(a)) = f(g(b)) = b.$$

Thus, g is injective.

(c) $\Rightarrow$ (a): Assume that there is an injection $g : A \to \mathbb{N}$. Then g is a bijection from A onto $g(A) := \{n \in \mathbb{N} \mid g(a) = n \text{ for some } a \in A\}$. Since $\mathbb{N}$ is countable and $g(A) \subset \mathbb{N}$, it follows from Theorem 2.5.10 that $g(A)$ is countable. Thus A is countable. ■

2.5.13 Theorem

$\mathbb{N} \times \mathbb{N}$ is countable.

≺Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$f(n, m) = 2^n \cdot 3^m.$$

We show that f is an injection. To that end, let (n, m) and (k, ℓ) be elements of $\mathbb{N} \times \mathbb{N}$ such that

$$f(n, m) = f(k, \ell).$$

Then

$$2^n \cdot 3^m = 2^k \cdot 3^\ell \iff 2^{n-k} = 3^{\ell - m}.$$

Hence, $n - k = 0$ and $\ell - m = 0$ and, consequently, $n = k$ and $m = \ell$. That is, $(n, m) = (k, \ell)$. This shows that f is injective. By Theorem 2.5.12(c), we conclude that $\mathbb{N} \times \mathbb{N}$ is countable. ■

2.5.14 Corollary

If A and B are countable sets, then $A \times B$ is also countable.

≺Since A and B are countable, there are bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Define $h : \mathbb{N} \times \mathbb{N} \to A \times B$ by

$$h(n, m) = (f(n), g(m)) \text{ for all } (n, m) \in \mathbb{N} \times \mathbb{N}.$$

Clearly, h is well-defined.

Claim 1: h is injective. Assume that $h(n, m) = h(k, \ell)$. Then, by definition of h, $(f(n), g(m)) = (f(k), g(\ell))$. Therefore $f(n) = f(k)$ and $g(m) = g(\ell)$. Since f and g are injective, it follows that $n = k$ and $m = \ell$ and, consequently, $(n, m) = (k, \ell)$.

Claim 2: h is surjective. Let $(a, b) \in A \times B$. Since f and g are surjective, there are natural numbers i and j such that $f(i) = a$ and $g(j) = b$. Hence, $(i, j) \in \mathbb{N} \times \mathbb{N}$ and

$$h(i, j) = (f(i), g(j)) = (a, b).$$

Thus, h is surjective. ■

We give another proof that the set $\mathbb{Q}$ of rational numbers is countable.

2.5.15 Corollary

The set $\mathbb{Q}$ of rational numbers is countable.

≺Since $\mathbb{Z}$ and $\mathbb{N}$ are countable, we have, by Corollary 2.5.14, that $\mathbb{Z} \times \mathbb{N}$ is countable. So there is a surjection $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$. Define $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ by

$$g(p, q) = \frac{p}{q}.$$

Clearly, g is surjective (by the definition of rational numbers). We have the following diagram:

$$\mathbb{N} \xrightarrow{f} \mathbb{Z} \times \mathbb{N} \xrightarrow{g} \mathbb{Q}.$$

Since the function $g \circ f$ is a surjection from $\mathbb{N}$ onto $\mathbb{Q}$, it follows from Theorem 2.5.12 that $\mathbb{Q}$ is countable.■

2.5.16 Theorem
A countable union of countable sets is countable.

≺Let $\{A_n \mid n \in \mathbb{N}\}$ be a collection of sets such that A_n is countable for each $n \in \mathbb{N}$ and let $A = \bigcup_{n=1}^{\infty} A_n$. We show that A is countable. Since A_n is countable for each $n \in \mathbb{N}$, there is a surjection $f_n : \mathbb{N} \to A_n$ for each $n \in \mathbb{N}$. Define $f : \mathbb{N} \times \mathbb{N} \to A$ by

$$f(n, m) = f_n(m).$$

We show that f is surjective. Indeed, if $a \in A$, then $a \in A_n$ for some $n \in \mathbb{N}$. Since f_n is surjective, there is an $m \in \mathbb{N}$ such that $f_n(m) = a$. Therefore $(n, m) \in \mathbb{N} \times \mathbb{N}$ and $f(n, m) = f_n(m) = a$. Thus, f is surjective. Since $\mathbb{N} \times \mathbb{N}$ is countable, there is a surjection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. We have the following diagram:

$$\mathbb{N} \xrightarrow{g} \mathbb{N} \times \mathbb{N} \xrightarrow{f} A.$$

Thus, $g \circ f$ is a surjection from $\mathbb{N}$ onto A. By Theorem 2.5.12, A is countable. ■

2.5.17 Exercise
[1] Show that the set of irrational numbers is uncountable.

2.5.1 The Cantor-Schröder-Bernstein Theorem

When we started the section on cardinality, we said that two sets A and B have the same cardinality if there is a bijection (one-to-one and onto function) between them. It is usually easier to find an injection than a bijection between two sets. The Cantor-Schröder-Bernstein Theorem asserts that if A and B are sets for which we can find an injection from A into B and an injection from B into A, then there is a bijection between A and B.

2.5.18 Lemma
Let A and B be sets such that $B \subseteq A$. If there is an injective function $f : A \to B$, then there is a bijective function $g : A \to B$.

Proof. If $A = B$, then the identity function i_A works. Assume that $B \subsetneqq A$. We inductively define a sequence (C_n) of sets as follows:

$$\begin{aligned}
C_0 &= A \setminus B \\
C_1 &= f(C_0) = f(A \setminus B) \\
C_2 &= f(C_1) = f^2(A \setminus B) \\
C_3 &= f(C_2) = f^3(A \setminus B) \\
\vdots \quad &\vdots \quad \vdots \\
C_n &= f(C_{n-1}) = f^n(A \setminus B) \\
\vdots \quad &\vdots \quad \vdots
\end{aligned}$$

Let $C = \bigcup_{n=0}^{\infty} C_n = \bigcup_{n=0}^{\infty} f^n(A \setminus B)$, where f^0 is the identity map on A. Note that $A \setminus B = C_0 \subset C$ and $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} f^n(A \setminus B) \subseteq B$.

Claim 1: If $j,\ k \in \mathbb{N}_0$ and $j \neq k$, then $C_j \cap C_k = \emptyset$; that is, the sets C_n are pairwise disjoint. To prove the claim, assume that $j < k$ and that $C_j \cap C_k \neq \emptyset$. Let $z \in C_j \cap C_k$; that is, $z \in f^j(A \setminus B) \cap f^k(A \setminus B)$. Then there are x and y in $A \setminus B$ such that $f^j(x) = z = f^k(y)$. Therefore

$$f^j(x) = f^k(y) = f^{k-j}\left(f^j(y)\right) = f^j\left(f^{k-j}(y)\right).$$

Since f is injective, so is f^j. Hence $x = f^{k-j}(y)$. But, since $x \in A \setminus B$ and $f^{k-j}(y) \in B$, the equality $x = f^{k-j}(y)$ means that $x = f^{k-j}(y) \in (A \setminus B) \cap B = \emptyset$. This is a contradiction. Hence, $C_j \cap C_k = \emptyset$.

Claim 2: $f(C) \subset C$. Indeed,

$$f(C) = f\left(\bigcup_{n=0}^{\infty} C_n\right) = \bigcup_{n=0}^{\infty} f(C_n) = \bigcup_{n=0}^{\infty} C_{n+1} \subset C.$$

Define $g : A \to B$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in A \setminus C. \end{cases}$$

Claim 3: g is injective. Let $x, y \in A$ such that $g(x) = g(y)$. If $x, y \in C$, then $f(x) = f(y)$. Since f is injective, it follows that $x = y$. If $x \notin C$ and $y \notin C$, then $x = g(x) = g(y) = y$. That is, $x = y$. If $x \in C$ and $y \in A \setminus C$, then $x \neq y$ and $f(x) \in f(C) \subset C$. Therefore $g(x) = f(x) \in C$ and $g(y) = y \in A \setminus C$. Hence $g(x) \neq g(y)$.

Claim 4: g is surjective. Let $y \in B$. If $y \in C$, then $y \in f^n(A \setminus B)$ for some $n = \{1,\ 2,\ \ldots\}$. Hence, there is an $z \in A \setminus B$ such that $y = f^n(z)$. Let $x = f^{n-1}(z)$. Then $x \in f^{n-1}(A \setminus B) \subset C$. Hence, by definition of g,

$$g(x) = f(x) = f\left(f^{n-1}(z)\right) = f^n(z) = y.$$

If $y \in A \setminus C$, then, by definition of g, $g(y) = y$. ∎

2.5.19 Theorem

(Cantor-Schröder-Bernstein Theorem). *Let A and B be sets. If there exist two injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.*

Proof. Since f and g are injective functions, the composite function $g \circ f$ is an injection from A into $g(B)$. Also, $g(B) \subseteq A$. By Lemma 2.5.18, there is a bijection $k : A \to g(B)$. Since g is an injection from B into A, it is a bijection from B onto $g(B)$. The inverse function g^{-1} is a bijection from $g(B)$ onto B. We now have the diagram

$$A \xrightarrow{k} g(B) \xrightarrow{g^{-1}} B.$$

The composite function $h := g^{-1} \circ k$ is a bijection from A onto B. ∎

2.5.20 Example

We use the Cantor-Schröder-Bernstein Theorem to show that the sets $[-1, 1]$ and $\mathbb{R}^+$ have the same cardinality. Let $f : [-1, 1] \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to [-1, 1]$ be given by $f(x) = x + 3$ and $g(x) = \dfrac{1}{x+1}$ respectively. The function f is clearly injective and maps the interval $[-1, 1]$ onto the interval $[2, 4]$. This function is not onto - for example, for 5, which is in $\mathbb{R}^+$, there is no $x \in [-1, 1]$ such that $f(x) = 5$.

The function g is also injective and maps $\mathbb{R}$ onto the interval $(0, 1)$. This function is not onto - for example, for 0, which is in $[-1, 1]$, there is no $x \in \mathbb{R}$ such that $g(x) = 0$.

By the Cantor-Schröder-Bernstein Theorem, there is a bijection between $[-1, 1]$ and $\mathbb{R}$. Hence, these sets have the same cardinality.

Chapter 3

Real Numbers and their Properties

3.1 Real Numbers as a Complete Ordered Field

FIELD AXIOMS

3.1.1 Definition
A **field** *is a set* $\mathbb{F}$ *together with two binary operations* $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (*called* **addition**) *and* $\times : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (*called* **multiplication**) *such that for all* $x, y, z \in \mathbb{F}$,

A1. $x + y = y + x$;

A2. $(x + y) + z = x + (y + z)$;

A3. *There is an element* $0 \in \mathbb{F}$, *called the* additive identity, *such that* $x + 0 = x$ *for each* $x \in \mathbb{F}$;

A4. *For each* $x \in \mathbb{F}$, *there is an element* $-x \in \mathbb{F}$, *called the* additive inverse of x, *such that* $x + (-x) = 0$;

M1. $x \times y = y \times x$;

M2. $(x \times y) \times z = x \times (y \times z)$;

M3. *There is an element* $1 \in \mathbb{F}$, *called the* multiplicative identity, *such that* $x \times 1 = x$;

M4. *For each* $x \in \mathbb{F} \setminus \{0\}$, *there is an element* $x^{-1} \in \mathbb{F}$, *called the* multiplicative inverse of x, *such that* $x \times x^{-1} = 1$;

D1. $x \times (y + z) = x \times y + x \times z$.

Note that a field is a triple $(\varphi, +, \times)$, where φ is a set, $+$ and $\times$ are binary operations satisfying the above properties. We shall abuse notation by simply writing φ for a field. To simplify notation, we shall write xy instead of $x \times y$ and $\frac{x}{y}$ for $x \times y^{-1}$.

3.1.2 Exercise

[1] Let $\mathbb{F}$ be a field.

(a) Show that the additive and multiplicative identities are unique.

(b) Let $x \in \mathbb{F}$ and $y \in \mathbb{F} \setminus \{0\}$. Show that $-x$ and y^{-1} are unique.

ORDER AXIOMS

3.1.3 Definition
An **ordered field** *is a field* $\mathbb{F}$ *on which an order relation* $<$ *is defined such that*

(i) (trichotomy) *for every $x, y \in \varphi$, exactly one of the following holds:*

$$x < y, \qquad x = y, \qquad y < x;$$

(ii) (transitivity) *for all x, y and z, $x < y \wedge y < z \Rightarrow x < z$;*

(iii) *For all x, y, and z in φ, $x < y \Rightarrow x + z < y + z$. Furthermore, if $z > 0$, then $xz < yz$.*

The sets $\mathbb{R}$ and $\mathbb{Q}$ are examples of ordered fields (under the usual $<$ relation).

3.1.4 Exercise

Show that $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ is a field and that $\mathbb{Z}_7$ is not an ordered field under the usual $<$ relation.

COMPLETENESS AXIOM

3.1.5 Definition

Let B be a subset of an ordered field $\mathbb{F}$.

(a) *An element $u \in \varphi$ is an* **upper bound** *for B if $x \leq u$ for all $x \in B$.*

(b) *An element $\ell \in \varphi$ is a* **lower bound** *for B if $\ell \leq x$ for all $x \in B$.*

(c) *B is said to be* **bounded** *if it has both an upper and a lower bound.*

(d) *An element $M \in \varphi$ is the* **least upper bound** *for B if*

  (i) *M is an upper bound for B and,*

  (ii) *for all upper bounds α for B, we have $M \leq \alpha$.*

  The least upper bound for B is also called the **supremum** *for B and is usually abbreviated as $\operatorname{lub}(B)$ or $\sup B$.*

(e) *An element $m \in \varphi$ is the* **greatest lower bound** *for B if*

  (i) *m is a lower bound for B and,*

  (ii) *for all lower bounds β for B, we have $\beta \leq m$.*

  The greatest lower bound for B is also called the **infimum** *for B and is usually abbreviated as $\operatorname{glb}(B)$ or $\inf B$.*

3.1.6 Proposition

A nonempty subset S of an ordered field φ can have at most one least upper bound.

Proof. Assume that λ and ν are both least upper bounds for S. Then, by definition of the least upper bound, $\lambda \leq \nu \leq \lambda$. Hence, $\lambda = \nu$.

3.1.7 Definition

An ordered field φ is said to be **complete** *if every nonempty subset S of φ which is bounded above has the least upper bound.*

3.1.8 Exercise

Show that in a complete field φ, every nonempty subset S of φ which is bounded below has the greatest lower bound.

3.1.9 Theorem

(Characterization of supremum) *Let S be a nonempty subset of an ordered field φ, and $M \in \varphi$. Then $M = \sup S$ if and only if* *(i)* M *is an upper bound for* S*, and*

(ii) for any $\epsilon \in \varphi$ *with* $\epsilon > 0$*, there is an element* $s \in S$ *such that* $M - \epsilon < s$

Proof. Assume that M is the supremum for S, i.e., $M = \sup S$. Then, by definition, M is an upper bound for S. If there is an $\epsilon' \in \varphi$ with $\epsilon' > 0$ for which $M - \epsilon' \geq s$ for all $s \in S$, then $M - \epsilon'$ is an upper bound for S which is smaller than M, a contradiction.

For the converse, assume that (i) and (ii) hold. Since S is bounded above, it has a supremum, A (say). Since M is an upper bound for S, we must have that $A \leq M$. If $A < M$, then with $\epsilon = M - A$, there is an element s in S such that

$$M - (M - A) < s \leq A, \quad \text{i.e., } A < A,$$

which is absurd. Therefore $A = M$, i.e., M is the supremum of S. ■

3.1.10 Definition
Let $(\varphi, <)$ *and* $(\mathbb{G}, <)$ *be ordered fields. An* **order isomorphism** *between* φ *and* $\mathbb{G}$ *is a bijection* $\phi : \varphi \to \mathbb{G}$ *such that for all* $x, y \in \varphi$*,*

(i) $\phi(x + y) = \phi(x) + \phi(y)$*;*

(ii) $\phi(xy) = \phi(x)\phi(y)$*;*

(iii) if $x < y$*, then* $\phi(x) < \phi(y)$*.*

◇ A complete ordered field exists. We denote it by $\mathbb{R}$ and call it the field of real numbers.

◇ There is an order isomorphism between any two complete ordered field.

◇ It follows from the above two statements that there is essentially one complete ordered field, viz. $\mathbb{R}$. "Essentially" here means that there is an order isomorphism between any complete ordered field and the field $\mathbb{R}$ of real numbers.

3.1.11 Theorem
Let A *and* B *be nonempty subsets of* $\mathbb{R}$ *which are bounded above. Then the set*

$$S = \{a + b : \ a \in A, \ b \in B\}$$

is bounded above and $\sup S = \sup A + \sup B$*.*

Proof. Let $c \in S$. Then $c = a + b$ for some $a \in A$ and $b \in B$. Thus, $c = a + b \leq \sup A + \sup B$. Therefore $\sup A + \sup B$ is an upper bound for S. Since $\sup S$ is the *least* upper bound for S, we have that $\sup S \leq \sup A + \sup B$.

It now remains to show that $\sup A + \sup B \leq \sup S$. To that end, let $\epsilon > 0$ be given. By Theorem 3.1.9, there exist elements $x_\epsilon \in A$ and $y_\epsilon \in B$ such that

$$\begin{aligned} \sup A - \frac{\epsilon}{2} &< x_\epsilon, \quad \text{and} \\ \sup B - \frac{\epsilon}{2} &< y_\epsilon. \end{aligned}$$

Thus, $\sup A + \sup B - \epsilon < x_\epsilon + y_\epsilon \leq \sup S$. Since this is true for any $\epsilon > 0$, we have that $\sup A + \sup B \leq \sup S$, whence $\sup A + \sup B = \sup S$. ■

3.1.12 Exercise
[1] Let $S \subset \mathbb{R}$ be bounded above and let $x \in \mathbb{R}$. Show that if $x < \sup S$, then there exists an $s \in S$ such that $x < s$.

[2] Let S be a subset of $\mathbb{R}$ which is bounded below. Show that the set

$$L = \{\ell \in \mathbb{R} : \ell \text{ is a lower bound for } S\}$$

is bounded above and $\sup L = \inf S$.

[3] Show that if a set S of real numbers is bounded below, then $\inf S$ exists.

[4] Let $S \subset T \subset \mathbb{R}$, where $S \neq \emptyset$.

(i) Show that if T is bounded above, then so is S and $\sup S \leq \sup T$.

(ii) Show that if T is bounded below, then so is S and $\inf T \leq \inf S$.

[5] For a subset S of $\mathbb{R}$, let $-S = \{-s : s \in S\}$. Show that if S is bounded below, then $-S$ is bounded above and $\sup(-S) = -\inf S$.

[6] Formulate and prove the characterization of infimum analogous to Theorem 3.1.9.

[7] Let A and B be non-empty bounded subsets of $\mathbb{R}$.

(i) Show that the set $S = \{a+b : a \in A,\ b \in B\}$ is bounded below and $\inf S = \inf A + \inf B$.

(ii) Show that the set $D = \{a - b : a \in A,\ b \in B\}$ is bounded above and $\sup D = \sup A - \inf B$.

(iii) Show that the set $D = \{a-b : a \in A,\ b \in B\}$ is bounded below and $\inf D = \inf A - \sup B$.

(iv) Show that the set $A \cup B$ is bounded above and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

[8] Let A and B be non-empty bounded subsets of $\mathbb{R}^+$, the set of positive real numbers. Show that the set $P = \{ab : a \in A,\ b \in B\}$ is bounded and $\sup P = \sup A \cdot \sup B$, $\inf P = \inf A \cdot \inf B$.

3.1.1 The Archimedean Property of the Real Numbers

The following property of real numbers is one of the major consequences of the Completeness Axiom.

3.1.13 Theorem
(Archimedean Property). *The set $\mathbb{N}$ of natural numbers is not bounded above.*

Proof. Assume that $\mathbb{N}$ is bounded above. By the Completeness Axiom, $\sup \mathbb{N}$ exists. Let $m = \sup \mathbb{N}$. Then, by Theorem 3.1.9, with $\epsilon = 1$, there is an element $k \in \mathbb{N}$ such that $m - 1 < k$. This implies that $m < k + 1 \leq m$, which is absurd. ■

There are several equivalent formulations of the Archimedean Property. We shall mention just a few of them as corollaries of Theorem 3.1.13.

3.1.14 Corollary
For every real number b there exists an integer m such that $m < b$.

Proof. For every real number b there is a natural number n such that $n > -b$. Hence, $m = -n < b$. ■

3.1.15 Corollary
Given any real number x, there exists an integer k such that $x - 1 \leq k < x$.

$\prec$Let $x \in \mathbb{R}$. By Corollary 3.1.14, there is an integer m such that $m < x$. By the Archimedean Property, there is a natural number n such that $x < n$. Hence, $m < x < n$. Choose the largest integer k from the finite collection $m, m+1, \ldots, n$ such that $k < x$. Then $k + 1 \geq x$, and consequently, $x - 1 \leq k < x$. ■

3.1.16 Corollary
If x and y are two positive real numbers, then there exists a natural number n such that $nx > y$.

$\prec$Assume that $nx \le y$ for all $n \in \mathbb{N}$. Then $n \le \frac{y}{x}$ for all $n \in \mathbb{N}$. This means that $\mathbb{N}$ is bounded above (by $\frac{y}{x}$), contradicting Theorem 3.1.13. ■

3.1.17 Corollary
If $\epsilon > 0$ then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. Take $x = \epsilon$ and $y = 1$ in Corollary 3.1.16. ■

Since $\mathbb{R}$ is a field in which order and completeness axioms and the Archimedean Property hold, $\mathbb{R}$ is called a **complete ordered Archimedean field**.

The following theorem asserts that we can approximate any real number as closely as we wish by a rational number. A similar statement also holds for irrational numbers.

3.1.18 Theorem
(Density of Rationals in Reals). *If x and y are real numbers such that $x < y$, then there exists a rational number r such that $x < r < y$. That is, between any two distinct real numbers there is a rational number.*

Proof. By Corollary 3.1.17, there is a natural number n such that $\frac{1}{n} < y - x$. That is,

$$x < y - \frac{1}{n}. \tag{3.1}$$

Also, by Corollary 3.1.15, there is an integer k such that

$$ny - 1 \le k < ny, \text{ i.e. } y - \frac{1}{n} \le \frac{k}{n} < y. \tag{3.2}$$

Combining (3.1) and (3.2), we have that

$$x < y - \frac{1}{n} \le \frac{k}{n} < y.$$ ■

3.1.19 Corollary
(Density of Irrationals in Reals). *If x and y are real numbers such that $x < y$, then there exists an irrational number z such that $x < z < y$. That is, between any two distinct real numbers there is an irrational number.*

Proof. By Theorem 3.1.18, there are rational numbers r_1 and r_2 such that

$$x < r_1 < r_2 < y.$$

Then $z = r_1 + \frac{r_2 - r_1}{\sqrt{2}}$ is an irrational number such that $x < z < y$. ■

3.1.20 Corollary
Let b be any real number and let $S = \{q \in \mathbb{Q} : q < b\}$. Then $b = \sup S$. That is, every real number is a supremum of a set of rational numbers.

Proof. Of course, b is an upper bound for the set S. By the Completeness Axiom, S has a supremum, $c = \sup S$ say. By definition of supremum, $c \le b$. If $c < b$, then by Theorem 3.1.18, there exists an $q \in \mathbb{Q}$ such that $c < q < b$. But then $q < b$ implies that $q \in S$, and $c < q$ contradicts the fact that c is the supremum of S. Therefore $c = b$. ■

It is now easy to see why the completeness axiom fails in $\mathbb{Q}$. Indeed, if η is an irrational number and $S = \{q \in \mathbb{Q} : 0 \le q < \eta\}$, then S is bounded, with $\sup S = \eta$. However η does not belong to the set $\mathbb{Q}$.

3.2 Topology of the Real Numbers

In this section we briefly discuss some elementary topological properties of the set $\mathbb{R}$ of real numbers.

The Absolute Value Function

3.2.1 Definition
Let $x \in \mathbb{R}$. The **absolute value** *of x is defined by*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

It is clear from the definition that the absolute value of any real number is always nonnegative.

3.2.2 Theorem
(Properties of the Absolute Value Function). *Let $x, y \in \mathbb{R}$. Then,*

[1] $|x| \geq x$ *and* $|x| \geq -x$.

[2] $|x| = |-x|$.

[3] $|xy| = |x||y|$.

[4] $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ *for* $y \neq 0$.

[5] $|x+y| \leq |x| + |y|$. *(Triangle Inequality.)*

$\prec$Exercise.

If we think of the real numbers as points on the real line, then $|x-y|$ is just the distance between the real numbers x and y.

3.2.3 Exercise
Let $x, y, z \in \mathbb{R}$. Show that

[1] $|x-y| + |y-z| \geq |x-z|$.

[2] $|x| + |y| \geq |x-y|$.

[3] $|x| - |y| \leq |x-y|$.

[4] $|x| = \max\{-x, x\}$.

[5] $|x-y| < \epsilon$ for all $\epsilon > 0$ if and only if $x = y$ if and only if $|x-y| = 0$.

Open Sets and Closed Sets

3.2.4 Definition
Let $a \in \mathbb{R}$ and $\epsilon > 0$.

*[1] An $\epsilon-$***neighbourhood** *of a is the set*

$$N(a, \epsilon) = \{x \in \mathbb{R} : |x-a| < \epsilon\}.$$

[2] A **deleted** ϵ**−neighbourhood** *of a is the set*

$$N^*(a, \epsilon) = \{x \in \mathbb{R} : 0 < |x - a| < \epsilon\}.$$

It is clear that

$$N(a, \epsilon) = (a - \epsilon, a + \epsilon) \text{ and } N^*(a, \epsilon) = (a - \epsilon, a) \cup (a, a + \epsilon).$$

3.2.5 Definition

A subset U of $\mathbb{R}$ is said to be **open** *if for each $s \in U$ there is an $\epsilon > 0$ such that $(s - \epsilon, s + \epsilon) \subset U$.*

3.2.6 Examples

[1] For any $a, b \in \mathbb{R}$ with $a < b$, the set (a, b) is open. Indeed, if $s \in (a, b)$, then $a < s < b$. Take $\epsilon = \min\{s - a, b - s\}$. We claim that $(s - \epsilon, s + \epsilon) \subset (a, b)$. To prove the claim, let $t \in (s - \epsilon, s + \epsilon)$. Then,

$$s - \epsilon < t < s + \epsilon \quad \Rightarrow \quad s - (s - a) \leq s - \epsilon < t < s + \epsilon \leq s + (b - s),$$

whence $a < t < b$, which proves the claim.

[2] The sets $(-1, 0) \cup (3, 7)$ and $(-\infty, 4) \cup (6, 9) \cup (12, 20)$ are open.

[3] The sets $\mathbb{R}$, $\emptyset$ are open.

[4] The set $\mathbb{Q}$ of rational numbers is not open in $\mathbb{R}$. Indeed, if $r \in \mathbb{Q}$, then for any $\epsilon > 0$ the interval $(r - \epsilon, r + \epsilon)$ contains an irrational number. Thus, $\mathbb{Q}$ is not open.

[5] The set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not open. The reasoning is the same as in the previous example.

3.2.7 Theorem

(1) A union of an arbitrary collection of open sets in $\mathbb{R}$ is an open set.

(2) An intersection of a finite collection of open sets in $\mathbb{R}$ is an open set.

$\prec$(1) Let $\{U_i \mid i \in I\}$ be a collection of open sets in $\mathbb{R}$. We want to show that the set $\bigcup_{i \in I} U_i$ is open. Let $x \in \bigcup_{i \in I} U_i$. Then $x \in U_k$ for some $k \in I$. Since U_k is open, we can find an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U_k$. Since $U_k \subset \bigcup_{i \in I} U_i$, we have that $(x - \epsilon, x + \epsilon) \subset \bigcup_{i \in I} U_i$, which proves the desired result.

(2) Let $U_1, U_2, \ldots U_n$ be open sets in $\mathbb{R}$. We want to show that the set $\bigcap_{k=1}^{n} U_k$ is open. Let $x \in \bigcap_{k=1}^{n} U_k$. Then $x \in U_k$ for all $k = 1, 2, \ldots, n$. Since U_k is open for each $k = 1, 2, \ldots, n$, we can find an $\epsilon_k > 0$ such that $(x - \epsilon_k, x + \epsilon_k) \subset U_k$ for each $k = 1, 2, \ldots, n$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Then $(x - \epsilon, x + \epsilon) \subset U_k$ for each $k = 1, 2, \ldots, n$, and consequently, $(x - \epsilon, x + \epsilon) \subset \bigcap_{k=1}^{n} U_k$. That is, the set $\bigcap_{k=1}^{n} U_k$ is open. ■

An arbitrary intersection of open sets need not be open. For example, if $I_k = \left(\frac{-1}{k}, \frac{1}{k}\right)$, then $\bigcap_{k=1}^{\infty} I_k = \{0\}$, which is not open.

The following theorem completely characterises open subsets of $\mathbb{R}$.

3.2.8 Theorem
A subset of $\mathbb{R}$ is open if and only if it is a countable union of disjoint open intervals in $\mathbb{R}$.

$\prec$Let G be an open subset of $\mathbb{R}$ and $x \in G$. Denote by I_x the union of all open intervals in G that contain x. Note that since G is open, there is an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset G$. There is therefore at least one open interval that contains x and is contained in G. Clearly $I_x \subseteq G$. Let

$$\alpha_x = \inf\{a \in \mathbb{R} \mid (a, x] \subset G\} \text{ and } \beta_x = \sup\{b \in \mathbb{R} \mid [x, b) \subset G\}.$$

Claim 1: $I_x = (\alpha_x, \beta_x)$.
$I_x \subseteq (\alpha_x, \beta_x)$: Let $y \in I_x$. Then there is an open interval $(a, b) \subset G$ with $x \in (a, b)$ such that $y \in (a, b)$. Since $(a, b) \subset G$ and $x \in (a, b)$, it follows that $(a, x] \subset G$ and $[x, b) \subset G$. Thus, $\alpha_x \leq a$ and $b \leq \beta_x$. Therefore

$$\alpha_x \leq a < y < b \leq \beta_x$$

and consequently $y \in (\alpha_x, \beta_x)$.
$(\alpha_x, \beta_x) \subseteq I_x$: It is clear that (α_x, β_x) is an open interval and $x \in (\alpha_x, \beta_x)$. It remains to show that $(\alpha_x, \beta_x) \subset G$. To that end, let $y \in (\alpha_x, \beta_x)$. The either $y \leq x$ or $x \leq y$. Without loss of generality, we assume that $y \leq x$. By characterization of α_x (as the infimum), given any $\epsilon > 0$, there is an $a_\epsilon \in \mathbb{R}$ such that $(a_\epsilon, x] \subset G$ and $a_\epsilon < \alpha_x + \epsilon$. In particular, taking $\epsilon = \frac{y-\alpha_x}{2}$, we have that

$$a_\epsilon < \alpha_x + \frac{y - \alpha_x}{2} = \frac{\alpha_x + y}{2}.$$

Since $\alpha_x < y$, it follows that

$$a_\epsilon < \frac{\alpha_x + y}{2} < \frac{y + y}{2} = y \leq x.$$

Therefore $y \in (a_\epsilon, x] \subset G$, and so $y \in G$.
If $x \leq y$, then we can similarly show that $y \in G$.
Claim 2: If $x, y \in G$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Assume that $z \in I_x \cap I_y$. Then $z \in I_x$ and $z \in I_y$. Therefore, I_x is an open interval in G which contains z, and so $I_x \subseteq I_z$. Since $x \in I_x \subseteq I_z$, I_z is an open interval in G which contains x. Hence $I_z \subseteq I_x$. It now follows that $I_x = I_z$. Similarly, $I_z = I_y$, whence $I_x = I_y$. This shows that $\{I_x \mid x \in G\}$ is a collection of disjoint open intervals.
Claim 3: $G = \bigcup_{x \in G} I_x$. Since $I_x \subseteq G$ for each $x \in G$, it follows that $\bigcup_{x \in G} I_x \subseteq G$. On the other hand, if $x \in G$, then, since G is open, there is an open interval $(a, b) \subset G$ such that $x \in (a, b)$ Since $(a, b) \subseteq I_x$, it follows that $x \in I_x$, whence $G \subseteq \bigcup_{x \in G} I_x$.

Claim 4: We can replace the intervals I_x by a countable collection of disjoint intervals. For each $x \in G$, $\alpha_x < x < \beta_x$. Since rationals are dense in reals, there are rational numbers r_x and s_x such that $\alpha_x < r_x < x < s_x < \beta_x$ for each $x \in G$. Thus,

$$x \in (r_x, s_x) \subseteq (\alpha_x, \beta_x) = I_x \text{ for each } x \in G.$$

It now follows that

$$G \subseteq \bigcup_{x \in G} (r_x, s_x) \subseteq \bigcup_{x \in G} I_x = G.$$

That is, $G = \bigcup_{x \in G} (r_x, s_x)$. The intervals $\{(r_x, s_x) \mid x \in G\}$ are clearly disjoint.
Conversely, assume that G is a countable union of disjoint open intervals. Since an open interval is an open set and an arbitrary union of open sets is open, it follows that G is also open. ■

3.2.9 Definition
Let S be a subset of $\mathbb{R}$, and $x \in \mathbb{R}$. Then

(a) *$x \in S$ is called an* **interior point** *of S if there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. The set of all interior points of a set S is denoted by S° or int(S).*

(b) *x is called a* **boundary point** *of S if for every $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains points of S as well as points of $\mathbb{R} \setminus S$. The set of boundary points of S is denoted by ∂S or bd(S).*

(c) *$x \in S$ is called an* **isolated point** *of S if there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap S = \{x\}$.*

It is clear from the definition that each point of an open set S is an interior point of S. Also, every isolated point of a set S is a boundary point of S.

3.2.10 Examples

[1] Let $S = \{x \in \mathbb{R} \ : \ 0 \le x < 1 \}$. Then $S^o = \{x \in \mathbb{R} \ : \ 0 < x < 1 \}$, $\partial S = \{0, 1\}$. S does not have isolated points.

[2] Let $S = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\}$. Then each point of S is an isolated point of S. Therefore $S \subset \partial S$.

[3] Let $S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots, 0 \right\}$. Then $\partial S = S$.

[4] The set $\mathbb{N}$ of natural numbers consists of isolated points only. Therefore, every point of $\mathbb{N}$ is a boundary point. Clearly, $\mathbb{N}^\circ = \emptyset$.

[5] $\mathbb{Z}^\circ = \emptyset$, and $\mathbb{R}^\circ = \mathbb{R}$.

[6] Each set with only finitely many elements consists entirely of isolated points.

The following Theorem asserts that elements of a set $S \subset \mathbb{R}$ can be divided into two groups: those that are interior to the set and those that are on the boundary of the set S.

3.2.11 Theorem

Let $S \subset \mathbb{R}$. Then each point of S is either an interior point of S or is a boundary point of S.

$\prec$Let $s \in S$. If s is not an interior point of S, then for each $\epsilon > 0$ the interval $(s - \epsilon, s + \epsilon)$ contains a point in $\mathbb{R} \setminus S$. Since $(s - \epsilon, s + \epsilon)$ already contains a point s of the set S, we have that this interval contains a point in S as well as a point in $\mathbb{R} \setminus S$. Hence s is a boundary point of S. ■

3.2.12 Definition

A subset S of $\mathbb{R}$ is said to be **closed** *if its complement $\mathbb{R} \setminus S$ is open.*

3.2.13 Examples

[1] The interval $[a, b]$ is closed since its complement $(-\infty, a) \cup (b, \infty)$ is open.

[2] The sets $\mathbb{R}$ and $\emptyset$ are closed.

[3] The set $\mathbb{Q}$ of rational numbers is not closed.

[4] The set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not closed.

[5] The set $\{1, \ 3, \ 4, \ 7 \}$ is closed since its complement $(-\infty, 1) \cup (1, 3) \cup (3, 4) \cup (4, 7) \cup (7, \infty)$ is open.

[6] The set $S = \left\{ \dfrac{1}{n} \ : \ n \in \mathbb{N} \right\} \cup \{0\}$ is closed since its complement

$$(-\infty, 0) \cup \left[\bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k} \right) \right] \cup (1, \infty)$$

is open.

3.2.14 Definition

Let S be a subset of $\mathbb{R}$.

(1) A point $x \in \mathbb{R}$ is an **accumulation point** *of S if for every $\epsilon > 0$ there exists an element $s \in S$ such that $0 < |x - s| < \epsilon$; i.e., $[\,(x-\epsilon, x) \cup (x, x+\epsilon)\,] \cap S \neq \emptyset$. In other words, x is an accumulation point of S if every deleted ϵ-neighbourhood $N^*(x, \epsilon)$ of x contains a point of S.*

The set of all accumulation points of S is called the **derived** *set of S and is denoted by S'.*

(2) S is said to be **dense in itself** *if $S \subset S'$.*

(3) S is called **perfect** *if $S = S'$.*

(4) The **closure** *of S is the set $\overline{S} = S \cup S'$.*

3.2.15 Remarks

(1) An accumulation point of a set S need not be an element of S.

(2) A real number x is an accumulation point of a set $S \subset \mathbb{R}$ if for each $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains infinitely many elements of S. Indeed, if x is an accumulation point of S then, for any $\epsilon > 0$, there exists an element $s_1 \in S$ with $s_1 \neq x$, such that $0 < |x - s_1| < \epsilon$. Taking $\epsilon_1 = |x - s_1|$, there exists an element $s_2 \in S$ with $s_2 \neq x$, such that $0 < |x - s_2| < \epsilon_1 < \epsilon$. Taking $\epsilon_2 = |x - s_2|$, there exists $s_3 \in S$ with $s_3 \neq x$ such that $0 < |x - s_3| < \epsilon_2 < \epsilon$. Continuing in this way we obtain a sequence (s_n) with the property that $s_n \neq x$ and $|s_n - x| < \epsilon$ for all n.

[1] Elements of a set $S \subset \mathbb{R}$ can be divided into two groups: **isolated points** = those points that can be separated from the rest of the set with an open interval, and **accumulation points** = those points that cannot be separated from the rest of the set with an open interval.

3.2.16 Examples

[1] Let $S = \{x \in \mathbb{R} \ : \ 0 < x \le 1\,\}$. Then $S' = \{x \in \mathbb{R} \ : \ 0 \le x \le 1\,\}$. Therefore $\overline{S} = S \cup S' = S'$.

[2] If $S = \{x \in \mathbb{R} \ : \ a \le x \le b\,\}$, then $S' = S$. Therefore $\overline{S} = S$.

[3] If $S = \left\{ \dfrac{1}{n} \ : \ n \in \mathbb{N} \right\}$, then $S' = \{0\}$. Indeed, if $\epsilon > 0$, then there is an $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < \epsilon$. Therefore $-\epsilon < \frac{1}{m} < \epsilon$; i.e., $\frac{1}{m} \in (-\epsilon, \epsilon)$. Since $\frac{1}{m} \in S$, it follows that for each $\epsilon > 0$, $N^*(0, \epsilon) \cap S \neq \emptyset$. Also $\overline{S} = S \cup S' = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\} \cup \{0\}$. Note that $0 \notin S$.

[4] Every real number is an accumulation point of the set $\mathbb{Q}$ of rational numbers; that is, $\mathbb{Q}' = \mathbb{R}$. Indeed, if $x \in \mathbb{R}$ and $\epsilon > 0$, then the interval $(x - \epsilon, x + \epsilon)$ contains infinitely many rational numbers. It now follows that $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

[5] If $\mathbb{Z}$ is the set of integers, then $\mathbb{Z}' = \emptyset$. Indeed, for any $x \in \mathbb{R}$ we can find an $\epsilon > 0$ small enough such that $(x - \epsilon, x + \epsilon)$ contains no integer, except possibly when x is itself an integer. It thus follows that $\overline{\mathbb{Z}} = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z} \cup \emptyset = \mathbb{Z}$.

[6] A finite set has no accumulation points. Indeed, if $S = \{ s_1, s_2, \dots, s_n \}$, and $x \in \mathbb{R}$, then taking $\epsilon = \min\{ |s_j - x|, \; j = 1, 2, \dots, n \}$, we have that $\epsilon > 0$ and

$$[(x - \epsilon, x) \cup (x, x + \epsilon)] \cap S = \emptyset.$$

Thus $S' = \emptyset$, and so $\overline{S} = S$.

Another way of seeing that a finite set S has no accumulation points is simply that no ϵ-neighbourhood can contain infinitely many points of S since S is finite!

3.2.17 Theorem
Let $S \subset \mathbb{R}$. Then S is closed if and only if S contains all its accumulation points.

$\prec$Suppose that S is closed and let $x \in S'$. We want to show that $x \in S$. If $x \notin S$, then $x \in \mathbb{R} \setminus S$. Since S is closed, $\mathbb{R} \setminus S$ is open. Therefore there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus S$. This then implies that $(x - \epsilon, x + \epsilon) \cap S = \emptyset$. But this contradicts the fact that $x \in S'$. Thus $S' \subset S$.

To prove the converse, assume that $S' \subset S$. We want to show that S is closed, or equivalently, that $\mathbb{R} \setminus S$ is open. To this end, let $x \in \mathbb{R} \setminus S$. Then $x \notin S'$, and so there is an $\epsilon > 0$ such that

$$[(x - \epsilon, x) \cup (x, x + \epsilon)] \cap S = \emptyset.$$

Since $x \notin S$, we have that $(x - \epsilon, x + \epsilon) \cap S = \emptyset$. Thus $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus S$, whence $\mathbb{R} \setminus S$ is open. ■

3.2.18 Corollary
Let $S \subset \mathbb{R}$. Then S is closed if and only if $\overline{S} = S$.

$\prec$Assume that S is closed. Then, by Theorem 3.2.17, $S' \subset S$. Therefore $\overline{S} = S \cup S' \subset S \cup S = S$. But $S \subset S \cup S' = \overline{S}$. Thus $S = \overline{S}$.

Conversely, assume that $S = \overline{S}$. Then $S' \subset S \cup S' = \overline{S} = S$. Thus S contains all its accumulation points and, consequently, S is closed. ■

It follows from Theorem 3.2.17 that the sets $[a, b]$, $\mathbb{Z}$, $S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0 \right\}$ are all closed, as is any finite set.

3.2.19 Theorem
If $S \subset \mathbb{R}$ is closed and bounded, then $\sup S$ and $\inf S$ belong to S.

$\prec$Let $s = \sup S$. ($\sup S$ exists because S is bounded above.) Then, for any $\epsilon > 0$, there is an $x_\epsilon \in S$ such that $s - \epsilon < x_\epsilon \le s < s + \epsilon$. If $x_\epsilon = s$, then $s \in S$ and we are done. If $x_\epsilon < s$, then $0 < |s - x_\epsilon| < \epsilon$. That is, for every $\epsilon > 0$, there is an $x_\epsilon \in S$ such that $0 < |s - x_\epsilon| < \epsilon$. Thus $s \in S'$. Since S is closed, $s \in S$.

A similar argument shows that $\inf S \in S$. ■

3.2.20 Exercise

[1] Let $S = \left\{ n + \frac{1}{m}, \; n, m \in \mathbb{Z}, \; m > 0 \right\}$. Find S'.

[2] Let S and T be subsets of $\mathbb{R}$.

(a) Show that if $S \subset T$, then $S' \subset T'$.

(b) Show that if $S \subset T$, then $\overline{S} \subset \overline{T}$.

(c) Show that $\overline{S}$ is a closed subset of $\mathbb{R}$.

(d) Show that if F is a closed subset of $\mathbb{R}$ and $S \subset F$, then $\overline{S} \subset F$.

(e) Show that $\overline{S} = \bigcap \{ F \subset \mathbb{R} \mid F \text{ is closed and } S \subset F \}$. Deduce that $\overline{S}$ is the smallest closed set containing S.

(f) Show that $\overline{\overline{S}} = \overline{S}$.

(g) Show that $(S \cup T)' = S' \cup T'$. Deduce that $\overline{S \cup T} = \overline{S} \cup \overline{T}$.

(h) Show that $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$.

(i) Is it true that $\overline{S \cap T} = \overline{S} \cap \overline{T}$?

[3] (a) Show that a union of a finite collection of closed sets in $\mathbb{R}$ is a closed set.

(b) Show that an intersection of an arbitrary collection of closed sets in $\mathbb{R}$ is a set.

[4] Let S be a subset of $\mathbb{R}$.

(a) Show that S' is closed.

(b) Show that if S is bounded, then so is S'.

(c) Show that if S is bounded, then so is $\overline{S}$.

[5] Let S and T be subsets of $\mathbb{R}$.

(a) Show that if $S \subset T$, then $S^\circ \subset T^\circ$.

(b) Show that T° is an open subset of $\mathbb{R}$.

(c) Show that T is open if and only if $T = T^\circ$.

(d) Show that if G is an open subset of $\mathbb{R}$ and $G \subset T$, then $G \subset T^\circ$.

(e) Show that $T^\circ = \bigcup\{G \subset \mathbb{R} \mid G \text{ is open and } G \subset T\}$. Deduce that T° is the largest open set contained in T.

(f) Show that $T^{\circ\circ} = T^\circ$.

(g) Show that $(S \cap T)^\circ = S^\circ \cap T^\circ$.

(h) Show that $S^\circ \cup T^\circ \subseteq (S \cup T)^\circ$.

(i) Is it true that $(S \cup T)^\circ = S^\circ \cup T^\circ$?

3.3 Compactness

3.3.1 Definition
An **open cover** *of a set $S \subset \mathbb{R}$ is a collection $\mathcal{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ of open sets such that*

$$S \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha.$$

If $\mathcal{G}' \subseteq \mathcal{G}$ and $\mathcal{G}'$ is also an open cover for S, then $\mathcal{G}'$ is called a **subcover** *for S. If, in addition, $\mathcal{G}'$ has a finite number of elements, then $\mathcal{G}'$ is called a* **finite subcover** *of S.*

3.3.2 Examples

[1] Let $S = [0, \infty)$ and, for each $n \in \mathbb{N}$, let $G_n = (-1, n)$. Then $\mathcal{G} = \{G_n \mid n \in \mathbb{N}\}$ is an open cover for S.

[2] Let $S = [0, 1]$ and for each $n \in \mathbb{N}$, let $A_n = (-\frac{1}{n}, 1 + \frac{1}{n})$. Then $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$ is an open cover for S.

[3] Let $S = (0, 1)$ and $\mathcal{U} = \{(\frac{1}{n}, 2) \mid n \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover for S. Indeed, let $x \in (0, 1)$. Then, by the Archimedean Property, there is a natural number m such that $0 < \frac{1}{m} < x$. Therefore $x \in (\frac{1}{m}, 2)$, whence, $(0, 1) \subset \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 2)$.

3.3.3 Definition
A subset S of $\mathbb{R}$ is said to be **compact** *if every open cover for S has a finite subcover.*

3.3.4 Examples

[1] A finite subset of $\mathbb{R}$ is compact. Indeed, let $S = \{x_1, x_2, \ldots, x_n\}$ and let $\mathcal{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ be an open cover for S. Then each x_i belongs to some G_{α_i} in $\mathcal{G}$. The set $\mathcal{G}' = \{G_{\alpha_i} \mid 1 \leq i \leq n\}$ is a finite subcover for S.

[2] The set $\mathbb{R}$ of real numbers is not compact since the open cover $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ does not have a finite subcover.

3.3.5 Theorem
Let S be a compact subset of $\mathbb{R}$. If F is a closed subset of S, then F is compact.

$\prec$Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover for F. Then $\mathcal{G} = \mathcal{U} \cup \{F^c\}$ is an open cover for S. Since S is compact, the cover $\mathcal{G}$ is reducible to a finite subcover. That is, there are indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$S \subset \bigcup_{i=1}^{n} U_{\alpha_i} \cup F^c.$$

Since $F \subset S$ and $F \cap F^c = \emptyset$, it follows that $F \subset \bigcup_{i=1}^{n} U_{\alpha_i}$. Hence F is compact. ■

3.3.6 Theorem
Let a and b be real numbers such that $-\infty < a < b < \infty$. Then the interval $[a, b]$ is compact.

$\prec$Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover for the interval $[a, b]$ and let

$$A = \{x \in [a, b] \mid \ [a, x] \text{ has a finite subcover in } \mathcal{U}\}.$$

Clearly $A \neq \emptyset$ since $a \in A$. Also, A is bounded above as $x \leq b$ for all $x \in A$. By the Completeness Axiom, $\sup A$ exists. Let $c = \sup A$. Then $a \leq c \leq b$.

Claim 1: The element c belongs to A.

Proof of Claim 1: Since $\mathcal{U}$ is an open cover for the interval $[a, b]$, there is an index i_0 such that $c \in U_{i_0}$. Then, since U_{i_0} is open, there is an $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subseteq U_{i_0}$. In particular, $(c - \epsilon, c] \subseteq U_{i_0}$. By characterization of the supremum, there is an element $x_\epsilon \in A$ such that $c - \epsilon < x_\epsilon \leq c$. Therefore $a \leq x_\epsilon \leq c$ and so the interval $[a, c] = [a, x_\epsilon] \cup [x_\epsilon, c]$ has a finite subcover in $\mathcal{U}$. [The interval $[a, x_\epsilon]$ is finitely covered in $\mathcal{U}$ since $x_\epsilon \in A$. The interval $[x_\epsilon, c]$ is covered by the set U_{i_0}.] It now follows that $c \in A$.

Claim 2: $c = b$.

Proof of Claim 2: If $c < b$, then, since $\mathcal{U}$ is an open cover for $[a, b]$, there is a U_{i_0} in $\mathcal{U}$ such that $c \in \mathcal{U}$. Since $\mathcal{U}$ is open, there is a $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq U_{i_0}$. Choose $\delta > \delta' > 0$ small enough such that $[c, c + \delta'] \subseteq U_{i_0}$ and $c + \delta' < b$. Then $c + \delta' \in [a, b]$ and $[a, c + \delta'] = [a, c] \cup [c, c + \delta']$ has a finite subcover in $\mathcal{U}$. [The interval $[a, c]$ is finitely covered in $\mathcal{U}$ since $c \in A$ and the interval $[c, c + \delta']$ is covered by the open set $U_{i_0} \in \mathcal{U}$.] This shows that $c + \delta'$ is in A. But this contradicts the fact that c is the supremum of A. Hence, $c = b$.

By definition of the set A, we conclude that $[a, b]$ can be covered by finitely many elements of $\mathcal{U}$. That is, $[a, b]$ is compact. ■

3.3.7 Theorem
(Heine-Borel Theorem). *A subset K of $\mathbb{R}$ is compact if and only if K is closed and bounded.*

Assume that K is compact. We show that K is closed and bounded.

Closedness of K: It suffices to show that the complement, $\mathbb{R} \setminus K$, of K is open. To that end, let $x_0 \in \mathbb{R} \setminus K$ and for each $k \in \mathbb{N}$, let

$$U_k = \{x \in \mathbb{R} \mid |x - x_0| > \frac{1}{k}\} = (-\infty, x_0 - \frac{1}{k}) \cup (x_0 + \frac{1}{k}, \infty).$$

Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R} \setminus \{x_0\}$ and $\mathcal{U} = \{U_k \mid k \in \mathbb{N}\}$ is an open cover for K. Since K is compact, this cover of K is reducible to a finite subcover. That is, there are indices $k_1,\ k_2,\ \ldots,\ k_n$ such that $K \subset \bigcup_{j=1}^{n} U_{k_j}$. Let $k_{\max} = \max\{k_1,\ k_2,\ \ldots,\ k_n\}$. Then

$$K \subset \bigcup_{j=1}^{n} U_{k_j} = (-\infty, x_0 - \frac{1}{k_{\max}}) \cup (x_0 + \frac{1}{k_{\max}}, \infty) = \{x \in \mathbb{R} \mid |x - x_0| > \frac{1}{k_{\max}}\}.$$

Hence,

$$\{x \in \mathbb{R} \mid |x - x_0| < \frac{1}{k_{\max}}\} \subset \{x \in \mathbb{R} \mid |x - x_0| \le \frac{1}{k_{\max}}\} \subset \mathbb{R} \setminus K,$$

whence $\mathbb{R} \setminus K$ is open and so K is closed.

Boundedness of K: Let $\mathcal{U} = \{(-k, k) \mid k \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover for K. Indeed,

$$K \subset \mathbb{R} = \bigcup_{k \in \mathbb{N}} (-k, k).$$

Since K is compact, there are natural numbers $k_1,\ k_2,\ \ldots,\ k_n$ such that $K \subset \bigcup_{j=1}^{n} (-k_j, k_j)$. Let $k_{\max} = \max\{k_1,\ k_2,\ \ldots,\ k_n\}$. Then

$$K \subset \bigcup_{j=1}^{n} (-k_j, k_j) = (-k_{\max}, k_{\max}).$$

It now follows that K is bounded since it is contained in the bounded interval $(-k_{\max}, k_{\max})$.

Conversely, assume that K is a closed and bounded subset of $\mathbb{R}$. Then there are real numbers a and b such that $K \subset [a, b]$. It now follows from Theorem 3.3.6 and Theorem 3.3.5 that K is compact. ■

We now apply the Heine-Borel Theorem to prove another important result: the Bolzano-Weierstrass Theorem (for sets).

3.3.8 Theorem

(Bolzano-Weierstrass Theorem for sets). *Every bounded infinite set of real numbers has at least one accumulation point.*

$\prec$Let S be a bounded infinite set of real numbers. Suppose that S has no accumulation points. That is, the derived set of S, S', is empty. Therefore $\overline{S} = S \cup S' = S$, and so S is closed. Thus S is a closed and bounded subset of $\mathbb{R}$. By the Heine-Borel Theorem (Theorem 3.3.7), S is compact. Since S has no accumulation points, given any $x \in S$, there is an $\epsilon > 0$ such that $S \cap N(x, \epsilon) = \{x\}$. Therefore, the collection $\{N(x, \epsilon) \mid x \in S,\ \epsilon > 0\}$ is an open cover for S. Since S is compact, there exist $x_1,\ x_2,\ \ldots,\ x_n$ in S and positive numbers $\epsilon_1,\ \epsilon_2,\ \ldots,\ \epsilon_n$ such that $S \subset \bigcup_{k=1}^{n} N(x_k, \epsilon_k)$. But then

$$S = S \cap \bigcup_{k=1}^{n} N(x_k, \epsilon_k) = \bigcup_{k=1}^{n} (S \cap N(x_k, \epsilon_k)) = \bigcup_{k=1}^{n} \{x_k\} = \{x_1,\ x_2,\ \ldots,\ x_n\}.$$

That is, $S = \{x_1,\ x_2,\ \ldots,\ x_n\}$, a finite set. This is a contradiction since S has infinitely many points. ■

3.3.9 Exercise

[1] We showed in Theorem 3.3.5 that a closed subset F of a compact set K is compact. Supply another proof to this statement by using the Heine-Borel Theorem.

[2] Show that the interval $(0, 1)$ is not compact by

(a) showing that $\mathcal{U} = \{(\frac{1}{n}, 2) \mid n \in \mathbb{N}\}$ is an open cover for $(0, 1)$ with no finite subcover.

(b) using the Heine-Borel Theorem.

[3] Show that the interval $[0, \infty)$ is not compact by

(a) showing that $\mathcal{G} = \{(-1, n) \mid n \in \mathbb{N}\}$ is an open cover for $[0, \infty)$ with no finite subcover.

(b) using the Heine-Borel Theorem.

[4] Show that the set $\mathbb{N}$ is not compact by

(a) finding an open cover for $\mathbb{N}$ that has no finite subcover.

(b) using the Heine-Borel Theorem.

[5] Show that if F is closed and K compact, then $F \cap K$ is compact by

(a) using the definition of compactness.

(b) applying the Heine-Borel Theorem.

[6] Show that an arbitrary intersection of compact subsets of $\mathbb{R}$ is compact.

[7] Show that if A and B are compact subsets of $\mathbb{R}$ then so is $A \cup B$ by

(a) using the definition of compactness.

(b) applying the Heine-Borel Theorem.

[8] Find an infinite collection $\{K_n \mid n \in \mathbb{N}\}$ of compact sets in $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} K_n$ is **not** compact.

This shows that an arbitrary union of compact sets is not compact.

[9] Show that a subset K of $\mathbb{R}$ is compact if and only of every infinite subset of K has an accumulation point in K.

Chapter 4

Sequences of Real Numbers

4.1 Introduction

In this chapter we study convergence of sequences of real numbers. We prove, among others, the Monotone Convergence Theorem, the Bolzano-Weierstrass Theorem for sequences, and the Cauchy Criterion for sequences of real numbers.

4.1.1 Definition
A **sequence** *is a function whose domain is the set* $\mathbb{N}$ *of natural numbers. If* f *is such a sequence, let* $f(n) = x_n$ *denote the value of the function* f *at* $n \in \mathbb{N}$*. In this case, we denote the sequence* f *by* $(x_n)_{n=1}^{\infty}$ *(or simply by* (x_n)*).*

4.1.2 Examples

[1] $\left(\dfrac{n}{n+1}\right)$ is the sequence $(\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots)$.

[2] $(\ (-1)^n\ )$ is the sequence $(-1,\ 1,\ -1, \ldots)$.

[3] $(\ 2^n\ )$ is the sequence $(2,\ 4,\ 8,\ \ldots)$.

Sequences are also frequently specified by giving a recursion formula. For example, if $x_{n+2} = \dfrac{x_n + x_{n+1}}{3}$, where $x_1 = 0$ and $x_2 = 1$, then the terms of the sequence (x_n) are: $(0,\ 1,\ \frac{1}{3},\ \frac{4}{9},\ \frac{7}{27},\ \ldots)$.

4.1.3 Remarks

[1] The order of the terms of a sequence is an important part in the definition of a sequence. For example, the sequence $(1,\ 5,\ 7,\ \ldots)$ is *not* the same as the sequence $(1,\ 7,\ 5,\ \ldots)$.

[2] There is a distinction between the *terms* of a sequence and the *values* of a sequence. A sequence has infinitely many terms while its values may or may not be finite.

[3] It is not necessary for the terms of a sequence to be different. For example, $(1,\ 2,\ 2,\ 2, \ldots)$ is a perfectly good sequence.

4.1.4 Exercise
Write down the first five terms of each of the following sequences.

(i) $\left(\dfrac{n^2+2n}{3n}\right)$ (ii) $\left(\dfrac{\cos n\pi}{n^2}\right)$ (iii) $\left(\dfrac{1}{\sqrt{n}}\right)$ (iv) $\left(\sin\dfrac{n\pi}{2}\right)$.

4.1.5 Definition
A sequence (x_n) *is said to be*

[1] **bounded above** *if there is a real number* K *such that* $x_n \le K$ *for all* $n \in \mathbb{N}$*;*

[2] **bounded below** *if there is a real number k such that $k \leq x_n$ for all $n \in \mathbb{N}$;*

[3] **bounded** *if it is bounded above and bounded below; otherwise it is* **unbounded**.

It is easy to see that a sequence $(x_n)_n$ is bounded if and only if there is a positive real number M such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

4.1.6 Examples

[1] The sequence $\left(\frac{1}{n}\right)$ is bounded since $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$.

[2] The sequence $\left(n + \frac{1}{n}\right)$ is bounded below by 2 but is not bounded above.

[3] The sequence $((-1)^n n)$ is not bounded above and it is not bounded below.

4.1.7 Definition

[1] A sequence (x_n) is said to **converge** *to a real number ℓ if, given $\epsilon > 0$, there exists a natural number N (which depends on ϵ) such that*

$$|x_n - \ell\,| < \epsilon \text{ for all } n \geq N.$$

Symbolically,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})[(\forall n \geq N) \Rightarrow |x_n - \ell| < \epsilon].$$

If (x_n) converges to ℓ, then we say that ℓ is the **limit** *of the sequence (x_n) as n increases without bound, and we write*

$$\lim_{n\to\infty} x_n = \ell \text{ or } x_n \to \ell \text{ as } n \to \infty.$$

[2] If the sequence (x_n) does not converge to a real number, we say that it **diverges**.

[3] A sequence (x_n) is said to **diverge to** *∞, denoted by $x_n \to \infty$ as $n \to \infty$, if for any positive real number M, there is an $N \in \mathbb{N}$ such that*

$$x_n > M \text{ for all } n \geq N.$$

Similarly, (x_n) **diverges to** *$-\infty$, denoted by $x_n \to -\infty$ as $n \to \infty$, if for any negative real number K, there is an $N \in \mathbb{N}$ such that*

$$x_n < K \text{ for all } n \geq N.$$

It is clear from the definition that convergence or divergence of a sequence is about the behaviour of the 'tail-end' of a sequence. Therefore, altering a finite number of terms of a sequence does not affect its convergence or divergence.

4.1.8 Examples

[1] Show that a sequence (x_n) converges to zero if and only if the sequence $(|x_n|)$ converges to zero.

Solution: Assume that the sequence (x_n) converges to zero. Then, given $\epsilon > 0$, there exists a natural number N (which depends on ϵ) such that

$$|x_n - 0\,| = |x_n| < \epsilon \text{ for all } n \geq N.$$

Now, for all $n \geq N$, we have

$$||x_n| - 0| = |x_n| < \epsilon.$$

That is, the sequence $(|x_n|)$ converges to zero.

For the converse, assume that the sequence $(|x_n|)$ converges to zero. That is, given $\epsilon > 0$, there exists a natural number N (which depends on ϵ) such that

$$||x_n| - 0\,| = |x_n| < \epsilon \text{ for all } n \geq N.$$

It now follows that the sequence (x_n) converges to zero.

[2] Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Solution: Let $\epsilon > 0$ be given. We must find an $N \in \mathbb{N}$ such that

$$\left|\frac{1}{n} - 0\right| < \epsilon \text{ for all } n \geq N.$$

By the Archimedean Property, there is an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \epsilon$. Thus, if $n \geq N$, then we have that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

That is, $\lim_{n\to\infty} \frac{1}{n} = 0$.

[3] Show that $\lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$.

Solution: Let $\epsilon > 0$ be given. We need to find an $N \in \mathbb{N}$ such that

$$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \epsilon \text{ for all } n \geq N.$$

Noting that

$$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| = \frac{1}{2^n} = \frac{1}{(1+1)^n}, \text{ and}$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \geq \binom{n}{0} + \binom{n}{1} = 1 + n,$$

we have that

$$\frac{1}{2^n} = \frac{1}{(1+1)^n} \leq \frac{1}{n+1} < \frac{1}{n}.$$

Now, by the Archimedean Property, there is an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \epsilon$. Therefore, for all $n \geq N$ we have

$$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Thus, $\lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$.

[4] Find $\lim_{n\to\infty} \frac{n}{n^2 - 2}$.

Solution: By trying a few values of n, we conjecture that $\lim_{n\to\infty} \frac{n}{n^2-2} = 0$. Let us prove this conjecture. Let $\epsilon > 0$ be given. We need to find an $N \in \mathbb{N}$ such that

$$\left|\frac{n}{n^2 - 2} - 0\right| < \epsilon \text{ for all } n \geq N.$$

We first note that
$$\left|\frac{n}{n^2-2}\right| = \frac{n}{|n^2-2|}.$$
If we take $N \geq 2$, then for all $n \geq N$, we have that
$$\frac{n}{|n^2-2|} = \frac{n}{n^2-2} = \frac{1}{n-\frac{2}{n}} < \frac{1}{n-1} \leq \frac{1}{N-1}.$$
If we can choose N so that $N \geq 2$ and $\frac{1}{N-1} < \epsilon$, then we are done. That is, we need to choose N so that $N \geq 2$ and $N > 1 + \frac{1}{\epsilon}$. Choose $N > \max\left(2,\ 1+\frac{1}{\epsilon}\right)$. Then, by working backwards, we have that
$$\left|\frac{n}{n^2-2} - 0\right| < \epsilon \ \text{ for all } \ n \geq N.$$

[5] Show that the sequence $((-1)^n)$ diverges.

Solution: Assume that this sequence converges to some real number ℓ. Then, with $\epsilon = \frac{1}{2}$, there is an $N \in \mathbb{N}$ such that
$$|(-1)^n - \ell\,| < \frac{1}{2} \ \text{ for all } \ n \geq N.$$
In particular,
$$|(-1)^{n+1} - \ell\,| < \frac{1}{2}\,.$$
Therefore, for all $n \geq N$,
$$2 = |(-1)^n - (-1)^{n+1}| \leq |(-1)^n - \ell\,| + |\ell - (-1)^{n+1}\,| < \frac{1}{2} + \frac{1}{2} = 1,$$
which is absurd.

[6] Show that the sequence $(1 + (-1)^n)$ diverges.

Solution: Assume that this sequence converges to some real number ℓ. Then, with $\epsilon = 1$, there exists a number $N \in \mathbb{N}$ such that
$$|(1+(-1)^n) - \ell\,| < 1 \text{ for all } n \geq N.$$
Now, if $n \geq N$ is odd, then we have
$$|(1+(-1)^n) - \ell\,| = |\ell| < 1, \text{ whence } -1 < \ell < 1,$$
and if $n \geq N$ is even, we have that
$$|(1+(-1)^n) - \ell\,| = |2-\ell| < 1, \text{ whence } 1 < \ell < 3.$$
But this is impossible.

[7] Show that if $x \in \mathbb{R}$ and $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Solution: If $x = 0$, then there is nothing to prove. Assume that $x \neq 0$. Since $|x| < 1$, we have that $\frac{1}{|x|} > 1$. Thus, there is a positive real number a such that
$$\frac{1}{|x|} = 1 + a.$$

Let $\epsilon > 0$ be given. We want to find an $N \in \mathbb{N}$ such that

$$|x^n - 0| < \epsilon \text{ for all } n \geq N.$$

Now,

$$\frac{1}{|x|} = 1 + a \quad \Rightarrow \quad |x^n| = |x|^n = \frac{1}{(1+a)^n}.$$

Using the binomial theorem, we have that

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = 1 + na + \frac{n(n-1)a^2}{2} + \cdots + a^n > na,$$

and consequently,

$$|x^n| = \frac{1}{(1+a)^n} < \frac{1}{na}.$$

If we can find an $N \in \mathbb{N}$ such that $\frac{1}{na} < \epsilon$ for all $n \geq N$, then we are done. Since $a\epsilon > 0$, we have, by the Archimedean Property, that there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < a\epsilon$. Hence, for all $n \geq N$, we have that

$$\frac{1}{n} \leq \frac{1}{N} \Rightarrow \frac{1}{na} \leq \frac{1}{Na} < \epsilon,$$

and so $|x^n| < \epsilon$.

[8] Suppose that (x_n) is a sequence such that $x_n > 0$ for all $n \in \mathbb{N}$. Show that $x_n \to \infty$ as $n \to \infty$ if and only if $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

Solution: By definition, $x_n \to \infty$ as $n \to \infty$, if and only if for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$x_n > \frac{1}{\epsilon} \text{ for all } n \geq N.$$

This is equivalent to the statement that

$$\frac{1}{x_n} < \epsilon \text{ for all } n \geq N, \text{ which, in turn, is equivalent to } \lim_{n\to\infty} \frac{1}{x_n} = 0.$$

4.1.9 Theorem

Let (s_n) and (t_n) be sequences of real numbers and let $s \in \mathbb{R}$. If for some positive real number k and some $N_1 \in \mathbb{N}$, we have

$$|s_n - s| \leq k|t_n| \text{ for all } n \geq N_1$$

and if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} s_n = s$.

≺Let $\epsilon > 0$ be given. Since $t_n \to 0$ as $n \to \infty$, there exists an $N_2 \in \mathbb{N}$ such that

$$|t_n| < \frac{\epsilon}{k} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ we have

$$|s_n - s| \leq k|t_n| < \frac{\epsilon}{k} \cdot k = \epsilon.$$

That is, $\lim_{n\to\infty} s_n = s$. ■

4.1.10 Example

Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Solution: Since $\sqrt[n]{n} \geq 1$ for each $n \in \mathbb{N}$, there is a nonnegative real number a_n such that

$$\sqrt[n]{n} = 1 + a_n.$$

Thus, by the binomial theorem, we have

$$n = (1 + a_n)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k = 1 + na_n + \frac{n(n-1)a_n^2}{2} + \cdots + a_n^n \geq 1 + \frac{n(n-1)a_n^2}{2}.$$

Therefore,

$$n - 1 \geq \frac{n(n-1)a_n^2}{2}, \text{ whence } a_n^2 \leq \frac{2}{n}, \text{ or } a_n \leq \sqrt{\frac{2}{n}} \text{ for all } n \geq 2.$$

Now, since

$$\left| \sqrt[n]{n} - 1 \right| = |a_n| = a_n \leq \sqrt{\frac{2}{n}}$$

and $\lim_{n\to\infty} \sqrt{\frac{2}{n}} = 0$, we have, by Theorem 4.1.9, that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. □

The next theorem says that a convergent sequence has only one limit. It is therefore unambiguous to talk of *the* limit of a convergent sequence.

4.1.11 Theorem

Let (s_n) be a sequence of real numbers. If $\lim_{n\to\infty} s_n = \ell_1$ and $\lim_{n\to\infty} s_n = \ell_2$, then $\ell_1 = \ell_2$.

≺Let $\epsilon > 0$ be given. Then there exist natural numbers N_1 and N_2 such that

$$|s_n - \ell_1| < \frac{\epsilon}{2} \text{ for all } n \geq N_1, \text{ and}$$

$$|s_n - \ell_2| < \frac{\epsilon}{2} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ we have that

$$|\ell_1 - \ell_2| = |(s_n - \ell_2) + (\ell_1 - s_n)| \leq |s_n - \ell_2| + |s_n - \ell_1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that $\ell_1 = \ell_2$. ■

4.1.12 Proposition

A sequence (x_n) converges to $\ell \in \mathbb{R}$ if and only if for each $\epsilon > 0$, the set $\{n \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\}$ is finite.

≺Assume that the sequence (x_n) converges to ℓ. Then, given $\epsilon > 0$, there is a natural number N such that, for all $n \geq N$,

$$|x_n - \ell| < \epsilon \iff \{x_n \mid n \geq N\} \subset (\ell - \epsilon, \ell + \epsilon).$$

It now follows that $\{n \in \mathbb{N} \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\} \subset \{1, 2, \ldots, N - 1\}$, a finite set.

Conversely, let $\epsilon > 0$ be given and assume that the set $\{n \in \mathbb{N} \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\}$ is finite. Let $N = \max\{n \in \mathbb{N} \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\} + 1$. If $n \geq N$, then

$$x_n \in (\ell - \epsilon, \ell + \epsilon) \iff |x_n - \ell| < \epsilon,$$

and so $x_n \to \ell$ as $n \to \infty$. ■

4.1.13 Theorem
Every convergent sequence of real numbers is bounded.

≺Let (s_n) be a sequence of real numbers which converges to s, say. Then, with $\epsilon = 1$, there exists an $N \in \mathbb{N}$ such that

$$|s_n - s| < 1 \text{ for all } n \geq N.$$

By the triangle inequality, we have that

$$|s_n| \leq |s_n - s| + |s| < 1 + |s| \text{ for all } n \geq N.$$

Let $M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s| + 1\}$. Then $|s_n| \leq M$ for all $n \in \mathbb{N}$. That is, the sequence (s_n) is bounded. ■

The converse of Theorem 4.1.13 is not necessarily true. That is, there are sequences which are bounded but do not converge. One such example is the sequence $((-1)^n)$. We shall however see later that every bounded sequence which is *monotone* will always converge.

4.1.14 Exercise

[1] Show that if $s_n \to s$, then $|s_n| \to |s|$. Does the converse hold?

[2] Show that if (s_n) and (t_n) are sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$ and if $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $s \leq t$.

[3] Let (s_n) and (t_n) be sequences such that (s_n) is bounded and $\lim_{n\to\infty} t_n = 0$. Show that $\lim_{n\to\infty} s_n t_n = 0$.

[4] Which of the following sequences are bounded?

(i) $\left(\frac{n}{n+4}\right)$ (ii) $\left(\frac{n^2+n-4}{n+5}\right)$ (iii) $\left(1 - \frac{1}{2^n}\right)$ (iv) $(1 + (-1)^n)$

(v) $(\sqrt[n]{n})$ (vi) $\left(\frac{1+(-1)^n}{n}\right)$ (vii) $\left(\frac{1+2^n}{2^n}\right)$.

4.1.15 Theorem
(Squeeze Theorem). *Suppose that (s_n), (t_n) and (u_n) are sequences such that $s_n \leq t_n \leq u_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} s_n = \ell = \lim_{n\to\infty} u_n$, then $\lim_{n\to\infty} t_n = \ell$.*

≺Let $\epsilon > 0$ be given. Then there exist N_1 and N_2 in $\mathbb{N}$ such that

$$\begin{aligned} |s_n - \ell| &< \epsilon \text{ for all } n \geq N_1 \quad \text{and} \\ |u_n - \ell| &< \epsilon \text{ for all } n \geq N_2. \end{aligned}$$

That is,

$$\begin{aligned} \ell - \epsilon &< s_n < \ell + \epsilon \text{ for all } n \geq N_1 \quad \text{and} \\ \ell - \epsilon &< u_n < \ell + \epsilon \text{ for all } n \geq N_2. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$\ell - \epsilon < s_n \leq t_n \leq u_n < \ell + \epsilon,$$

and consequently,

$$|t_n - \ell| < \epsilon \text{ for all } n \geq N.$$

That is, $\lim_{n\to\infty} t_n = \ell$. ■

4.1.16 Examples

[1] Show that $\lim_{n\to\infty} \frac{\cos\frac{n\pi}{2}}{n^2} = 0$.

Solution: Since

$$0 \le \left|\frac{\cos\frac{n\pi}{2}}{n^2} - 0\right| = \left|\frac{\cos\frac{n\pi}{2}}{n^2}\right| \le 1\cdot\frac{1}{n^2}, \text{ and } \lim_{n\to\infty}\frac{1}{n^2} = 0,$$

it follows that $\lim_{n\to\infty} \frac{\cos\frac{n\pi}{2}}{n^2} = 0$. □

[2] Show that for any x, with $|x| < 1$, $\lim_{n\to\infty} nx^n = 0$.

Solution: Without loss of generality, we assume that $x \neq 0$ and $n > 1$. Since $|x| < 1$, there there is a positive real number a such that

$$\frac{1}{|x|} = 1 + a.$$

Then

$$\frac{1}{|x^n|} = (1+a)^n = \sum_{r=0}^{n}\binom{n}{r}a^r \ge \frac{n(n-1)}{2}a^2$$

for some $a > 0$. Thus,

$$\begin{aligned} |x^n| &\le \frac{2}{n(n-1)a^2} \\ \Rightarrow \quad |nx^n| &\le \frac{2}{(n-1)a^2} \\ \Rightarrow \quad \frac{-2}{(n-1)a^2} &\le nx^n \le \frac{2}{(n-1)a^2}. \end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{-2}{(n-1)a^2} = 0 = \lim_{n\to\infty}\frac{2}{(n-1)a^2},$$

we have, by the Squeeze Theorem, that

$$\lim_{n\to\infty} nx^n = 0.$$

[3] Show that for any $x \in \mathbb{R}$, $\lim_{n\to\infty}\frac{x^n}{n!} = 0$.

Solution: Let N be the first integer greater than $|x|$. Then, if $n \ge N$,

$$\left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!} = \frac{|x|^{N-1}|x|^{n-N+1}}{(N-1)!N(N+1)\cdots n} = \frac{|x|^{N-1}}{(N-1)!}\cdot\frac{|x|^{n-N+1}}{N(N+1)\cdots n}.$$

Let $K = \frac{|x|^{N-1}}{(N-1)!}$. Then K is a constant which is independent of n. Thus,

$$\left|\frac{x^n}{n!}\right| = K\cdot\frac{|x|^{n-N+1}}{N(N+1)\cdots n} < K\cdot\frac{|x|^{n-N+1}}{N\cdot N\cdot N\cdots N} = K\cdot\left(\frac{|x|}{N}\right)^{n-N+1},$$

whence

$$-K\left(\frac{|x|}{N}\right)^{n-N+1} < \frac{x^n}{n!} < K\left(\frac{|x|}{N}\right)^{n-N+1}.$$

Since $\frac{|x|}{N} < 1$ and $n - N + 1 \to \infty$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \left(\frac{|x|}{N}\right)^{n-N+1} = 0,$$

and consequently,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0. \qquad \square$$

4.1.17 Exercise

[1] Show that $\lim_{n\to\infty} \frac{n}{2^n} = 0.$

[2] Show that $\lim_{n\to\infty} \frac{1}{n!} = 0.$

4.1.18 Theorem

Let S be a subset of $\mathbb{R}$ which is bounded above. Then there exists a sequence (s_n) in S such that

$$\lim_{n\to\infty} s_n = \sup S.$$

$\prec$Let $c = \sup S$. By the characterisation of supremum (Theorem 3.1.9), for each $n \in \mathbb{N}$ there exists $s_n \in S$ such that

$$c - \frac{1}{n} < s_n \leq c.$$

Since $\lim_{n\to\infty}\left(c - \frac{1}{n}\right) = c = \lim_{n\to\infty} c$, we have, by the Squeeze Theorem, that

$$\lim_{n\to\infty} s_n = c = \sup S. \qquad \blacksquare$$

4.2 Algebra of Limits

The following lemma asserts that if the sequence (t_n) converges to $t \neq 0$, then the sequence (t_n) is "bounded away from zero".

4.2.1 Lemma

If the sequence (t_n) converges to $t \neq 0$, then there is an $N \in \mathbb{N}$ such that $|t_n| > \frac{|t|}{2}$ for all $n \geq N$.

$\prec$Since $t \neq 0$, $|t| > 0$. Let $\epsilon = \frac{|t|}{2}$. Then there exists an $N \in \mathbb{N}$ such that

$$|t_n - t| < \epsilon \text{ for all } n \geq N.$$

Thus,

$$|t_n| = |t - (t - t_n)| \geq |t| - |t - t_n| > |t| - \frac{|t|}{2} = \frac{|t|}{2} \quad \text{for all } n \geq N.$$

4.2.2 Theorem

Let (s_n) and (t_n) be sequences of real numbers which converge to s and t respectively. Then *(i)* $\lim_{n\to\infty}(s_n + t_n) = s + t.$

(ii) $\lim_{n\to\infty} s_n t_n = st.$

(iii) $\lim_{n\to\infty}\frac{s_n}{t_n} = \frac{s}{t}$ *if* $t_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $t \neq 0$.

$\prec$

(i) Let $\epsilon > 0$ be given. Then there exist N_1 and N_2 in $\mathbb{N}$ such that

$$\begin{aligned} |s_n - s| &< \frac{\epsilon}{2} \text{ for all } n \geq N_1 \quad \text{and} \\ |t_n - t| &< \frac{\epsilon}{2} \text{ for all } n \geq N_2. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$\begin{aligned} |(s_n + t_n) - (s + t)| &= |(s_n - s) + (t_n - t)| \\ &\leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ . \end{aligned}$$

Hence, $\lim_{n\to\infty}(s_n + t_n) = s + t$.

(ii) Let $\epsilon > 0$ be given. Now,

$$\begin{aligned} |s_n t_n - st| = |s_n t_n - s t_n + s t_n - st| &= |(s_n - s)t_n + (t_n - t)s| \\ &\leq |s_n - s||t_n| + |t_n - t||s|\ . \end{aligned}$$

Since (t_n) is convergent, it is bounded. Therefore there is a positive real number K such that

$$|t_n| \leq K \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$|s_n t_n - st| \leq |s_n - s||t_n| + |t_n - t||s| \leq |s_n - s|K + |t_n - t||s|\ .$$

Let $M = \max\{K, |s|\}$. Then

$$|s_n t_n - st| \leq M(|s_n - s| + |t_n - t|).$$

Since $s_n \to s$ and $t_n \to t$ as $n \to \infty$, there exist N_1 and N_2 in $\mathbb{N}$ such that

$$\begin{aligned} |s_n - s| &< \frac{\epsilon}{2(M+1)} \text{ for all } n \geq N_1 \quad \text{and} \\ |t_n - t| &< \frac{\epsilon}{2(M+1)} \text{ for all } n \geq N_2. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$|s_n t_n - st| \leq M(|s_n - s| + |t_n - t|) < (M+1)\left(\frac{\epsilon}{2(M+1)} + \frac{\epsilon}{2(M+1)}\right) = \epsilon\ .$$

Hence, $\lim_{n\to\infty} s_n t_n = st$.

(iii) It suffices to show that $\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{t}$ if $t_n \neq 0$ for all $n \in \mathbb{N}$ and $t \neq 0$. Once this has been shown we can then apply (ii). By Lemma 4.2.1, there is an $N_1 \in \mathbb{N}$ such that

$$|t_n| > \frac{|t|}{2} \text{ for all } n \geq N_1.$$

Again, there exists an $N_2 \in \mathbb{N}$ such that

$$|t_n - t| < \frac{\epsilon|t|^2}{2} \quad \text{for all} \quad n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t t_n}\right| = \frac{|t - t_n|}{|t||t_n|} < \frac{2|t - t_n|}{|t|^2} < \frac{2}{|t|^2} \cdot \frac{\epsilon|t|^2}{2} = \epsilon.$$

Hence, $\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{t}$. ■

4.2.3 Exercise

[1] Let (s_n) and (t_n) be sequences of real numbers. Prove or disprove the following statements.

(i) If (s_n) converges and (t_n) diverges, then the sequence $(s_n + t_n)$ diverges.

(ii) If both (s_n) and (t_n) diverge, then the sequence $(s_n + t_n)$ also diverges.

(iii) If both (s_n) and (t_n) diverge, then the sequence $(s_n t_n)$ also diverges.

(iv) If both (s_n) and (t_n) diverge and $t_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $\left(\frac{s_n}{t_n}\right)$ diverges.

(v) If both (s_n) and $(s_n t_n)$ converge, then the sequence (t_n) converges.

(vi) If both (s_n) and $(s_n t_n)$ diverge, then the sequence (t_n) diverges.

[2] (a) Show that if the sequence (s_n) converges to s, then the sequence $\{s_n^2\}$ converges to s^2.

(b) Use (a) and the fact that for all $x, y \in \mathbb{R}$, $xy = \frac{1}{4}[(x + y)^2 - (x - y)^2]$ to give an alternative proof of Theorem 4.2.2(ii).

4.2.1 Monotone Sequences

4.2.4 Definition

Let (s_n) be a sequence of real numbers. We say that (s_n) is

(a) **increasing** *if for each $n \in \mathbb{N}$, $s_n \leq s_{n+1}$.*

(b) **strictly increasing** *if or each $n \in \mathbb{N}$, $s_n < s_{n+1}$.*

(c) **decreasing** *if or each $n \in \mathbb{N}$, $s_{n+1} \leq s_n$.*

(d) **strictly decreasing** *if or each $n \in \mathbb{N}$, $s_{n+1} < s_n$.*

(e) **monotone** *if (s_n) is increasing or decreasing.*

(f) **strictly monotone** *if (s_n) is strictly increasing or strictly decreasing.*

4.2.5 Remark

An increasing sequence (s_n) is bounded below by s_1; a decreasing sequence (t_n) is bounded above by t_1. It therefore follows that an increasing sequence is bounded if and only if it is bounded above. Similarly, a decreasing sequence is bounded if and only if it is bounded below.

4.2.6 Examples

[1] The sequence $(1, 1, 2, 3, 5, \ldots)$ is increasing.

[2] The sequence $(3, 1, 0, 0, -3, -7, \ldots)$ is decreasing.

[3] The sequence (n^2) is strictly increasing.

[4] The sequence $(-n)$ is strictly decreasing.

The regular behaviour of monotone sequences makes it easier to determine its convergence or divergence.

4.2.7 Theorem

(Monotone Convergence Theorem). *A monotone sequence converges if and only if it is bounded.*

$\prec$We have already proved in Theorem 4.1.13 that if a sequence converges then it is bounded.

To prove the converse, let (s_n) be a bounded increasing sequence and let $S = \{s_n \mid n \in \mathbb{N}\}$. Since S is bounded above, it has a supremum, $\sup S = s$, say. We claim that $\lim_{n\to\infty} s_n = s$. Let $\epsilon > 0$ be given. By the characterisation of supremum (Theorem 3.1.9), there exists $s_N \in S$ such that

$$s - \epsilon < s_N \leq s_n \leq s < s + \epsilon \quad \text{for all} \quad n \geq N.$$

Thus, $|s_n - s| < \epsilon$ for all $n \geq N$.

The proof for the case when the sequence (s_n) is decreasing is similar. ∎

4.2.8 Examples

[1] Show that $\left(\frac{n+1}{n}\right)$ is a convergent sequence.

Solution: We show that the sequence $\left(\frac{n+1}{n}\right)$ is (1) monotone, and (2) bounded. Its convergence will then follow from the Monotone Convergence Theorem (Theorem 4.2.7).

Monotonicity: Let $s_n = \frac{n+1}{n}$. Then

$$\begin{aligned} \frac{s_{n+1}}{s_n} &= \frac{n+2}{n+1} \div \frac{n+1}{n} = \frac{n+2}{n+1} \times \frac{n}{n+1} = \frac{n^2+2n}{(n+1)^2} \\ &< \frac{n^2+2n+1}{(n+1)^2} = \frac{(n+1)^2}{(n+1)^2} = 1. \end{aligned}$$

Thus,

$$s_n = \frac{n+1}{n} > \frac{n+2}{n+1} = s_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the sequence $\left(\frac{n+1}{n}\right)$ is monotone decreasing.

Another proof of monotonicity: Consider $f(x) = \frac{x+1}{x}$ for all $x \in [1, \infty)$. Then,

$$f'(x) = \frac{x - (x+1)}{x^2} = \frac{-1}{x^2} < 0 \quad \text{for all} \quad x \in [1, \infty).$$

Thus, f is decreasing on $[1,\infty)$. Therefore

$$f(n) > f(n+1) \text{ i.e., } \frac{n+1}{n} > \frac{n+2}{n+1} \text{ for all } n \in \mathbb{N}.$$

<u>Boundedness</u>:

$$\frac{n+1}{n} = 1 + \frac{1}{n} > 1 \text{ for all } n \in \mathbb{N}.$$

Thus, the sequence $\left(\frac{n+1}{n}\right)$ is bounded below by 1. □

[2] Show that $\left(\left(1+\frac{1}{n}\right)^n\right)$ is a convergent sequence.

Solution: We need an easy preliminary result:

$$r! \geq 2^{r-1} \text{ for } r \geq 2.$$

This follows from the fact that

$$r! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots r \geq 1\cdot 2\cdot 2\cdot 2\cdot 2\cdots 2 = 2^{r-1} \text{ for } r \geq 2.$$

We establish the existence of the limit by showing that

(a) $\left(1+\frac{1}{n}\right)^n < 3$ for each $n \in \mathbb{N}$ (boundedness),

(b) $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$ for each $n > 2$ (monotonicity).

Now, by the binomial theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{r=0}^{n}\binom{n}{r}\frac{1}{n^r} = \sum_{r=0}^{n}\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\frac{1}{n^r} \\ &= 1+\frac{n}{n}+\sum_{r=2}^{n}\frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right); \qquad (*)\end{aligned}$$

the first two terms correspond to $r=0$ and $r=1$, respectively. Then we obtain (a) from (*) by noting that

$$\left(1+\frac{1}{n}\right)^n < 2+\sum_{r=2}^{n}\frac{1}{r!} < 2+\sum_{r=2}^{n}\frac{1}{2^{r-1}}$$

and so

$$\left(1+\frac{1}{n}\right)^n < 2+\frac{\frac{1}{2}(1-\frac{1}{2^{n-1}})}{1-\frac{1}{2}} < 3.$$

We obtain (b) from (*) by noting that if we replace n by $n+1$, each of the brackets in (*) becomes bigger, so that each of the terms under the $\sum$ sign becomes bigger; and there is also one more positive term in the series. That's all we want! □

(The limit that the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)$ converges to is denoted by the letter e.)

4.2.9 Theorem
(Nested Intervals Theorem). *For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$, where $-\infty < a_n < b_n < \infty$. If $I_{n+1} \subset I_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point.*

≺Since $I_{n+1} \subset I_n$, we have that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Thus, (a_n) is an increasing sequence of real numbers. Let A be the set of all endpoints a_n, i.e., $A = \{a_n \ : \ n \in \mathbb{N}\}$.
Claim: $a_k \leq b_\ell$ for all $k, \ \ell \in \mathbb{N}$. Indeed, if $k \leq \ell$, then $a_k \leq a_\ell \leq b_\ell$. On the other hand, if $\ell \leq k$, then $a_k \leq b_k \leq b_\ell$.

It now follows that A is a nonempty set which is bounded above (by every b_n). By the Completeness Axiom, A has a supremum, $a = \sup A$ (say). Clearly, $a_n \leq a \leq b_n$ for all $n \in \mathbb{N}$. Hence $a \in I_n$ for each $n \in \mathbb{N}$, and consequently, $a \in \bigcap_{n=1}^{\infty} I_n$. We showed in the Monotone Convergence Theorem (Theorem 4.2.7) that $\lim_{n\to\infty} a_n = a$. Assume that $b \in \bigcap_{n=1}^{\infty} I_n$. Then $a_n \leq b \leq b_n$ for all $n \in \mathbb{N}$, and so $0 \leq b - a_n \leq b_n - a_n$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty}(b_n - a_n)$, we have, by the Squeeze Theorem, that $\lim_{n\to\infty}(b - a_n) = 0$, whence $\lim_{n\to\infty} a_n = b$. Thus, $a = b$. ■

4.2.10 Remark
The Nested Intervals Theorem (Theorem 4.2.9) may fail for a decreasing sequence of open or half-open intervals. For example, if $I_n = \left(0, \dfrac{1}{n+1}\right]$ or $I_n = [n, \infty)$ for each $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

4.2.2 Subsequences

4.2.11 Definition
Let (s_n) be a sequence of real numbers and let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\left(s_{n_k}\right)$ is called a **subsequence** *of (s_n). That is, a subsequence $\left(s_{n_k}\right)$ of the sequence (s_n) is a strictly increasing function $\phi : k \ \mapsto \ s_{n_k}$.*

4.2.12 Example
Let (s_n) be the sequence $\left(1, 2, \frac{1}{2}, 3, \frac{1}{3}, \ldots\right)$. Then $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ and $\left(1, 2, \frac{1}{3}, \ldots\right)$ are subsequences of (s_n).

4.2.13 Theorem
Let (s_n) be a sequence which converges to s. Then any subsequence of (s_n) converges to s.

≺Let $\left(s_{n_k}\right)$ be a subsequence of (s_n) and let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ such that

$$|s_n - s| < \epsilon \text{ for each } n \geq N.$$

Thus, when $k \geq N$ we have that $n_k \geq k \geq N$ and so

$$|s_{n_k} - s| < \epsilon \text{ for all } k \geq N.$$

That is, $\lim_{k\to\infty} s_{n_k} = s$. ■

Now we state and prove the version of the Bolzano-Weierstrass Theorem that applies to sequences. Compare this with Theorem 3.3.8 (Bolzano-Weierstrass Theorem for sets).

4.2.14 Theorem
(Bolzano-Weierstrass Theorem for sequences). *Every bounded infinite sequence (s_n) of real numbers has a convergent subsequence.*

$\prec$Since (s_n) is bounded, there exists $M > 0$ such that $|s_n| < M$ for all $n \in \mathbb{N}$, i.e., $-M < s_n < M$ for all $n \in \mathbb{N}$. If (s_n) has a finite range $S = \{s_n : n \in \mathbb{N}\}$, then there is (at least) one term of the sequence (s_n) which occurs infinitely many times in (s_n). Call this term x, and let

$$s_{n_1} = s_{n_2} = s_{n_3} = \cdots = x.$$

Therefore, there exists a strictly increasing sequence $(n_k)_k$ of natural numbers such that $s_{n_k} = x$ for all $k \in \mathbb{N}$. Thus, $\{s_{n_k}\}$ is a subsequence of (s_n) which converges to x.

Now, suppose that S is infinite. Then either $[-M, 0]$ or $[0, M]$ contains s_n for infinitely many $n \in \mathbb{N}$. Call such an interval $I_1 = [a_1, b_1]$. Note that $|I_1| = |b_1 - a_1| = M$. Now, bisect I_1. One of the two subintervals of I_1 contains s_n for infinitely many $n \in \mathbb{N}$. Call that subinterval $I_2 = [a_2, b_2]$. Clearly, $I_1 \supset I_2$ and $|I_2| = |b_2 - a_2| = \frac{M}{2}$. Bisect I_2. One of the two subintervals of I_2 contains s_n for infinitely many $n \in \mathbb{N}$. Call that subinterval $I_3 = [a_3, b_3]$. Clearly, $I_1 \supset I_2 \supset I_3$ and $|I_3| = |b_3 - a_3| = \frac{M}{2^2}$. Continue in this manner to obtain a sequence of intervals $I_1, I_2, I_3, \ldots$ with $I_1 \supset I_2 \supset I_3 \supset \cdots$ and

$$|I_n| = |b_n - a_n| = \frac{M}{2^{n-1}} \to 0 \text{ as } n \to \infty.$$

By the Nested Intervals Theorem (Theorem 4.2.9), we have that $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point, ℓ say. We obtain a convergent subsequence as follows: choose $s_{n_1} \in I_1$. Next, choose $s_{n_2} \in I_2$ with $n_2 > n_1$. Next, choose $s_{n_3} \in I_3$ with $n_3 > n_2$. Continue in this manner. (Such a selection is possible since I_n contains infinitely many terms of the sequence (s_n).) Then (s_{n_k}) is a subsequence of (s_n) with $s_{n_k} \in I_k$ for all $k \in \mathbb{N}$. Since ℓ is also in I_k, we have that

$$|s_{n_k} - \ell| < \frac{M}{2^{k-1}} \to 0 \text{ as } k \to \infty.$$

That is, $\lim_{k\to\infty} s_{n_k} = \ell$. ■

4.2.3 Cauchy Sequences

4.2.15 Definition
A sequence (s_n) is called a **Cauchy sequence** *if, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that*

$$|s_n - s_m| < \epsilon \text{ for all } n, m \geq N.$$

Symbolically,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n, m \in \mathbb{N})[(n \geq N) \wedge (m \geq N) \Rightarrow (|x_n - x_m| < \epsilon)].$$

Equivalently, (s_n) is a Cauchy sequence if $\lim_{n,m\to\infty} |s_n - s_m| = 0$.

4.2.16 Examples
[1] Show that the sequence (s_n), where $s_n = \dfrac{n+1}{n}$, is a Cauchy sequence.

Solution: For all $n, m \in \mathbb{N}$,

$$|s_n - s_m| = \left|\left(\frac{n+1}{n}\right) - \left(\frac{m+1}{m}\right)\right| = \left|\frac{mn + m - nm - n}{nm}\right| = \left|\frac{m-n}{nm}\right| \leq \frac{m+n}{nm}.$$

Therefore, if $m \geq n$, then

$$|s_n - s_m| \leq \frac{m+n}{nm} \leq \frac{2m}{nm} = \frac{2}{n}.$$

Let $\epsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Thus, for all $n \geq N$, we have

$$|s_n - s_m| = \left|\left(\frac{n+1}{n}\right) - \left(\frac{m+1}{m}\right)\right| < \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

Hence (s_n) is a Cauchy sequence. □

[2] Show that the sequence (s_n), where $s_n = 1 - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}$, is a Cauchy sequence.

Solution: For $n, m \in \mathbb{N}$ with $m \geq n$, we have that

$$\begin{aligned}
|s_n - s_m| &= \left|\left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}\right) - \left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{m+1}}{m!}\right)\right| \\
&= \left|\frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \cdots + \frac{(-1)^{m+1}}{m!}\right| \\
&\leq \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} = \frac{1}{2^n}\left[1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right] \\
&= \frac{2}{2^n}\left[1 - \left(\frac{1}{2}\right)^{m-n}\right] < \frac{2}{2^n} = \frac{1}{2^{n-1}}.
\end{aligned}$$

Since $\frac{1}{2^{n-1}} \to 0$ as $n \to \infty$, given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\frac{1}{2^{n-1}} = \left|\frac{1}{2^{n-1}} - 0\right| < \epsilon \text{ for all } \quad n \geq N.$$

Thus,

$$|s_n - s_m| = \left|\left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}\right) - \left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{m+1}}{m!}\right)\right| < \frac{1}{2^{n-1}} < \epsilon$$

for all $m \geq n \geq N$. That is, (s_n) is a Cauchy sequence. □

4.2.17 Theorem
Every convergent sequence (s_n) is a Cauchy sequence.

≺Assume that (s_n) converges to s. Then, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|s_n - s| < \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Now, for all $n, m \geq N$, we have that

$$|s_n - s_m| = |(s_n - s) + (s - s_m)| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

That is, (s_n) is a Cauchy sequence. ■

4.2.18 Theorem
Every Cauchy sequence (s_n) is bounded.

$\prec$Let $\epsilon = 1$. Then there exists an $N \in \mathbb{N}$ such that

$$|s_n - s_m| < 1 \text{ for all } n, m \geq N.$$

Choose a $k \geq N$ and observe that

$$|s_n| = |s_n - s_k + s_k| \leq |s_n - s_k| + |s_k| < 1 + |s_k| \text{ for all } n \geq N.$$

Let $M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s_k| + 1\}$. Then $|s_n| \leq M$ for all $n \in \mathbb{N}$, and therefore (s_n) is bounded. ■

4.2.19 Theorem
Every Cauchy sequence (s_n) of real numbers converges.

$\prec$By Theorem 4.2.18, (s_n) is bounded, and therefore, by the Bolzano-Weierstrass Theorem (Theorem 4.2.14), (s_n) has a subsequence $\{s_{n_k}\}$ which converges to some real number ℓ. We claim that the sequence (s_n) converges to ℓ. Let $\epsilon > 0$ be given. Then there exist natural numbers N_1 and N_2 such that

$$\begin{aligned} |s_n - s_m| &< \frac{\epsilon}{2} \text{ for all } n, m \geq N_1 \text{ and} \\ |s_{n_k} - \ell| &< \frac{\epsilon}{2} \text{ for all } k \geq N_2 . \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$|s_n - \ell| \leq |s_n - s_{n_k}| + |s_{n_k} - \ell| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\lim_{n\to\infty} s_n = \ell$. ■

Combining Theorem 4.2.17 and Theorem 4.2.19, we get:
Cauchy's Convergence Criterion for sequence: *A sequence (s_n) of real numbers converges if and only if it is a Cauchy sequence.*

4.2.20 Examples
[1] Use Cauchy's Criterion to show that the sequence $\left(\frac{(-1)^n}{n}\right)$ converges.

Solution: We must show that the sequence $\left(\frac{(-1)^n}{n}\right)$ is Cauchy. To that end, let $\epsilon > 0$ and $s_n = \frac{(-1)^n}{n}$. Then, for all $n, m \in \mathbb{N}$ with $m \geq n$,

$$|s_n - s_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.$$

Now, there is an $N \in \mathbb{N}$ such that $\frac{2}{N} < \epsilon$. Thus, for all $n \geq N$, we have

$$|s_n - s_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

Thus, $\left(\frac{(-1)^n}{n}\right)$ is a Cauchy sequence. □

[2] Show that the sequence (s_n), where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, diverges.

Solution: It suffices to show that (s_n) is *not* a Cauchy sequence. Now, for $n, m \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned}
|s_n - s_m| &= \left|\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right| \\
&= \left|\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}\right| = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n} \\
&> \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n-m \text{ terms}} = \frac{n-m}{n}.
\end{aligned}$$

In particular, if we take $n = 2m$, we get

$$|s_n - s_m| = \left|\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right| > \frac{n-m}{n} = \frac{1}{2}.$$

Thus, (s_n) is not Cauchy. □

4.2.21 Exercise

[1] Show that if a (x_n) contains two subsequences that converge to different limits, then (x_n) diverges.

[2] Show that every subsequence of a bounded sequence is bounded.

[3] Show that if (x_n) is a Cauchy sequence, then so is $\{|x_n|\}$.

4.2.4 Limit Superior and Limit Inferior

Let (x_n) be a bounded sequence of real numbers and for each $n \in \mathbb{N}$, let

$$E_n = \{x_n, x_{n+1}, \ldots\} = \{x_k \mid k \geq n\}.$$

Then E_n is a bounded subset of $\mathbb{R}$. Set

$$s_n = \inf E_n \text{ and } S_n = \sup E_n.$$

Clearly, $E_j \subset E_i$ for all $i, j \in \mathbb{N}$ such that $i < j$. In particular, $E_{k+1} \subset E_k$ for each $k \in \mathbb{N}$. Therefore

$$s_k \leq s_{k+1} \leq S_{k+1} \leq S_k.$$

These inequalities show that (s_n) is a monotone increasing sequence of real numbers and (S_n) is a monotone decreasing sequence of real numbers. Note further that, for each $k \in \mathbb{N}$,

$$s_1 \leq s_k \leq s_{k+1} \leq S_{k+1} \leq S_k \leq S_1.$$

Therefore, the increasing sequence (s_n) is bounded above by S_1 and the decreasing sequence (S_n) is bounded below by s_1. In fact, for each $n \in \mathbb{N}$, S_n is a upper bound for the sequence (s_n) and for each $n \in \mathbb{N}$, s_n is a lower bound for the sequence (S_n). Therefore, by the Monotone Convergence Theorem

(Theorem 4.2.7), the sequence (s_n) converges to the supremum $s = \sup_n s_n$ and the sequence (S_n) converges to the infimum $S = \inf_n S_n$. That is,

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \inf_{k\geq n} x_k = \sup_n s_n \text{ and } S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sup_{k\geq n} x_k = \inf_n S_n.$$

The number s is called the **limit inferior** of the sequence (x_n) and the number S is called the **limit superior** of the sequence (x_n). We write

$$s = \liminf_{n\to\infty} x_n \text{ and } S = \limsup_{n\to\infty} x_n.$$

We also use the notation $\underline{\lim}_{n\to\infty} x_n$ for the limit inferior and $\overline{\lim}_{n\to\infty} x_n$ for the limit superior.

4.2.22 Definition

Let (x_n) be a bounded sequence of real numbers. The **limit inferior** *of the sequence (x_n), denoted by $\liminf_{n\to\infty} x_n$, is defined by*

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right) = \sup_{n\geq 1} \left(\inf_{k\geq n} x_k \right).$$

Similarly, we define the **limit superior** *of the sequence (x_n), denoted by $\limsup_{n\to\infty} x_n$, as*

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right) = \inf_{n\geq 1} \left(\sup_{k\geq n} x_k \right).$$

4.2.23 Examples

[1] Consider the sequence (x_n), where, for each $n \in \mathbb{N}$, $x_n = (-1)^n$. Then

$$\inf_{k\geq n} x_k = \inf_{k\geq n} (-1)^k = -1 \text{ and } \sup_{k\geq n} x_k = \sup_{k\geq n} (-1)^k = 1.$$

Therefore

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right) = -1 \text{ and}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right) = 1.$$

[2] Consider the sequence (x_n), where, for each $n \in \mathbb{N}$, $x_n = (-1)^n + \frac{1}{n}$. Then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right) = -1 \text{ and}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right) = 1.$$

[3] Consider the sequence (x_n), where, for each $n \in \mathbb{N}$, $x_n = \frac{1}{n}$. Then

$$\begin{aligned} E_1 &= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \\ E_2 &= \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \\ E_3 &= \{\frac{1}{3}, \frac{1}{4}, \ldots\} \\ &\vdots \\ E_n &= \{\frac{1}{n}, \frac{1}{n+1}, \ldots\} \\ &\vdots \end{aligned}$$

Therefore

$$\sup_{k \geq n} x_k = \sup E_n = \frac{1}{n} \text{ and } \inf_{k \geq n} x_k = \inf E_n = 0.$$

It now follows that

$$\begin{aligned} \limsup_{n \to \infty} x_n &= \lim_{n \to \infty} \left(\sup_{k \geq n} x_k \right) = \lim_{n \to \infty} \left(\frac{1}{n} \right) = 0 \text{ and} \\ \liminf_{n \to \infty} x_n &= \lim_{n \to \infty} \left(\inf_{k \geq n} x_k \right) = \lim_{n \to \infty} (0) = 0. \end{aligned}$$

4.2.24 Lemma

Let (x_n) be a bounded sequence of real numbers, $\alpha = \liminf_{n \to \infty} x_n$, and $\beta = \limsup_{n \to \infty} x_n$. Then there is a subsequence of (x_n) which converges to α and a subsequence of (x_n) which converges to β.

$\prec$Let $E_k = \{x_k, x_{k+1}, \ldots\} = \{x_n \mid n \geq k\}$, $s_k = \inf E_k$ and $S_k = \sup E_k$. Then

$$\limsup_{n \to \infty} x_n = \beta = \lim_{k \to \infty} S_k. \tag{4.1}$$

Since S_k is the supremum of the set E_k, given $\epsilon > 0$, there is an index $n_k \geq k$ such that

$$S_k - \frac{\epsilon}{2} < x_{n_k} \leq S_k < S_k + \frac{\epsilon}{2} \iff \left| S_k - x_{n_k} \right| < \frac{\epsilon}{2}. \tag{4.2}$$

Also, since $\lim_{k \to \infty} S_k = \beta$, there is a natural number N such that, for all $k \geq N$,

$$|S_k - \beta| < \frac{\epsilon}{2}. \tag{4.3}$$

From (4.2) and (4.3), we have that for all $n_k \geq k \geq N$,

$$\left| x_{n_k} - \beta \right| \leq \left| x_{n_k} - S_k \right| + |S_k - \beta| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, (x_{n_k}) is a subsequence of (x_n) which converges to β.
The proof of the second part is similar. ■

4.2.25 Theorem
Let (x_n) be a bounded sequence of real numbers. Then

(i) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

(ii) $\limsup_{n\to\infty} x_n = -\liminf_{n\to\infty}(-x_n)$.

(iii) The sequence (x_n) converges if and only $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. *In this case,*

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

$\prec$

(i) For each $n \in \mathbb{N}$, $\inf_{k\geq n} x_k \leq \sup_{k\geq n} x_k$. Therefore

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right) \leq \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right) = \limsup_{n\to\infty} x_n.$$

(ii) For each $n \in \mathbb{N}$, $\inf_{k\geq n}(-x_k) = -\sup_{k\geq n} x_k$. Therefore

$$\liminf_{n\to\infty}(-x_n) = \lim_{n\to\infty}\left(\inf_{k\geq n}(-x_k)\right) = \lim_{n\to\infty}\left(-\sup_{k\geq n} x_k\right) = -\limsup_{n\to\infty} x_n.$$

(iii) Assume that the sequence (x_n) converges to x. By Lemma 4.2.24, there are subsequences $\left(x_{n_k}\right)$ and $\left(y_{n_\ell}\right)$ of (x_n) which converge to $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ respectively. Since every subsequence of a convergent sequence converges to the same limit as the sequence itself, it follows that

$$\liminf_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} = x = \lim_{\ell\to\infty} y_{n_\ell} = \limsup_{n\to\infty} x_n.$$

Conversely, assume that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. Then for each $n \in \mathbb{N}$,

$$s_n = \inf\{x_n,\ x_{n+1},\ \ldots\} \leq x_n \leq S_n = \sup\{x_n,\ x_{n+1},\ \ldots\}.$$

Since

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} S_n,$$

it follows, by the Squeeze Theorem, that the sequence (x_n) converges and $\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. ■

4.2.26 Exercise

[1] For each of the following, find the limit superior, limit inferior and the limit of the sequence.

(a) (x_n), where $x_n = \dfrac{(-1)^n}{n+1}$.

(b) (x_n), where $x_n = (-1)^n + (-1)^{n+2}$.

(c) (x_n), where $x_n = 2(-1)^n + \dfrac{n}{n+1}$.

4.2.5 Sequential Characterization of Closed Sets

4.2.27 Theorem
Let K be a nonempty subset of $\mathbb{R}$ and $x \in \mathbb{R}$. Then

(a) *$x \in \overline{K}$ if and only if there is a sequence $(x_n) \subset K$ such that $x_n \to x$ as $n \to \infty$.*

(b) *K is closed if and only if K contains the limit of every convergent sequence in K.*

$\prec$

(a) Assume that $x \in \overline{K}$. Then either $x \in K$ or $x \in K'$. If $x \in K$, then the constant sequence $(x, x, x, \ldots)$ in K converges to x. If $x \in K'$, then, for each $n \in \mathbb{N}$, the interval $(x - \frac{1}{n}, x + \frac{1}{n})$ contains a point $x_n \in K$ distinct from x. It now follows that $|x_n - x| < \frac{1}{n}$. Clearly, $(x_n) \subset K$ and $x_n \to x$ as $n \to \infty$.

Conversely, assume that there is a sequence $(x_n) \subset K$ such that $x_n \to x$ as $n \to \infty$. Then, either $x \in K$ or every ϵ-neighbourhood of x contains a point $x_n \neq x$, in which case $x \in K'$ Thus $x \in \overline{K}$.

(b) By Corollary 3.2.18, K is closed if and only if $K = \overline{K}$. Hence, (b) follows from (a). ∎

4.2.6 Sequential Compactness

4.2.28 Definition
A subset K of $\mathbb{R}$ is said to be **sequentially compact** *if every sequence in K has a subsequence that converges to a point in K.*

The following theorem asserts that for subsets of $\mathbb{R}$ compactness and sequential compactness are equivalent. In fact, this is true in any "metric space". We shall discuss metric spaces later.

4.2.29 Theorem
A subset K of $\mathbb{R}$ is compact if and only it is sequentially compact.

$\prec$Assume that K is compact and let (x_n) be a sequence in K. Then, by the Heine-Borel Theorem (Theorem 3.3.7), K is closed and bounded. Therefore the sequence (x_n) is bounded. By Bolzano-Weierstrass Theorem (Theorem 4.2.14) (x_n) has a subsequence $\left(x_{n_k}\right)$ which converges to some $x \in \mathbb{R}$. Since K is closed, we have by Theorem 7.3.6, that $x \in K$. Hence, K is sequentially compact.

Conversely, assume that K is *not* compact. Then, by the Heine-Borel Theorem (Theorem 3.3.7), either K is not closed or K is not bounded. If K is not closed, then there is a sequence (x_n) in K that converges to a point outside of K. But then every convergent subsequence of (x_n) will converge to a point outside of K. Therefore K is not sequentially compact. If K is not bounded, then there is a sequence (x_n) in K such that $|x_n| > n$ for each $n \in \mathbb{N}$. Thus, every subsequence of (x_n) is unbounded, and so, by Theorem 4.1.13, no subsequence of (x_n) converges (to a point in K). Hence K is not sequentially compact. ∎

4.2.30 Exercise

[1] Let a sequence (x_n) of real numbers be defined recursively by

$$x_1 = 0, \qquad x_{n+1} = \frac{3x_n + 1}{x_n + 3} \quad \text{for all} \quad n \geq 1.$$

(a) Show, by induction, that $0 \leq x_n \leq 1$ for all $n \in \mathbb{N}$.

(b) Show that the sequence (x_n) is monotonically increasing.

(c) Does the sequence (x_n) converge? If so, find its limit.

(d) Does $\sup\{x_n \mid n \in \mathbb{N}\}$ exist? If so, find it.

Chapter 5

Limits and Continuity

5.1 Limits of Functions

5.1.1 Definition

Suppose that a and ℓ are real numbers and let f be a real-valued function whose domain D includes all points in some open interval about a (except possibly the point a itself). Then ℓ is called the **limit** *of the function f at a if, given any $\epsilon > 0$, there exists a $\delta > 0$ (depending on a and ϵ) such that*

$$|f(x) - \ell| < \epsilon \text{ for all } x \in D \text{ satisfying } 0 < |x - a| < \delta.$$

In this case, we write $\lim_{x \to a} f(x) = \ell$ or $f(x) \to \ell$ as $x \to a$.

Note that the existence of the limit of $f(x)$ as x tends to a does not depend on $f(a)$. Indeed, $f(a)$ may or may not be defined since a is not necessarily in the domain of f. If $f(a)$ and $\lim_{x \to a} f(x)$ both exist, they may or may not be equal. We are only interested in the behaviour of f as x gets *closer to a*. It is implicit in the definition of the limit that a is an accumulation point of the domain D of f.

We can reformulate the above definition in the ϵ-neighbourhood language as follows: $\lim_{x \to a} f(x) = \ell$, if for each ϵ-neighbourhood $N(\ell, \epsilon)$ of ℓ there exists a deleted δ-neighbourhood $N^*(a, \delta)$ of a such that $f(x) \in N(\ell, \epsilon)$ whenever $x \in N^*(a, \delta) \cap D$.

5.1.2 Definition

(1) Suppose that f is defined for all real numbers $x > k$, where $k \in \mathbb{R}$. Then $\ell \in \mathbb{R}$ is the **limit of** *f* **as** *x* **tends to** *∞ if, given $\epsilon > 0$, there exists a real number K such that*

$$|f(x) - \ell| < \epsilon \text{ whenever } x > K.$$

In this case we write $\lim_{x \to \infty} f(x) = \ell$.

(2) Suppose that f is defined for all real numbers $x < k$, where $k \in \mathbb{R}$. Then $\ell \in \mathbb{R}$ is the **limit of** *f* **as** *x* **tends to** *$-\infty$, denoted by $\lim_{x \to -\infty} f(x) = \ell$, if, given $\epsilon > 0$, there exists a real number k such that*

$$|f(x) - \ell| < \epsilon \text{ whenever } x < k.$$

5.1.3 Examples

[1] Show that $\lim_{x \to 2} x^2 = 4$.

Solution: Let $\epsilon > 0$ be given. We need to produce a $\delta > 0$ such that

$$|x^2 - 4| < \epsilon \text{ whenever } 0 < |x - 2| < \delta.$$

Now,

$$|x^2 - 4| = |(x-2)(x+2)| = |x-2||x+2|.$$

Consider all x which satisfy the inequality $|x-2| < 1$. Then, for all such x, we have $1 < x < 3$. Thus,

$$|x+2| \leq |x| + 2 < 3 + 2 = 5,$$

and so

$$|x^2 - 4| = |x-2||x+2| < 5|x-2|.$$

Choose $\delta = \min\left\{1, \frac{\epsilon}{5}\right\}$. Then, whenever $0 < |x-2| < \delta$, we have that

$$|x^2 - 4| < \epsilon\ . \qquad \square$$

[2] Show that $\lim_{x \to 3} (x^2 + 2x) = 15$.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$|(x^2 + 2x) - 15| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x-3| < \delta.$$

Note that

$$|(x^2 + 2x) - 15| = |(x+5)(x-3)| = |x+5||x-3|.$$

Since we are interested in the values of x near 3, we may consider those values of x which satisfy the inequality $|x-3| < 1$, i.e., $2 < x < 4$. For all these values we have that $|x+5| < 9$. Therefore, if $|x-3| < 1$, we have that

$$|(x^2 + 2x) - 15| < 9|x-3|.$$

Choose $\delta = \min\left\{1, \frac{\epsilon}{9}\right\}$. Then, working backwards, we have that

$$|(x^2 + 2x) - 15| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x-3| < \delta. \qquad \square$$

[3] Show that $\lim_{x \to -1} \frac{2x+3}{x+2} = 1$.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left|\frac{2x+3}{x+2} - 1\right| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x - (-1)| = |x+1| < \delta.$$

By elementary algebraic manipulation, we have that

$$\left|\frac{2x+3}{x+2} - 1\right| = \left|\frac{(2x+3)-(x+2)}{x+2}\right| = \left|\frac{x+1}{x+2}\right| = \frac{|x+1|}{|x+2|}\ .$$

Since we are interested in the values of x near -1, we may consider those values of x which satisfy the inequality $|x+1| < \frac{1}{2}$, i.e., $\frac{-3}{2} < x < \frac{-1}{2}$. Recognising $|x+2| = |x-(-2)|$ as the distance of x from -2, we have that

$$|x+2| = |x-(-2)| > \left|\frac{-3}{2} - (-2)\right| = \frac{1}{2}.$$

Therefore

$$\left|\frac{2x+3}{x+2} - 1\right| = \frac{|x+1|}{|x+2|} < 2|x+1|.$$

Choose $\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{2}\right\}$. Then whenever $0 < |x+1| < \delta$, we have that

$$\left|\frac{2x+3}{x+2} - 1\right| < \epsilon \; . \qquad \square$$

[4] Show that $\lim_{x \to 0} f(x)$, where $f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$ does not exist.

Solution: Assume that the limit exists and $\lim_{x \to 0} f(x) = \ell$. Then, with $\epsilon = 1$, there is a $\delta > 0$ such that

$$|f(x) - \ell| < 1 \text{ for all } x \text{ satisfying } 0 < |x| < \delta.$$

Taking $x = \frac{-\delta}{2}$, we have that $|x| = \frac{\delta}{2} < \delta$, and so

$$1 > |f(x) - \ell| = |-1-\ell| = |1+\ell|.$$

Thus, $-2 < \ell < 0$.

On the hand, if $x = \frac{\delta}{2}$, we have that $|x| = \frac{\delta}{2} < \delta$, and so

$$1 > |f(x) - \ell| = |1 - \ell|.$$

Therefore, $0 < \ell < 2$. But there is no real number that can simultaneously satisfy the inequalities $-2 < \ell < 0$ and $0 < \ell < 2$. Therefore $\lim_{x \to 0} f(x)$ does not exist. $\square$

[5] Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left|x \sin \frac{1}{x} - 0\right| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x - 0| < \delta.$$

Now,

$$\left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \le |x|.$$

Choose $0 < \delta \le \epsilon$. Then, whenever $0 < |x - 0| = |x| < \delta$, we have that

$$\left|x \sin \frac{1}{x} - 0\right| \le |x| < \epsilon,$$

which proves that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$. $\square$

[6] Consider the function $f : \mathbb{R} \to \{0, 1\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \; . \end{cases}$$

Show that if $a \in \mathbb{R}$, then $\lim_{x \to a} f(x)$ does not exist.

Solution: Assume that there is an $\ell \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = \ell$. Then, with $\epsilon = \frac{1}{4}$, there exists $\delta > 0$ such that

$$|f(x) - \ell| < \frac{1}{4} \text{ for all } x \text{ satisfying } 0 < |x - a| < \delta.$$

If $x \in \mathbb{Q}$, then

$$|1 - \ell| < \frac{1}{4} \text{ whenever } 0 < |x - a| < \delta, \text{ and}$$

if $x \in \mathbb{R} \setminus \mathbb{Q}$, then

$$|\ell| < \frac{1}{4} \text{ whenever } 0 < |x - a| < \delta.$$

Since the set $\{x \in \mathbb{R} \,:\, 0 < |x - a| < \delta\}$ contains both rationals and irrationals, we have that

$$1 = |1 - 0| = |1 - \ell + \ell| \le |1 - \ell| + |\ell| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

which is absurd. □

The following theorem highlights the relationship between convergence of sequences and limits of functions.

5.1.4 Theorem
Let f be a function which is defined in some open interval I containing $a \in \mathbb{R}$, except possibly at a. Then $\lim_{x\to a} f(x) = \ell$ if and only if for every sequence $(a_n) \subset I \setminus \{a\}$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = \ell$.

$\prec$Assume that $\lim_{x\to a} f(x) = \ell$ and let $(a_n) \subset I \setminus \{a\}$ be a sequence such that $\lim_{n\to\infty} a_n = a$. Then, given $\epsilon > 0$, there is a $\delta > 0$ and an $N \in \mathbb{N}$ such that

$$|f(x) - \ell| < \epsilon \text{ for all } x \in I \text{ satisfying } 0 < |x - a| < \delta \text{ and } |a_n - a| < \delta \text{ for all } n \ge N.$$

Now, $0 < |a_n - a| < \delta$ since $a_n \ne a$ for all $n \ge N$. Therefore

$$|f(a_n) - \ell| < \epsilon \text{ for all } n \ge N.$$

That is, $\lim_{n\to\infty} f(a_n) = \ell$.

For the converse, assume that for every sequence $(a_n) \subset I \setminus \{a\}$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = \ell$.

Claim: $\lim_{x\to a} f(x) = \ell$. If the claim were false, then there would exist an $\epsilon_0 > 0$ such that for every $\delta > 0$ with $0 < |x - a| < \delta$, we have

$$|f(x) - \ell| \ge \epsilon_0.$$

Let $n \in \mathbb{N}$ and take $\delta = \frac{1}{n}$. Then we can find $a_n \in I \setminus \{a\}$ such that $0 < |a_n - a| < \frac{1}{n}$ and

$$|f(a_n) - \ell| \ge \epsilon_0.$$

Clearly, (a_n) is a sequence in $I \setminus \{a\}$ with the property that $\lim_{n\to\infty} a_n = a$ and

$$|f(a_n) - \ell| \ge \epsilon_0 \text{ for all } n \in \mathbb{N}.$$

That is, $\lim_{n\to\infty} f(a_n) \ne \ell$, a contradiction. ■

The condition that $a_n \neq a$ for all $n \in \mathbb{N}$ in Theorem 5.1.4 is essential. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0. \end{cases}$$

Let (a_n) be the sequence where $a_n = 0$ for all $n \in \mathbb{N}$. Then $a_n \in \mathbb{R}(=$ domain of $f)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = 0$. Since $f(a_n) = f(0) = \dfrac{1}{2}$ for all $n \in \mathbb{N}$, and $\lim_{x\to 0} f(x) = 1$, it follows that

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 1 = \lim_{x\to 0} f(x).$$

5.1.5 Theorem
(Uniqueness of Limits). *Let f be a function which is defined on some open interval I containing a, except possibly at a. If $\lim_{x\to a} f(x) = \ell_1$ and $\lim_{x\to a} f(x) = \ell_2$, then $\ell_1 = \ell_2$.*

$\prec$If $\ell_1 \neq \ell_2$, let $\epsilon = \dfrac{|\ell_1 - \ell_2|}{3}$. Then, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} |f(x) - \ell_1| &< \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_1, \text{ and} \\ |f(x) - \ell_2| &< \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x-a| < \delta$, we have

$$0 < |\ell_1 - \ell_2| \leq |f(x) - \ell_1| + |f(x) - \ell_2| < \frac{|\ell_1 - \ell_2|}{3},$$

which is impossible. ■

5.1.1 Algebra of Limits

5.1.6 Theorem
Let $\ell_1, \ell_2, a \in \mathbb{R}$. Suppose that f and g are real-valued functions defined on some open interval I containing a, except possibly at a itself, and that $\lim_{x\to a} f(x) = \ell_1$ and $\lim_{x\to a} g(x) = \ell_2$. Then,

(1) $\lim_{x\to a}[f(x) \pm g(x)] = \ell_1 \pm \ell_2$.

(2) $\lim_{x\to a}[f(x)g(x)] = \ell_1 \ell_2$.

(3) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell_1}{\ell_2}$ *provided* $g(x) \neq 0$ *for all* $x \in I$ *and* $\ell_2 \neq 0$.

$\prec$

(1) Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} |f(x) - \ell_1| &< \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_1, \text{ and} \\ |g(x) - \ell_2| &< \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $x \in I$ and $0 < |x - a| < \delta$, we have

$$|\,[f(x)+g(x)]-[\ell_1+\ell_2]\,| = |\,[f(x)-\ell_1]+[g(x)-\ell_2]\,| \le |f(x)-\ell_1|+|g(x)-\ell_2| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

That is, $f(x) + g(x) \to \ell_1 + \ell_2$ as $x \to a$.

A similar argument shows that $f(x) - g(x) \to \ell_1 - \ell_2$ as $x \to a$.

(2) With $\epsilon = 1$, there exists a $\delta_1 > 0$ such that

$$|f(x) - \ell_1| < 1 \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1.$$

This implies that

$$|f(x)| \le |f(x) - \ell_1| + |\ell_1| < 1 + |\ell_1| \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1.$$

Now, for all $x \in I$ with $0 < |x - a| < \delta_1$, we have

$$\begin{aligned} |f(x)g(x) - \ell_1\ell_2| &= |f(x)g(x) - f(x)\ell_2 + f(x)\ell_2 - \ell_1\ell_2| \\ &\le |f(x)||g(x) - \ell_2| + |\ell_2||f(x) - \ell_1| \\ &< (1 + |\ell_1|)|g(x) - \ell_2| + |\ell_2||f(x) - \ell_1|\,. \end{aligned}$$

Given $\epsilon > 0$, there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$\begin{aligned} |f(x) - \ell_1| &< \frac{\epsilon}{2(1 + |\ell_2|)} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_2, \text{ and} \\ |g(x) - \ell_2| &< \frac{\epsilon}{2(1 + |\ell_1|)} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_3. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, whenever $x \in I$ and $0 < |x - a| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - \ell_1\ell_2| &< (1 + |\ell_1|)\left[\frac{\epsilon}{2(1 + |\ell_1|)}\right] + |\ell_2|\left[\frac{\epsilon}{2(1 + |\ell_2|)}\right] \\ &= \frac{\epsilon}{2} + \frac{\epsilon|\ell_2|}{2(1 + |\ell_2|)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

That is, $f(x)g(x) \to \ell_1\ell_2$ as $x \to a$.

(3) It is enough to show that $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{\ell_2}$ provided $g(x) \neq 0$ for all $x \in I$ and $\ell_2 \neq 0$. Since $\ell_2 \neq 0$, $\epsilon = \frac{|\ell_2|}{2} > 0$. Therefore there exists a $\delta_1 > 0$ such that

$$|g(x) - \ell_2| < \frac{|\ell_2|}{2} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1.$$

Now, for all $x \in I$ satisfying $0 < |x - a| < \delta_1$, we have

$$|\ell_2| \le |\ell_2 - g(x)| + |g(x)| < \frac{|\ell_2|}{2} + |g(x)|.$$

That is, $\frac{|\ell_2|}{2} < |g(x)|$ for all $x \in I$ satisfying $0 < |x - a| < \delta_1$. It now follows that for all $x \in I$ satisfying $0 < |x - a| < \delta_1$,

$$\left|\frac{1}{g(x)} - \frac{1}{\ell_2}\right| = \left|\frac{\ell_2 - g(x)}{g(x)\ell_2}\right| = \frac{|\ell_2 - g(x)|}{|g(x)\ell_2|} < \frac{2|\ell_2 - g(x)|}{|\ell_2||\ell_2|} = \frac{2|\ell_2 - g(x)|}{\ell_2^2}.$$

Given $\epsilon > 0$ there exist $\delta_2 > 0$ such that

$$|g(x) - \ell_2| < \frac{\epsilon \ell_2^2}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $x \in I$ and $0 < |x-a| < \delta$, we have

$$\left|\frac{1}{g(x)} - \frac{1}{\ell_2}\right| < \frac{2|\ell_2 - g(x)|}{\ell_2^2} < \frac{2}{\ell_2^2} \cdot \frac{\epsilon \ell_2^2}{2} = \epsilon.$$

That is, $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{\ell_2}$ provided $\ell_2 \neq 0$. ■

5.1.7 Theorem

Let $\ell_1, \ell_2, a \in \mathbb{R}$. Suppose that f and g are real-valued functions defined on some open interval I containing a, except possibly at a itself, and that $f(x) \leq g(x)$ for all $x \in I$. If $\lim_{x \to a} f(x) = \ell_1$ and $\lim_{x \to a} g(x) = \ell_2$, then $\ell_1 \leq \ell_2$.

≺If $\ell_2 < \ell_1$, let $\epsilon = \frac{\ell_1 - \ell_2}{2}$. Now, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} |f(x) - \ell_1| &< \epsilon \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_1, \text{ and} \\ |g(x) - \ell_2| &< \epsilon \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2. \end{aligned}$$

That is,

$$\begin{aligned} \ell_1 - \epsilon < f(x) &< \ell_1 + \epsilon \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_1, \text{ and} \\ \ell_2 - \epsilon < g(x) &< \ell_2 + \epsilon \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2. \end{aligned}$$

That is,

$$\begin{aligned} \frac{\ell_1 + \ell_2}{2} &< f(x) \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_1, \text{ and} \\ g(x) &< \frac{\ell_1 + \ell_2}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $x \in I$ and $0 < |x-a| < \delta$, we have

$$g(x) < \frac{\ell_1 + \ell_2}{2} < f(x),$$

and so $g(x) < f(x)$, a contradiction. ■

5.1.8 Theorem

(Squeeze Theorem). *Suppose that f, g and h are real-valued functions defined on some open interval I containing a, except possibly at a itself, and that $f(x) \leq g(x) \leq h(x)$ for all $x \in I$. If $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} h(x) = \ell$, then $\lim_{x \to a} g(x) = \ell$.*

≺Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} |f(x) - \ell| &< \epsilon \text{ whenever } 0 < |x-a| < \delta_1, \text{ and} \\ |h(x) - \ell| &< \epsilon \text{ whenever } 0 < |x-a| < \delta_2. \end{aligned}$$

That is,

$$\begin{aligned} \ell - \epsilon < f(x) &< \ell + \epsilon \text{ whenever } 0 < |x-a| < \delta_1, \text{ and} \\ \ell - \epsilon < h(x) &< \ell + \epsilon \text{ whenever } 0 < |x-a| < \delta_2. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x-a| < \delta$, we have

$$\ell - \epsilon < f(x) \le g(x) \le h(x) < \ell + \epsilon.$$

Thus,

$$|g(x) - \ell| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x-a| < \delta.$$

That is, $\lim_{x \to a} g(x) = \ell$. ■

5.1.9 Exercise

[1] Show that $f(x) \to 0$ as $x \to a$ if and only if $|f(x)| \to 0$ as $x \to a$.

[2] Let $a, \ell \in \mathbb{R},\ D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Show that if $f(x) \to \ell$ as $x \to a$, then $|f(x)| \to |\ell|$ as $x \to a$. Does the converse hold? Justify your answer.

[3] Let $a, \ell \in \mathbb{R},\ D \subset \mathbb{R},\ f : D \to \mathbb{R}$ and $\ell > 0$. Show that if $\lim_{x \to a} f(x) = \ell$, then there exists a deleted ϵ-neighbourhood $N^*(a, \epsilon)$ of a such that $f(x) > 0$ for all $x \in N^*(a, \epsilon) \cap D$.

[4] Let $a, \ell \in \mathbb{R},\ D \subset \mathbb{R},\ f : D \to \mathbb{R}$ and $\lim_{x \to a} f(x) = \ell$. Show that there exists a deleted ϵ-neighbourhood $N^*(a, \epsilon)$ and a positive real number M such that $|f(x)| \le M$ for all $x \in N^*(a, \epsilon) \cap D$.

5.2 Continuous Functions

When discussing the limit $\lim_{x \to a} f(x)$, we made no reference to $f(a)$, the value of the function f at a. In fact, we emphasized that $f(a)$ was unimportant in the analysis of $\lim_{x \to a} f(x)$. In this section we want to bring $f(a)$ into the picture; we want to relate the limit $\lim_{x \to a} f(x)$ to the value of f at a.

5.2.1 Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is said to be **continuous at** $a \in D$ *if, given any $\epsilon > 0$, there exists a $\delta > 0$ (which generally depends on ϵ and a) such that*

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in D \text{ and } |x - a| < \delta.$$

The function f is **continuous on** *D if it is continuous at each point of D. If f is not continuous at a, we say that f is* **discontinuous** *there.*

Let us reformulate this definition in the language of neighbourhoods:

5.2.2 Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is said to be **continuous at** $a \in D$ *if, for each ϵ-neighbourhood $N(f(a), \epsilon)$ of $f(a)$, there is a δ-neighbourhood $N(a, \delta)$ of a such that*

$$f(x) \in N(f(a), \epsilon) \text{ whenever } x \in N(a, \delta) \cap D.$$

Note that, in contrast to the *deleted* δ-neighbourhoods used in the definition of $\lim_{x\to a} f(x) = \ell$, for continuity we use δ-neighbourhoods.

5.2.3 Examples

[1] Show that the function $f(x) = x^2$ is continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$ be given and $a \in \mathbb{R}$. We need to produce a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta.$$

Now,

$$|f(x) - f(a)| = |x^2 - a^2| = |(x - a)(x + a)|.$$

Since we are interested in the behaviour of f near a, we may restrict our attention to those real numbers x that satisfy the inequality $|x - a| < 1$. These real numbers satisfy the inequalities $a - 1 < x < a + 1$. Therefore, for all these real numbers, we have

$$|x + a| \le |x| + |a| < |a + 1| + |a| \le 1 + 2|a|.$$

Now, take $\delta = \min\left\{1, \dfrac{\epsilon}{1 + 2|a|}\right\}$. Then, $|x - a| < \delta$ implies that

$$|f(x) - f(a)| = |x^2 - a^2| < \epsilon.$$

That is, f is continuous at a. Since a was arbitrarily chosen in $\mathbb{R}$, it follows that f is continuous on $\mathbb{R}$. □

[2] Show that the function $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ is continuous at 0.

Solution: Let $\epsilon > 0$ be given. We need to produce a $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon \text{ whenever } |x - 0| < \delta.$$

Now

$$|f(x) - f(0)| = \left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \le |x|.$$

Choose $0 < \delta \le \epsilon$. Then, $|x - 0| < \delta$ implies that

$$|f(x) - f(0)| = \left|x \sin \frac{1}{x} - 0\right| \le |x| < \delta \le \epsilon.$$

That is, f is continuous at 0. □

[3] Show that the function $f : \mathbb{R} \to \{-1, 1\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is discontinuous at every real number.

Solution: Assume that f is continuous at some $a \in \mathbb{R}$. Then, given $\epsilon = 1$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta.$$

Since rationals and irrationals are dense in $\mathbb{R}$, the interval $|x-a| < \delta$ contains both rationals and irrationals. If $x \in \mathbb{Q}$ and $|x - a| < \delta$, then

$$|1 - f(a)| < 1, \text{ whence } 0 < f(a) < 2.$$

On the other hand, if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $|x - a| < \delta$, then

$$|-1 - f(a)| < 1, \text{ whence } -2 < f(a) < 0.$$

But there is no real number that can simultaneously satisfy both the inequalities
$0 < f(a) < 2$ and $-2 < f(a) < 0$. Therefore f is discontinuous at every $a \in \mathbb{R}$. □

[4] Show that the function $f(x) = \dfrac{1}{x}$ is continuous at 1.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$|f(x) - f(1)| < \epsilon \text{ whenever } |x - 1| < \delta.$$

Since we are interested in the values of x near 1, we may consider those x for which $|x-1| < \dfrac{1}{2}$. These x satisfy the inequalities

$$\frac{1}{2} < x < \frac{3}{2}.$$

Now, for all the x which satisfy $|x - 1| < \dfrac{1}{2}$, we have

$$|f(x) - f(1)| = \left|\frac{1}{x} - 1\right| = \frac{|x-1|}{x} < 2|x-1|.$$

Choose $\delta = \min\left\{\dfrac{1}{2}, \dfrac{\epsilon}{2}\right\}$. Then, whenever $|x - 1| < \delta$, we have that

$$\left|\frac{1}{x} - 1\right| < \epsilon.$$

That is, f is continuous at $x = 1$. □

[5] Show that if a is an isolated point in the domain D of f, then f is continuous at a.

Solution: Since $a \in D$, f is defined at a. Let $\epsilon > 0$ be given. Since a is an isolated point of D, there is a δ-neighbourhood $N(a, \delta)$ of a such that $N(a, \delta) \cap D = \{a\}$. Assume that $x \in D$ and $|x - a| < \delta$; i.e., $x \in N(a, \delta)$. Then $x = a$ since $N(a, \delta) \cap D = \{a\}$. Hence

$$|f(x) - f(a)| = |f(a) - f(a)| = 0 < \epsilon.$$

Therefore f is continuous at a. □

[6] We can deduce from Example 5 that if $f : \mathbb{Z} \to \mathbb{R}$, then f is continuous at every point of $\mathbb{Z}$. □

The following theorem gives a criterion for continuity at a point in terms of sequences. On some occasions it is easier to apply this formulation than the $\epsilon - \delta$ definition. In particular, this formulation comes handy when one wants to use contradiction to prove discontinuity of a function.

5.2.4 Theorem
Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous at $a \in D$ if and only if for every sequence $(a_n) \subset D$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = f(a)$.

$\prec$Suppose that f is continuous at $a \in D$ and that (a_n) is a sequence in D such that $\lim_{n\to\infty} a_n = a$. Given $\epsilon > 0$, there is a $\delta > 0$ and an $N \in \mathbb{N}$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta \text{ and } |a_n - a| < \delta \text{ for all } n \geq N.$$

Therefore
$$|f(a_n) - f(a)| < \epsilon \text{ for all } n \geq N.$$

That is, $\lim_{n\to\infty} f(a_n) = f(a)$.

For the converse, assume that for every sequence $(a_n) \subset D$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = f(a)$ and that f is *not* continuous at a. Then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ with $0 < |x - a| < \delta$, we have
$$|f(x) - f(a)| \geq \epsilon_0.$$

For $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$. Then we can find $a_n \in D$ such that $0 < |a_n - a| < \frac{1}{n}$ and

$$|f(a_n) - f(a)| \geq \epsilon_0.$$

Clearly, (a_n) is a sequence in D with the property that $\lim_{n\to\infty} a_n = a$ and

$$|f(a_n) - f(a)| \geq \epsilon_0 \text{ for all } n \in \mathbb{N}.$$

That is, $\lim_{n\to\infty} f(a_n) \neq f(a)$, a contradiction. ■

5.2.5 Examples

[1] Find the limit of the sequence $\left\{\ell\text{n}\left(\frac{n+1}{n}\right)\right\}$, if it exists.

Solution: Since
$$\lim_{n\to\infty} \frac{n+1}{n} = 1$$
and the function $f(x) = \ell\text{n } x$ is continuous on $(0, \infty)$, it follows from Theorem 5.2.4 that

$$\lim_{n\to\infty} \ell\text{n}\left(\frac{n+1}{n}\right) = \ell\text{n } 1 = 0.$$

That is, the sequence $\left\{\ell\text{n}\left(\frac{n+1}{n}\right)\right\}$ converges to 0. □

[2] Show that the function $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is continuous only at $x = 0$.

Solution: Let us first show that f is continuous at $x = 0$. Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon \text{ whenever } |x - 0| < \delta.$$

Now, since $0 \in \mathbb{Q}$, we have that

$$|f(x) - f(0)| = |f(x) - 0| = |f(x)|.$$

If $x \in \mathbb{Q}$, then $|f(x)| = |x|$, and if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $|f(x)| = |-x| = |x|$. Choose $\delta = \epsilon$. Then whenever $|x| < \delta$, we have that $|f(x) - f(0)| < \epsilon$, i.e., f is continuous at $x = 0$.

Next, we show that f is discontinuous on $a \in \mathbb{R} \setminus \{0\}$. Assume that f is continuous at some $a \in \mathbb{R} \setminus \{0\}$. If $a \in \mathbb{Q}$, then for each $n \in \mathbb{N}$ there is an $a_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$|a_n - a| < \frac{1}{n}.$$

That is, the sequence (a_n) converges to a. Since $a_n \in \mathbb{R} \setminus \mathbb{Q}$ for each $n \in \mathbb{N}$, $f(a_n) = -a_n$, and since $a \in \mathbb{Q}$, $f(a) = a$. Therefore

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} (-a_n) = -\lim_{n \to \infty} a_n = -a \neq a = f(a).$$

Similarly, if $a \in \mathbb{R} \setminus \mathbb{Q}$, then for each $n \in \mathbb{N}$ there is an $a_n \in \mathbb{Q}$ such that

$$|a_n - a| < \frac{1}{n}.$$

Again, the sequence (a_n) converges to a. Since $a_n \in \mathbb{Q}$ for each $n \in \mathbb{N}$, $f(a_n) = a_n$, and since $a \in \mathbb{R} \setminus \mathbb{Q}$, $f(a) = -a$. Therefore

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = a \neq -a = f(a).$$

Thus f is discontinuous at a. □

5.2.6 Exercise

[1] Show that the function $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is continuous only at $x = 0$.

[2] Show that the function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is discontinuous at every point of $\mathbb{R}$.

The following theorem asserts that continuity is preserved by the standard algebraic operations on functions.

5.2.7 Theorem

Let f and g be functions with common domain $D \subset \mathbb{R}$, and let $a \in D$. If f and g are continuous at a, then so are the functions

(i) $f \pm g$,

(ii) cf for each $c \in \mathbb{R}$,

(iii) $|f|$,

(iv) fg,

(v) $\dfrac{f}{g}$, *provided* $g(a) \neq 0$.

≺Exercise. ■

5.2.8 Theorem
Let f be a function which is continuous at $a \in \mathbb{R}$. Suppose that g is a function which is continuous at the point $f(a)$. Then the composite function $g \circ f$ is continuous at a.

≺Let $\epsilon > 0$ be given. Then there exist $\eta > 0$ and $\delta > 0$ such that

$$\begin{aligned} |g(y) - g(f(a))| &< \epsilon \text{ whenever } |y - f(a)| < \eta, \text{ and} \\ |f(x) - f(a)| &< \eta \text{ whenever } |x - a| < \delta. \end{aligned}$$

(Now δ depends on η, which in turn, depends on ϵ. Therefore δ depends on ϵ.) Hence, for all $x \in \mathbb{R}$ with $|x - a| < \delta$, we have that

$$|g(f(x)) - g(f(a))| < \epsilon.$$

That is, $g \circ f$ is continuous at a. ■

The next theorem, called the Intermediate Value Theorem, asserts that if the domain of a continuous function is an interval, then so is its range.

5.2.9 Theorem
(Intermediate Value Theorem). *If f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for each number k between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = k$.*

≺For definiteness, assume that $f(a) < f(b)$. Let $S = \{x \in [a, b] \mid f(x) \leq k\}$. Then $S \neq \emptyset$ since $a \in S$. Thus, $c = \sup S$ exists as a real number in $[a, b]$. By Theorem 4.1.18, there exists a sequence (x_n) in S such that $\lim\limits_{n\to\infty} x_n = c$. Since $a \leq x_n \leq b$ for each $n \in \mathbb{N}$, we have that $a \leq c \leq b$, and so f is continuous at c. This then implies that $\lim\limits_{n \to \infty} f(x_n) = f(c)$. As $f(x_n) \leq k$ for each $n \in \mathbb{N}$, we deduce that $f(c) \leq k$, and so $c \in S$. It now remains to show that $f(c) \geq k$. To this end, we first observe that since $c \in S$ and $c = \sup S$, $c + \frac{1}{n} \notin S$ for each $n \in \mathbb{N}$. Also, since $k < f(b)$, we have that $c < b$. Therefore, by Corollary 3.1.17, there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < b - c$. Hence, for each $n \geq N$, we have that

$$\frac{1}{n} < b - c, \text{ i.e., } c + \frac{1}{n} < b.$$

This implies that for all $n \geq N$, $c + \frac{1}{n} \in [a, b]$ and $c + \frac{1}{n} \notin S$. Thus, $f(c + \dfrac{1}{n}) > k$ for all $n \geq N$. By continuity of f, we have that $f(c) \geq k$, whence $f(c) = k$. ■

One of the many interesting consequences of the Intermediate Value Theorem is the following *fixed-point theorem.*

5.2.10 Theorem
(Fixed-point Theorem). *If f is continuous on a closed interval $[a, b]$ and $f(x) \in [a, b]$ for each $x \in [a, b]$, then f has a fixed point; i.e., there exists a point $c \in [a, b]$ such that $f(c) = c$.*

≺If $f(a) = a$ or $f(b) = b$, then we are done. We therefore assume that $a < f(a)$ and $f(b) < b$. Let $g(x) = f(x) - x$ for every $x \in [a, b]$. Clearly, g is a continuous function on $[a, b]$, $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. That is, 0 is an intermediate value for g on $[a, b]$. Hence, by the Intermediate Value Theorem (Theorem 5.2.9), there exists a $c \in [a, b]$ such that $g(c) = 0$. This, of course, implies that $f(c) = c$. ■

5.2.1 Uniform Continuity

Before giving the formal definition of uniform continuity we need to look closely at the definition of continuity given earlier. We said that *a function f, with domain $D \subset \mathbb{R}$, is continuous at $a \in D$ if, given any $\epsilon > 0$ there exists a $\delta > 0$ (which depends on ϵ and a) such that*

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in D \text{ and } |x - a| < \delta.$$

For continuity at another point $b \in D$, for the same ϵ, a $\delta' > 0$ would exist such that

$$|f(x) - f(b)| < \epsilon \text{ whenever } x \in D \text{ and } |x - b| < \delta'.$$

The δ and δ' may not be the same. Therefore, δ *depends* on ϵ as well as the point a. For this reason, continuity is a local concept – it describes what happens to a function in a neighbourhood of a point.

We now define uniform continuity.

5.2.11 Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is said to be **uniformly continuous on** *D if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in D \text{ and } |x - y| < \delta.$$

The most important point to note here is that δ does not depend on any particular point of the domain D – the same δ works for all points of D. Therefore uniform continuity is a global concept.

5.2.12 Examples

[1] Show that the function $f(x) = x$ is uniformly continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$ be given. We must produce a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in \mathbb{R} \text{ and } |x - y| < \delta.$$

Since $|f(x) - f(y)| = |x - y|$, we may choose $0 < \delta \le \epsilon$. Then, for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have that

$$|f(x) - f(y)| = |x - y| < \delta \le \epsilon.$$

That is, f is uniformly continuous on $\mathbb{R}$. □

[2] Show that the function $f(x) = x^2$ is *not* uniformly continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$ be given. We must show that for every $\delta > 0$ there exist $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and

$$|f(x) - f(y)| = |x^2 - y^2| \ge \epsilon.$$

Choose $x, y \in \mathbb{R}$ with $x - y = \dfrac{\delta}{2}$ and $x + y = \dfrac{2\epsilon}{\delta}$. Then $|x - y| < \delta$ and

$$|x^2 - y^2| = |x + y||x - y| \ge \frac{2\epsilon}{\delta} \cdot \frac{\delta}{2} = \epsilon.$$

Thus, f is not uniformly continuous on $\mathbb{R}$. □

[3] Show that the function $f(x) = x^2$ is uniformly continuous on $[-1, 1]$.

Solution: Let $\epsilon > 0$ be given. Then for all $x, y \in [-1, 1]$ we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 2|x - y|.$$

Choose $\delta = \frac{\epsilon}{2}$. Then, for all $x, y \in [-1, 1]$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 2|x - y| < 2\frac{\epsilon}{2} = \epsilon.$$

Hence f is uniformly continuous on $[-1, 1]$. □

[4] Show that the function $f(x) = \frac{1}{x}$ is *not* uniformly continuous on $(0, 1]$.

Solution: Let $\epsilon = \frac{1}{2}$ and $\delta > 0$. Then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Take $x = \frac{1}{n}$, and $y = \frac{1}{n+1}$. Then

$$|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta \quad \text{and} \quad |f(x) - f(y)| = |n - (n+1)| = 1 > \frac{1}{2}.$$

Hence f is not uniformly continuous on $(0, 1]$. □

[5] Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$, where $a > 0$.

Solution: Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in [a, \infty) \text{ and } |x - y| < \delta.$$

Now, for all $x, y \in [a, \infty)$,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \frac{|x - y|}{xy} \leq \frac{|x - y|}{a^2}.$$

Take $\delta = a^2\epsilon$. Then, for all $x, y \in [a, \infty)$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| \leq \frac{|x - y|}{a^2} < \epsilon.$$

That is, f is uniformly continuous on $[a, \infty)$. □

5.2.13 Theorem

If $f : D \to \mathbb{R}$ is uniformly continuous on D, then it is continuous there.

≺Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in D \text{ and } |x - y| < \delta.$$

Let $y = a \in D$. Then

$$|f(x) - f(a)| < \epsilon \text{ for all } x, y \in D \text{ such that } |x - y| < \delta.$$

Thus, f is continuous at $a \in D$. Since $a \in D$ was arbitrarily chosen, f is continuous on D. ■

5.2.14 Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a **Lipschitz condition** *on an interval $I \subset \mathbb{R}$ if there is a positive real number M such that*

$$|f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in I.$$

If $M < 1$, then f is called a **contraction map**.

5.2.15 Examples

[1] Show that the function $f(x) = x^2$ satisfies a Lipschitz condition on $[0, 2]$.

Solution: For all $x, y \in [0, 2]$, we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 4|x - y|. \qquad \square$$

[2] Show that the function $f(x) = |x|$ satisfies a Lipschitz condition on $\mathbb{R}$.

Solution: For all $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y|. \qquad \square$$

[3] Show that the function $f(x) = \sin x$ satisfies a Lipschitz condition on $\mathbb{R}$.

Solution: Let $x, y \in \mathbb{R}$. Then

$$\begin{aligned} |f(x) - f(y)| = |\sin x - \sin y| &= \left|\sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right)\right| \\ &= \left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-y}{2}\right)\right| \\ &\leq |x - y|, \end{aligned}$$

where we have used the two facts:

$$\left|\cos\left(\frac{x+y}{2}\right)\right| \leq 1 \quad \text{and} \quad \left|\sin\left(\frac{x-y}{2}\right)\right| \leq \frac{|x-y|}{2}. \qquad \square$$

5.2.16 Theorem

If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition on an interval $I \subset \mathbb{R}$, then f is uniformly continuous there.

$\prec$Since f satisfies a Lipschitz condition on I, there exists a positive real number M such that

$$|f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in I.$$

Let $\epsilon > 0$ be given and take $\delta = \frac{\epsilon}{M}$. Then, whenever $x, y \in I$ and $|x - y| < \delta$, we have that

$$|f(x) - f(y)| \leq M|x - y| < M\frac{\epsilon}{M} = \epsilon.$$

That is, f is uniformly continuous on I. $\blacksquare$

5.2.17 Theorem
If f is contraction map on a closed interval $[a,b]$ such that $f(x) \in [a,b]$ for each $x \in [a,b]$, then f has a unique fixed point; i.e., there exists exactly one point $c \in [a,b]$ such that $f(c) = c$.

$\prec$Since f is a contraction map, it is (uniformly) continuous on $[a,b]$. Furthermore, f satisfies the hypotheses of the fixed-point theorem. Therefore there exists a point $c \in [a,b]$ such that $f(c) = c$.

To prove uniqueness, assume that there is a $d \in [a,b]$ such that $f(d) = d$. Since f is a contraction map, there exists an $M \in \mathbb{R}$ such that $0 < M < 1$ and

$$|c-d| = |f(c) - f(d)| \leq M|c-d| < |c-d|,$$

which is impossible. Thus $c = d$. ■

5.2.2 Continuous Functions and Compact Sets

5.2.18 Theorem
A continuous image of a compact set is compact, i.e., if K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous on K, then the set

$$f(K) := \{y \in \mathbb{R} \mid f(x) = y \text{ for some } x \in K\}$$

is compact.

$\prec$Let (y_n) be a sequence in $f(K)$. Then, for each $n \in \mathbb{N}$, there is an $x_n \in K$ such that $y_n = f(x_n)$. Since K is compact, the sequence (x_n) has a subsequence $\left(x_{n_k}\right)$ which converges to some $x \in K$. Using continuity of f, we have, by Theorem 5.2.4, that $f\left(x_{n_k}\right) \overset{k\to\infty}{\longrightarrow} f(x) \in f(K)$. Hence, the subsequence $\left(y_{n_k}\right) = \left(f\left(x_{n_k}\right)\right)$ of (y_n) converges to $y = f(x) \in f(K)$. ■

5.2.19 Definition
A real-valued function f with domain D is said to be **bounded** *on D if there exists a positive real number M such that*

$$|f(x)| \leq M \text{ for all } x \in D.$$

A continuous function may not be bounded even when its domain is a bounded set. One such example is the function $f(x) = \dfrac{1}{x}$ defined on $(0,1)$. As x approaches 0 from the right, f grows without bound.

The next theorem asserts that a *continuous* real-valued function defined on a compact set is always bounded there.

5.2.20 Corollary
If K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous on K, then f is bounded on K. That is,

$$M = \sup\{f(x) \mid x \in K\} \text{ and } m = \inf\{f(x) \mid x \in K\}$$

are finite. Moreover, there are points x_1 and x_2 in K such that $f(x_1) = M$ and $f(x_2) = m$.

$\prec$Since K is compact and f is continuous, it follows from Theorem 5.2.18 that $f(K)$ is a compact. By the Heine-Borel Theorem (Theorem 3.3.7), we have that $f(K)$ is closed and bounded. Therefore M and m are finite. Since $f(K)$ is closed, M and m belong to $f(K)$. Therefore there are points x_1 and x_2 such that $M = f(x_1)$ and $m = f(x_2)$. ■

While the Intermediate Value Theorem (Theorem 5.2.9) assures us that a continuous function takes an interval into an interval, the following theorem tells us that, in fact, a continuous function takes a closed and bounded interval into a closed and bounded one! Its proof is contained in Corollary 5.2.20.

5.2.21 Theorem
(Extreme Value Theorem). *If f is continuous on a closed interval $[a, b]$, then there exist points u and v in $[a, b]$ such that*

$$f(u) \leq f(x) \leq f(v) \ \text{ for all } \ x \in [a, b], \ \text{ i.e., } \ x \in [a, b] \ \Rightarrow \ f(x) \in [f(u), f(v)].$$

The following result asserts that a continuous function on a compact set is uniformly continuous on K.

5.2.22 Theorem
If K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous on K, then f is uniformly continuous.

$\prec$Assume that f is continuous on K but *not* uniformly continuous there. Then, there is an $\epsilon > 0$ such that, for each $\delta > 0$, there are points $x, y \in K$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. In particular, for each $n \in \mathbb{N}$, there are points $x_n, \ y_n \in K$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$. Since K is compact, the sequence (x_n) has a subsequence $\left(x_{n_k}\right)$ which converges to some $x \in K$. Similarly, the sequence (y_n) has a subsequence $\left(y_{n_k}\right)$ which converges to some $y \in K$. Since

$$0 \leq |x - y| \leq \left|x - x_{n_k}\right| + \left|x_{n_k} - y_{n_k}\right| + \left|y_{n_k} - y\right| \ \to \ 0 \ \text{ as } \ k \to \infty,$$

it follows that $x = y$. Since f is continuous on K, we have that $f\left(x_{n_k}\right) \overset{k\to\infty}{\longrightarrow} f(x)$ and $f\left(y_{n_k}\right) \overset{k\to\infty}{\longrightarrow} f(y)$. Hence, there are natural numbers N_1 and N_2 such that

$$\begin{aligned} \left|f(x_{n_k}) - f(x)\right| &< \frac{\epsilon}{2} \ \text{ for all } \ k \geq N_1, \ \text{ and} \\ \left|f(y_{n_k}) - f(x)\right| &< \frac{\epsilon}{2} \ \text{ for all } \ k \geq N_2. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then for all $k \geq N$ we have

$$0 < \epsilon \leq \left|f(x_{n_k}) - f(y_{n_k})\right| \leq \left|f(x_{n_k}) - f(x)\right| + \left|f(y_{n_k}) - f(x)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is absurd. ■

5.2.23 Exercise

[1] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that if $f(q) = 0$ for all $q \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$. More generally, show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that vanishes on a dense set, then f is identically zero.

[2] Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **linear** if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that the function $f(x) = cx$, where $c \in \mathbb{R}$, is a continuous linear function. Show that, in fact, every continuous linear function f is of this form.

[3] Let $S \subset \mathbb{R}$. The **inverse image** of S under f, denoted by $f^{-1}(S)$, is the set

$$f^{-1}(S) = \{x \in \mathbb{R} : f(x) \in S\}.$$

Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if the inverse image $f^{-1}(V)$ of every open set V is open.

[4] Show that the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.

[5] Show that the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \sin \dfrac{1}{x}$ is continuous but not uniformly continuous on $\mathbb{R}^+$.

[6] Let $S \subset \mathbb{R}$ and, for $x \in \mathbb{R}$, define

$$f(x) = \inf\{|x - s| : s \in S\}.$$

Show that if $x \notin \overline{S}$, then $f(x) > 0$. Also, show that

$$|f(x) - f(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R}.$$

This says that f satisfies a Lipschitz condition on $\mathbb{R}$.

Chapter 6

Riemann Integration

6.1 Basic Definitions and Theorems

In this section we briefly discuss the construction of Riemann integral. We also point out some of the shortcomings of the Riemann integral.

6.1.1 Definition
Let $[a, b]$ be a closed interval in $\mathbb{R}$. A **partition of** $[a, b]$ *is a set* $P = \{x_0, x_1, x_2, \ldots, x_n\}$ *of points in $\mathbb{R}$ such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Let f be a real-valued function which is bounded on $[a, b]$ and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. Denote by

$$M = \sup_{a \le x \le b} f(x) \quad \text{and} \quad m = \inf_{a \le x \le b} f(x).$$

Since f is bounded on $[a, b]$, it is bounded on each subinterval $[x_{i-1}, x_i]$ for each $i = 1, 2, \ldots, n$. Let

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} \quad \text{and} \quad m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$$

for each $i = 1, 2, \ldots, n$. Clearly,

$$m \le m_i \le M_i \le M \quad \text{for each } i = 1, 2, \ldots, n.$$

We now form the sums

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

6.1.2 Definition
The sums $U(f, P)$ and $L(f, P)$ are called, respectively, the **upper** *and the* **lower sum** *of f relative to the partition P.*

It is important to note that $U(f, P)$ and $L(f, P)$ *depend* on the partition P. If f is nonnegative on $[a, b]$, then the upper sum $U(f, P)$ is the sum of the areas of rectangles whose heights are M_i and whose bases are $[x_{i-1}, x_i]$. Similarly, $L(f, P)$ is the sum of the areas of rectangles whose heights are m_i and whose bases are $[x_{i-1}, x_i]$.

It is clear that $L(f, P) \le U(f, P)$.

6.1.3 Theorem
Let f be a real-valued function which is bounded on $[a, b]$ and let $P = \{x_0,\ x_1,\ x_2, \ldots,\ x_n\}$ be a partition of $[a, b]$. Then

$$m(b-a) \le L(f, P) \le U(f, P) \le M(b-a).$$

≺Since $M_i \le M$ for each $i = 1,\ 2,\ \ldots,\ n$, it follows that

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} M(x_i - x_{i-1}) = M \sum_{i=1}^{n}(x_i - x_{i-1}) = M(b-a).$$

Similarly, since $m \le m_i$ for each $i = 1,\ 2,\ \ldots,\ n$, it follows that

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \ge \sum_{i=1}^{n} m(x_i - x_{i-1}) = m \sum_{i=1}^{n}(x_i - x_{i-1}) = m(b-a).$$

This theorem says that the set $A = \{U(f, P) : P \text{ is a partition of } [a, b]\}$ is bounded below by $m(b-a)$. Hence, A has an infimum, $\Sigma(f)$, say. That is,

$$\Sigma(f) = \inf_P U(f, P),$$

where the infimum is taken over all possible partitions P of $[a, b]$. This theorem also shows that the set $B = \{L(f, P) : P \text{ is a partition of } [a, b]\}$ is bounded above by $M(b-a)$ and hence B has an supremum, $\sigma(f)$, say. That is,

$$\sigma(f) = \sup_P L(f, P),$$

where the supremum is taken over all possible partitions P of $[a, b]$. It is clear that

$$\begin{aligned} m(b-a) \le \Sigma(f) &\le M(b-a), \quad \text{and} \\ m(b-a) \le \sigma(f) &\le M(b-a). \end{aligned}$$

6.1.4 Definition
Let f be a real-valued function which is bounded on $[a, b]$. The **upper integral of** f **on** $[a, b]$ *is defined by*

$$\overline{\int_a^b} f(x)\, dx = \inf_P U(f, P),$$

and the **lower integral of** f **on** $[a, b]$ *is defined by*

$$\underline{\int_a^b} f(x)\, dx = \sup_P L(f, P),$$

where, of course, the infimum and the supremum are taken over all possible partitions P of $[a, b]$.

It is intuitively clear that $\underline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} f(x)\, dx$. We shall prove this fact shortly.

6.1.5 Definition
Let $P = \{x_0,\ x_1,\ x_2, \ldots,\ x_n\}$ be a partition of $[a, b]$. A partition P^* of $[a, b]$ is called a **refinement** *of P, denoted by $P \subseteq P^*$, if $x_i \in P^*$ for each $i = 0,\ 1,\ 2,\ \ldots,\ n$. A partition P^* is called a* **common refinement** *of the partitions P_1 and P_2 of $[a, b]$ if P^* is a refinement of both P_1 and P_2.*

The following theorem says that refining a partition decreases the upper sum and increases the lower sum.

6.1.6 Theorem

Let f be a real-valued function which is bounded on $[a, b]$. If P^* is a refinement of a partition $P = \{x_0,\ x_1,\ x_2, \ldots,\ x_n\}$ of $[a, b]$, then

$$L(f, P) \leq L(f, P^*) \quad \text{and} \quad U(f, P^*) \leq U(f, P).$$

$\prec$Suppose that P^* has one more point than P, say a point x^* which lies in the subinterval $[x_{r-1}, x_r]$. Let

$$L_1 = \sup\{f(x) : x_{r-1} \leq x \leq x^*\},\ \ L_2 = \sup\{f(x) : x^* \leq x \leq x_r\} \ \text{ and}$$
$$\ell_1 = \inf\{f(x) : x_{r-1} \leq x \leq x^*\},\ \ \ell_2 = \inf\{f(x) : x^* \leq x \leq x_r\}.$$

Recalling that

$$M_r = \sup\{f(x) : x_{r-1} \leq x \leq x_r\}, \ \text{ and } \ m_r = \inf\{f(x) : x_{r-1} \leq x \leq x_r\},$$

we observe that

$$m_r \leq \ell_1,\ \ m_r \leq \ell_2,\ \ L_1 \leq M_r,\ \text{ and }\ L_2 \leq M_r.$$

It now follows that

$$m_r(x_r - x_{r-1}) = m_r(x_r - x^*) + m_r(x^* - x_{r-1}) \leq \ell_2(x_r - x^*) + \ell_1(x^* - x_{r-1}).$$

Hence,

$$\begin{aligned}
L(f, P^*) &= \sum_{j=1}^{r-1} m_j(x_j - x_{j-1}) + \ell_1(x^* - x_{r-1}) + \ell_2(x_r - x^*) + \sum_{j=r+1}^{n} m_j(x_j - x_{j-1}) \\
&\geq \sum_{j=1}^{r-1} m_j(x_j - x_{j-1}) + m_r(x_r - x_{r-1}) + \sum_{j=r+1}^{n} m_j(x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} m_j(x_j - x_{j-1}) = L(f, P).
\end{aligned}$$

Similarly,

$$M_r(x_r - x_{r-1}) = M_r(x_r - x^*) + M_r(x^* - x_{r-1}) \geq L_2(x_r - x^*) + L_1(x^* - x_{r-1}),$$

and so

$$\begin{aligned}
U(f, P^*) &= \sum_{j=1}^{r-1} M_j(x_j - x_{j-1}) + L_1(x^* - x_{r-1}) + L_2(x_r - x^*) + \sum_{j=r+1}^{n} M_j(x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{r-1} M_j(x_j - x_{j-1}) + M_r(x_r - x_{r-1}) + \sum_{j=r+1}^{n} M_j(x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} M_j(x_j - x_{j-1}) = U(f, P).
\end{aligned}$$

The case where P^* contains $k \geq 2$ more points than P can be proved by repeating the above argument k times. ∎

6.1.7 Theorem
Let f be a real-valued function which is bounded on $[a, b]$. Then

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

≺Let P_1 and P_2 be any two partitions of $[a, b]$ and let P^* be their common refinement. Then, by Theorem 6.1.6,

$$L(f, P_1) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P_2).$$

Since P_1 is any partition of $[a, b]$, it follows that

$$\sup_P L(f, P) \leq U(f, P_2),$$

and since P_2 is any partition of $[a, b]$, we have that

$$\sup_P L(f, P) \leq \inf_P U(f, P),$$

where the infimum and the supremum are taken over all possible partitions P of $[a, b]$. Thus, $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$.■

6.1.8 Remark
Implicit in the proof of Theorem 6.1.7 is the fact that no lower sum can exceed an upper sum. That is, every lower sum is less than or equal to every upper sum.

6.1.9 Definition
Let f be a real-valued function on $[a, b]$. We say that f is **Riemann-integrable** *on $[a, b]$ if f is bounded on $[a, b]$ and*

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

If f is Riemann-integrable on $[a, b]$, we define **the integral of f on** *$[a, b]$ to be the common value of the upper and the lower integrals; i.e.,*

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

We shall denote by $\mathcal{R}[a, b]$ the set of all functions that are Riemann-integrable on $[a, b]$.

6.1.10 Remark
In the definition of the integral of f on $[a, b]$, we have tacitly assumed that $a < b$. If $a = b$, then $\int_a^b f(x)\,dx = \int_a^a f(x)\,dx = 0$. Also, if $b < a$, then we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

6.1.11 Examples

[1] Show that if f is a constant function on $[a, b]$, then $f \in \mathcal{R}[a, b]$ and find its integral.

Soln: Let $f(x) = k$ for all $x \in [a, b]$ and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a, b]$. Then,

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} = k \quad \text{and} \quad m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\} = k$$

for each $i = 1, 2, \ldots, n$. Thus,

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = k\sum_{i=1}^{n}(x_i - x_{i-1}) = k(b-a) \quad \text{and} \\ L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = k\sum_{i=1}^{n}(x_i - x_{i-1}) = k(b-a). \end{aligned}$$

Since P is *any* partition of $[a, b]$, it follows that $U(f, P) = L(f, P) = k(b - a)$ for all partitions P of $[a, b]$. Therefore

$$\underline{\int_a^b} f(x)\, dx = k(b-a) = \overline{\int_a^b} f(x)\, dx.$$

That is, f is integrable on $[a, b]$ and

$$\int_a^b f(x)dx = k(b-a). \qquad \square$$

[2] Let f be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a, b] \\ -1 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [a, b]. \end{cases}$$

Show that f is *not* Riemann-integrable on $[a, b]$.

Soln: We first observe that f is bounded on $[a, b]$. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a, b]$. Since for each $i = 1, 2, \ldots, n$ the subinterval $[x_{i-1}, x_i]$ contains both rational and irrational numbers, we have that

$$\begin{aligned} M_i &= \sup\{f(x) : x_{i-1} \le x \le x_i\} = \sup\{-1, 1\} = 1, \quad \text{and} \\ m_i &= \inf\{f(x) : x_{i-1} \le x \le x_i\} = \inf\{-1, 1\} = -1 \end{aligned}$$

for each $i = 1, 2, \ldots, n$. Therefore,

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = b - a \quad \text{and} \\ L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = (-1)\sum_{i=1}^{n}(x_i - x_{i-1}) = -1(b-a). \end{aligned}$$

Since P is *any* partition of $[a, b]$, it follows that $U(f, P) = b - a$ and $L(f, P) = -(b - a)$ for all partitions P of $[a, b]$. Therefore

$$\underline{\int_a^b} f(x)\, dx = b - a \quad \text{and} \quad \overline{\int_a^b} f(x)\, dx = -(b-a),$$

and so f is not Riemann-integrable on $[a, b]$. $\square$

6.1.12 Theorem
(Darboux's Integrability Condition). *Let f be a real-valued function which is bounded on $[a, b]$. Then f is integrable on $[a, b]$ if and only if, for any $\epsilon > 0$, there exists a partition P^* of $[a, b]$ such that*

$$U(f, P^*) - L(f, P^*) < \epsilon.$$

≺Assume that f is integrable on $[a, b]$ and let $\epsilon > 0$. Since

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \sup_P L(f, P),$$

there is a partition P_1 of $[a, b]$ such that

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} < L(f, P_1).$$

Again, since

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \inf_P U(f, P),$$

there is a partition P_2 of $[a, b]$ such that

$$U(f, P_2) < \int_a^b f(x)\,dx + \frac{\epsilon}{2}.$$

Let P^* be a common refinement of P_1 and P_2. Then

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} < L(f, P_1) \le L(f, P^*) \le U(f, P^*) \le U(f, P_2) < \int_a^b f(x)\,dx + \frac{\epsilon}{2}.$$

It now follows that

$$U(f, P^*) - L(f, P^*) < \epsilon.$$

For the converse, assume that given any $\epsilon > 0$, there is a partition P^* of $[a, b]$ such that

$$U(f, P^*) - L(f, P^*) < \epsilon.$$

Now,

$$\overline{\int_a^b} f(x)\,dx = \inf_P U(f, P) \le U(f, P^*), \quad \text{and} \quad \underline{\int_a^b} f(x)\,dx = \sup_P L(f, P) \ge L(f, P^*).$$

Thus,

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(f, P^*) - L(f, P^*) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

That is, $f \in \mathcal{R}[a,b]$. ■

Let us highlight the following important fact which is contained in the first part of the proof of Darboux's Integrability Condition:

6.1.13 Theorem
If f is integrable on $[a,b]$, then for each $\epsilon > 0$, there exists a partition P of $[a,b]$ such that

$$\int_a^b f(x)\,dx - \epsilon < L(f,P) \leq U(f,P) < \int_a^b f(x)\,dx + \epsilon.$$

6.1.14 Theorem
If f is continuous on $[a,b]$, then it is integrable there.

≺Since f is continuous on $[a,b]$, we have that f is bounded on $[a,b]$. Moreover, f is uniformly continuous on $[a,b]$. Hence, given $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \text{ whenever } x, y \in [a,b] \text{ and } |x-y| < \delta.$$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a,b]$ such that $x_i - x_{i-1} < \delta$ for each $i = 1, 2, \ldots, n$. By the Extreme-Value Theorem (applied to f on $[x_{i-1}, x_i]$ for each $i = 1, 2, \ldots, n$), there exist points t_i and s_i in $[x_{i-1}, x_i]$ for each $i = 1, 2, \ldots, n$ such that

$$f(t_i) = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = M_i \quad \text{and} \quad f(s_i) = \inf\{f(x) : x_{i-1} \leq x \leq x_i\} = m_i.$$

Since $x_i - x_{i-1} < \delta$, it follows that $|t_i - s_i| < \delta$, and so

$$M_i - m_i = f(t_i) - f(s_i) = |f(t_i) - f(s_i)| < \frac{\epsilon}{b-a} \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

Hence,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1}) = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n |f(t_i) - f(s_i)|(x_i - x_{i-1}) < \sum_{i=1}^n \left(\frac{\epsilon}{b-a}\right)(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}(b-a) = \epsilon. \end{aligned}$$

It now follows from Theorem 6.1.12 that f is integrable on $[a,b]$. ■

6.1.15 Theorem
If f is monotone on $[a,b]$, then f is integrable there.

≺Assume that f is monotone increasing on $[a,b]$ and $f(a) < f(b)$. Since $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, f is clearly bounded on $[a,b]$. We want to show that, given any $\epsilon > 0$, there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. Let $\epsilon > 0$ be given and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a,b]$ such that $x_i - x_{i-1} < \dfrac{\epsilon}{f(b) - f(a)}$ for each $i = 1, 2, \ldots, n$. Since f is increasing on $[a,b]$, we have that

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = f(x_i) \quad \text{and} \quad m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\} = f(x_{i-1})$$

for each $i = 1, 2, \ldots, n$. Hence,

$$\begin{aligned} U(f,P)-L(f,P) &= \sum_{i=1}^{n} M_i(x_i-x_{i-1})-\sum_{i=1}^{n} m_i(x_i-x_{i-1}) = \sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1}) \\ & \sum_{i=1}^{n}[f(x_i)-f(x_{i-1})](x_i-x_{i-1}) < \frac{\epsilon}{f(b)-f(a)}\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})] \\ &= \frac{\epsilon}{f(b)-f(a)}[f(b)-f(a)] = \epsilon. \end{aligned}$$

It now follows from Theorem 6.1.12 that f is Riemann-integrable on $[a, b]$. The case where f is monotone decreasing can be proved in exactly the same way. ■

6.1.1 Properties of the Riemann Integral

6.1.16 Theorem
If f is integrable on $[a, b]$ and $a \le c < d \le b$, then f is integrable on $[c, d]$.

≺Since f is integrable on $[a, b]$, it is bounded there. Hence f is bounded on $[c, d]$. Furthermore, given any $\epsilon > 0$, there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Let $P^* = P \cup \{c, d\}$. Then P^* is a refinement of P, and hence

$$U(f, P^*) - L(f, P^*) \le U(f, P) - L(f, P) < \epsilon.$$

Let $Q_1 = P^* \cap [a, c]$, $Q_2 = P^* \cap [c, d]$, $Q_3 = P^* \cap [d, b]$. Then $P^* = Q_1 \cup Q_2 \cup Q_3$, and so

$$\begin{aligned} U(f,P^*) &= U(f,Q_1)+U(f,Q_2)+U(f,Q_3), \quad \text{and} \\ L(f,P^*) &= L(f,Q_1)+L(f,Q_2)+L(f,Q_3). \end{aligned}$$

Hence,

$$[U(f,Q_1)-L(f,Q_1)]+[U(f,Q_2)-L(f,Q_2)]+[U(f,Q_3)-L(f,Q_3)] = U(f,P^*)-L(f,P^*) < \epsilon.$$

Note that all terms on the left are nonnegative. Therefore Q_2 is a partition of $[c, d]$ with the property that

$$U(f, Q_2) - L(f, Q_2) < \epsilon.$$

This implies that f is integrable on $[c, d]$. ■

6.1.17 Corollary
If f is integrable on $[a, b]$ and $a < c < b$, then f is integrable on both $[a, c]$ and $[c, b]$.

The following theorem says that the converse of Corollary 6.1.17 also holds.

6.1.18 Theorem
If $a < c < b$ and f is integrable on both $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$\prec$Let $\epsilon > 0$ be given. Since f is integrable on $[a, c]$ and on $[c, b]$, there are partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, such that

$$\begin{aligned} U(f, P_1) - L(f, P_1) &< \frac{\epsilon}{2}, \text{ and} \\ U(f, P_2) - L(f, P_2) &< \frac{\epsilon}{2}. \end{aligned}$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ and

$$\begin{aligned} U(f, P) - L(f, P) &= U(f, P_1) + U(f, P_2) - L(f, P_1) - L(f, P_2) \\ &= U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus f is integrable on $[a, b]$. Furthermore,

$$\begin{aligned} \int_a^b f(x)\, dx \le U(f, P) &= U(f, P_1) + U(f, P_2) \\ &< \left[L(f, P_1) + \frac{\epsilon}{2}\right] + \left[L(f, P_1) + \frac{\epsilon}{2}\right] = L(f, P_1) + L(f, P_2) + \epsilon \\ &\le \int_a^c f(x)\, dx + \int_c^b f(x)\, dx + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that

$$\int_a^b f(x)\, dx \le \int_a^c f(x)\, dx + \int_c^b f(x)\, dx. \tag{6.1}$$

Also,

$$\begin{aligned} \int_a^b f(x)\, dx \ge L(f, P) &= L(f, P_1) + L(f, P_2) \\ &> \left[U(f, P_1) - \frac{\epsilon}{2}\right] + \left[U(f, P_2) - \frac{\epsilon}{2}\right] = U(f, P_1) + U(f, P_2) - \epsilon \\ &\ge \int_a^c f(x)\, dx + \int_c^b f(x)\, dx - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that

$$\int_a^b f(x)\, dx \ge \int_a^c f(x)\, dx + \int_c^b f(x)\, dx. \tag{6.2}$$

Combining (6.1) and (6.2), we have that $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$ ■

6.1.19 Corollary
Let f be defined on $[a,b]$ and suppose that $a = c_0 < c_1 < \ldots < c_{n-1} < c_n = b$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[c_{k-1}, c_k]$ for each $k = 1, 2, \ldots, n$. In this case,

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{c_{k-1}}^{c_k} f(x)\,dx.$$

6.1.20 Corollary
If f is continuous at all but a finite set of points in $[a,b]$, then f is Riemann integrable on $[a,b]$.

6.1.21 Theorem
Let f and g be integrable functions on $[a,b]$ and $k \in \mathbb{R}$. Then

(1) $f + g \in \mathcal{R}[a,b]$ and $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$

(2) $kf \in \mathcal{R}[a,b]$ and $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx;$

(3) if $f(x) \le g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx;$

(4) $|f| \in \mathcal{R}[a,b]$ and $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx;$

(5) $f^2 \in \mathcal{R}[a,b];$

(6) $fg \in \mathcal{R}[a,b].$

$\prec$(1) Let $\epsilon > 0$ be given. Since $f, g \in \mathcal{R}[a,b]$, there are partitions P_1 and P_2 of $[a,b]$ such that

$$\begin{aligned} U(f,P_1) - L(f,P_1) &< \frac{\epsilon}{2}, \text{ and} \\ U(g,P_2) - L(g,P_2) &< \frac{\epsilon}{2}. \end{aligned}$$

Let P be a common refinement of P_1 and P_2. Then P is a partition of $[a,b]$ and

$$U(f+g,P) - L(f+g,P) \le U(f,P) + U(g,P) - L(f,P) - L(g,P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f+g$ is integrable on $[a,b]$. That is,

$$\int_a^b [f(x)+g(x)]\,dx = \underline{\int_a^b} [f(x)+g(x)]\,dx = \overline{\int_a^b} [f(x)+g(x)]\,dx.$$

Next, we show that $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. Since $f, g \in \mathcal{R}[a,b]$,

$$\begin{aligned}\int_a^b f(x)\,dx &= \sup_P L(f,P) = \inf_P U(f,P), \quad \text{and} \\ \int_a^b g(x)\,dx &= \sup_P L(g,P) = \inf_P U(g,P),\end{aligned}$$

where the supremum and infimum are taken over all possible partitions of $[a,b]$. By Theorem 6.1.13, there are partitions Q and R of $[a,b]$ such that

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} < L(f,Q) \le U(f,Q) < \int_a^b f(x)\,dx + \frac{\epsilon}{2}, \quad \text{and}$$

$$\int_a^b g(x)\,dx - \frac{\epsilon}{2} < L(g,R) \le U(g,R) < \int_a^b g(x)\,dx + \frac{\epsilon}{2}.$$

Let P^* be a common refinement of Q and R. Then

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} < L(f,Q) \le L(f,P^*), \quad \text{and}$$

$$\int_a^b g(x)\,dx - \frac{\epsilon}{2} < L(g,R) \le L(g,P^*).$$

Hence,

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \epsilon < L(f,P^*) + L(g,P^*) \le L(f+g,P^*) \le \int_a^b [f(x)+g(x)]\,dx.$$

Since $\epsilon > 0$ is arbitrary, we have that

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \le \int_a^b [f(x)+g(x)]\,dx. \tag{6.3}$$

Also,

$$U(f,P^*) \le U(f,Q) < \int_a^b f(x)\,dx + \frac{\epsilon}{2}, \quad \text{and}$$

$$U(g,P^*) \le U(g,R) < \int_a^b g(x)\,dx + \frac{\epsilon}{2}.$$

Thus,

$$\int_a^b [f(x)+g(x)]\,dx \le U(f+g,P^*) \le U(f,P^*)+U(g,P^*) < \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \ge \int_a^b [f(x)+g(x)]\,dx. \tag{6.4}$$

Combining 6.3 and 6.4, we get that $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(2) Exercise.
(3) Exercise.

(4) Let $\epsilon > 0$ be given. Then, there is a partition $P = \{x_0,\ x_1,\ x_2, \ldots,\ x_n\}$ of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon.$$

For $i = 1,\ 2,\ \ldots,\ n$, let

$$M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}, \qquad m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$$
$$L_i = \sup\{|f|(x) = |f(x)| \mid x_{i-1} \le x \le x_i\}, \qquad \ell_i = \inf\{|f|(x) = |f(x)| \mid x_{i-1} \le x \le x_i\}.$$

Now, since for all $x, y \in [x_{i-1}, x_i]$ we have

$$||f|(x) - |f|(y)| = ||f(x)| - |f(y)|| \le |f(x) - f(y)| \le M_i - m_i,$$

it follows that

$$L_i - \ell_i \le M_i - m_i$$

for each $i = 1,\ 2,\ \ldots,\ n$. This then implies that

$$\sum_{i=1}^n (L_i - \ell_i)(x_i - x_{i-1}) \le \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1})$$
$$\Rightarrow \quad \sum_{i=1}^n L_i(x_i - x_{i-1}) - \sum_{i=1}^n \ell_i(x_i - x_{i-1}) \le \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1})$$
$$\Rightarrow \quad U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P) < \epsilon.$$

Thus $|f|$ is integrable on $[a,b]$.

Since $-|f(x)| \le f(x) \le |f(x)|$ for each $x \in [a,b]$, we have by (3), that

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|dx,$$

and consequently

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|dx.$$

(5) Since f is integrable on $[a, b]$, it is bounded there. Therefore, there exists $K > 0$ such that $|f(x)| \le K$ for all $x \in [a, b]$. Note that for each $x \in [a, b]$, $|f^2(x)| = |f(x)|^2 \le K^2$, so that f^2 is bounded on $[a, b]$. Also, by (4), $|f|$ is integrable on $[a, b]$, and therefore, given $\epsilon > 0$ there is a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ such that

$$U(|f|, P) - L(|f|, P) < \frac{\epsilon}{2K}.$$

For $i = 1, 2, \ldots, n$, let

$$M_i = \sup\{f^2(x) \mid x_{i-1} \le x \le x_i\}, \qquad m_i = \inf\{f^2(x) \mid x_{i-1} \le x \le x_i\}$$
$$L_i = \sup\{|f|(x) = |f(x)| \mid x_{i-1} \le x \le x_i\}, \qquad \ell_i = \inf\{|f|(x) = |f(x)| \mid x_{i-1} \le x \le x_i\}.$$

Then $L_i^2 = M_i$ and $\ell_i^2 = m_i$ for each $i = 1, 2, \ldots, n$, and so

$$\begin{aligned}
U(f^2, P) - L(f^2, P) &= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n}(L_i^2 - \ell_i^2)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n}(L_i + \ell_i)(L_i - \ell_i)(x_i - x_{i-1}) \\
&\le 2K\sum_{i=1}^{n}(L_i - \ell_i)(x_i - x_{i-1}) \\
&= 2K\left[\sum_{i=1}^{n} L_i(x_i - x_{i-1}) - \sum_{i=1}^{n}\ell_i(x_i - x_{i-1})\right] \\
&= 2K[U(|f|, P) - L(|f|, P)] < 2K\left(\frac{\epsilon}{2K}\right) = \epsilon.
\end{aligned}$$

Thus, f^2 is integrable on $[a, b]$.

(6) This follows from (1),(2) and (5) and from the observation that

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}.$$

■

6.1.22 Exercise

[1] Let f be the function on $[0, 1]$ given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is not Riemann integrable on $[0, 1]$.

[2] Let f be the function on $[0, 1]$ given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ x - \frac{1}{2} & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

(a) Show, from first principles, that f is Riemann integrable on $[0, 1]$.

(b) Quote a result that assures us that f is Riemann integrable.

(c) Find $\int_0^1 f(x)dx$.

(d) Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of rational in the interval $[0, 1]$. For each $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, r_3, \ldots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Show that (f_n) is a nondecreasing sequence of functions that are Riemann-integrable on $[0, 1]$. Show also that the sequence (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

and that f is not Riemann-integrable.

Chapter 7

Introduction to Metric Spaces

7.1

7.1.1 Definition

Let X be a nonempty set. A **metric** *on X is a function $d : X \times X \to \mathbb{R}$ such that for every $x,\ y,\ z \in X$,*

M1. $d(x, y) \geq 0$;

M2. $d(x, y) = 0$ *if and only if* $x = y$;

M3. $d(x, y) = d(y, x)$;

M4. $d(x, z) \leq d(x, y) + d(y, z)$, *(triangle inequality).*

A **metric space** *is a pair (X, d), where X is a nonempty set and d a metric on X.*

The elements of a metric space (X, d) are usually referred to as **points**. If $x, y \in X$, then $d(x, y)$ is called the **distance** between x and y. A set can have more than one metric defined on it.

If condition M2 is replaced by the condition

M2′. $d(x, x) = 0$ for all $x \in X$,

then d is a pseudo-metric on X and (X, d) is a pseudo-metric space.

7.1.2 Examples

[1] Let $X = \mathbb{R}$ and for $x, y \in X$, define $d : X \times X \to \mathbb{R}$ by $d(x, y) = |x - y|$. Then $(\mathbb{R}, d)$ is a metric space. This metric is called the **usual metric** on $\mathbb{R}$.

[2] Let $X = \mathbb{C}$, the set of complex numbers. For $x, y \in X$, define $d : X \times X \to \mathbb{R}$ by $d(x, y) = |x - y|$. Then $(\mathbb{C}, d)$ is a metric space. This is metric called the **usual metric** on $\mathbb{C}$.

[3] Let $X = \mathbb{R}^n$, where n is a natural number. The elements of X are ordered n-tuples $x = (x_1,\ x_2,\ \ldots,\ x_n)$ of real numbers. For $x = (x_1,\ x_2,\ \ldots,\ x_n)$ and $y = (y_1,\ y_2,\ \ldots,\ y_n)$ in X, define

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^{n} |x_i - y_i|, \\ d_2(x,y) &= \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}, \\ d_\infty(x,y) &= \max_{1\le i\le n} |x_i - y_i|. \end{aligned}$$

Then each of $d_1,\ d_2$ and d_∞ defines a metric on $\mathbb{R}^n$.

These metrics have special names attached to them:

d_1 is called the **taxicab** metric. The reason for this name is that it measures the distance that a taxicab would have to travel from one point to another if the streets of the city were laid out in a grid-like pattern. This metric is also called the 1-metric.

d_2 is called the **Euclidean metric** or the **usual (standard) metric** on $\mathbb{R}^n$.

d_∞ is called the **maximum, supremum, or infinity metric**.

(i) We leave it as an easy exercise to show that $(\mathbb{R}^n, d_1)$ is a metric space.

(ii) We show that $(\mathbb{R}^n, d_2)$ is a metric space. Checking that d_2 satisfies properties M1, M2 and M3 is straightforward. We establish property M4. To that end, let $x = (x_1,\ x_2,\ \dots,\ x_n),\ y = (y_1,\ y_2,\ \dots,\ y_n)$ and $z = (z_1,\ z_2,\ \dots,\ z_n)$ be elements of $\mathbb{R}^n$. We want to show that $d_2(x,z) \le d_2(x,y) + d_2(y,z)$. This is equivalent to showing that

$$\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \le \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2}. \tag{7.1}$$

For each $i = 1,\ 2,\ \dots,\ n$, let $a_i = x_i - y_i$ and $b_i = y_i - z_i$. Then equation (7.1) can be rewritten as

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

Since both sides of the inequality are nonnegative, it suffices to show that the inequality holds for the squares of the left and right hand sides of the inequality. That is, we have to show that

$$\sum_{i=1}^{n}(a_i + b_i)^2 \le \sum_{i=1}^{n} a_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} + \sum_{i=1}^{n} b_i^2. \tag{7.2}$$

The left hand side of (7.2) can be expanded as

$$\sum_{i=1}^{n}(a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2.$$

It now follows that inequality (7.2) is equivalent to the inequality

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} \tag{7.3}$$

Equation (7.3) is called the Cauchy-Schwarz Inequality. We now prove the Cauchy-Schwarz Inequality.

Cauchy-Schwarz Inequality: *If* $(a_1,\ a_2,\ \ldots,\ a_n) \in \mathbb{R}^n$ *and* $(b_1,\ b_2,\ \ldots,\ b_n) \in \mathbb{R}^n$, *then*

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

Proof. If $a_i = 0$ for all $i = 1,\ 2,\ \ldots,\ n$ or $b_i = 0$ for all $i = 1,\ 2,\ \ldots,\ n$, then the inequality obviously holds. Assume that there is an $i \in \{1,\ 2,\ \ldots,\ n\}$ such that $a_i \neq 0$ and a $j \in \{1,\ 2,\ \ldots,\ n\}$ such that $b_j \neq 0$. For each $i = 1,\ 2,\ \ldots,\ n$, let

$$\alpha_i = \frac{a_i}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}} \quad \text{and} \quad \beta_i = \frac{b_i}{\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}}.$$

Recall that if $a, b \in \mathbb{R}$, then $2ab \le a^2 + b^2$. Therefore

$$\begin{aligned}
2\alpha_i\beta_i \le \alpha_i^2 + \beta_i^2 \quad &\Longleftrightarrow \quad \frac{2a_i b_i}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}} \le \frac{a_i^2}{\sum_{i=1}^{n} a_i^2} + \frac{b_i^2}{\sum_{i=1}^{n} b_i^2} \\
&\Longrightarrow \quad \frac{2\sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}} \le \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2} + \frac{\sum_{i=1}^{n} b_i^2}{\sum_{i=1}^{n} b_i^2} = 2 \\
&\Longrightarrow \quad \frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}} \le 1 \\
&\Longrightarrow \quad \sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2},
\end{aligned}$$

which proves the Cauchy-Schwarz Inequality. ■

(iii) We show that $(\mathbb{R}^n, d_\infty)$ is a metric space. Let $x = (x_1,\ x_2,\ \ldots,\ x_n)$, $y = (y_1,\ y_2,\ \ldots,\ y_n)$ and $z = (z_1,\ z_2,\ \ldots,\ z_n)$ be elements of $\mathbb{R}^n$.

M1. Since for each $i = 1,\ 2,\ \ldots,\ n$, $|x_i - y_i| \ge 0$, it follows that

$$d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i| \ge 0.$$

M2.

$$\begin{aligned}
d_\infty(x, y) = 0 \quad &\Longleftrightarrow \quad \max_{1\le i\le n} |x_i - y_i| = 0 \\
&\Longleftrightarrow \quad |x_i - y_i| \le 0 \text{ for each } i = 1,\ 2,\ \ldots,\ n \\
&\Longleftrightarrow \quad |x_i - y_i| = 0 \text{ for each } i = 1,\ 2,\ \ldots,\ n \\
&\Longleftrightarrow \quad x_i = y_i \text{ for each } i = 1,\ 2,\ \ldots,\ n \\
&\Longleftrightarrow \quad x = y.
\end{aligned}$$

M3. $d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i| = \max_{1\le i\le n} |y_i - x_i| = d_\infty(y, x)$.

M4. Since, for each $j = 1,\ 2,\ \ldots,\ n$, $|x_j - z_j| \le |x_j - y_j| + |y_j - z_j|$, it follows that

$$|x_j - z_j| \le \max_{1\le i\le n} |x_i - y_i| + \max_{1\le i\le n} |y_i - z_i| = d_\infty(x, y) + d_\infty(y, z).$$

Hence,

$$d_\infty(x,z) = \max_{1\le j\le n} |x_j - z_j| \le d_\infty(x,y) + d_\infty(y,z).$$

[4] For $1 \le p < \infty$, let $X = \ell_p$ be a set of sequences $(x_i)_{i=1}^\infty$ of real or complex numbers such that $\sum_{i=1}^\infty |x_i|^p < \infty$. That is,

$$\ell_p = \left\{ x = (x_i)_{i=1}^\infty \mid \sum_{i=1}^\infty |x_i|^p < \infty \right\}.$$

For $x = (x_i)_{i=1}^\infty$ and $y = (y_i)_{i=1}^\infty$ in ℓ_p, define $d_p : X \times X \to \mathbb{R}$ by

$$d_p(x,y) = \left(\sum_{i=1}^\infty |x_i - y_i|^p \right)^{1/p}.$$

Then (ℓ_p, d_p) is a metric space.

Properties M1, M2 and M3 are easy to prove. Property M4 requires Minkowski's Inequality: *If $p > 1$ and $(a_i)_{i=1}^\infty$ and $(b_i)_{i=1}^\infty$ are in ℓ_p, then*

$$\left(\sum_{i=1}^\infty |a_i + b_i|^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^\infty |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^\infty |b_i|^p \right)^{\frac{1}{p}}.$$

We now establish M4. Let $x = (x_i)_{i=1}^\infty$, $y = (y_i)_{i=1}^\infty$ and $z = (z_i)_{i=1}^\infty$ be elements of ℓ_p. For each $i \in \mathbb{N}$, let $a_i = x_i - y_i$ and $b_i = y_i - z_i$. Then, by Minkowski's Inequality, we have that

$$\begin{aligned} d_p(x,z) = \left(\sum_{i=1}^\infty |x_i - z_i|^p \right)^{1/p} &= \left(\sum_{i=1}^\infty |a_i + b_i|^p \right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^\infty |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^\infty |b_i|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^\infty |x_i - y_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^\infty |y_i - z_i|^p \right)^{\frac{1}{p}} \\ &= d_p(x,y) + d_p(y,z). \end{aligned}$$

[5] Let $X = \ell_\infty$ be a set of bounded sequences of real or complex numbers. For $x = (x_i)_{i=1}^\infty$ and $y = (y_i)_{i=1}^\infty$ in ℓ_∞, define $d_\infty : X \times X \to \mathbb{R}$ by

$$d_\infty(x,y) = \sup_{i\in\mathbb{N}} |x_i - y_i|.$$

Then (ℓ_∞, d_∞) is a metric space.

[6] Let X be a set of sequences of real or complex numbers. For $x = (x_i)_{i=1}^\infty$ and $y = (y_i)_{i=1}^\infty$ in X, define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \sum_{i=1}^\infty \frac{|x_i - y_i|}{2^i\,(1 + |x_i - y_i|)}.$$

Then (X, d) is a metric space.

[7] Let $X = C[a,b]$ be a set of continuous real-valued functions on the interval $[a,b]$ and $p \in [1,\infty)$. For $f, g \in C[a,b]$, define $d_p : X \times X \to \mathbb{R}$ and $d_\infty : X \times X \to \mathbb{R}$ respectively by

$$d_p(f,g) = \left(\int_a^b |f(t) - g(t)|^p \, dt \right)^{1/p}$$

and

$$d_\infty(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|.$$

Then (X, d_p) and (X, d_∞) are metric metric space.

[8] Let X be a set. For $x, y \in X$, define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then (X, d) is a metric space. This metric d is called the **discrete metric** on X.

7.1.3 Proposition
Let (X,d) be a metric space. Then for all $x, y, z \in X$,

$$|d(x,z) - d(y,z)| \leq d(x,y).$$

≺By the triangle inequality we have that

$$d(x,z) \leq d(x,y) + d(y,z) \iff d(x,z) - d(y,z) \leq d(x,y). \tag{7.1.3.1}$$

Interchanging the roles of x and y in (7.1.3.1),

$$d(y,z) - d(x,z) \leq d(x,y). \tag{7.1.3.2}$$

It now follows from equations (7.1.3.1) and (7.1.3.2) that

$$|d(x,z) - d(y,z)| \leq d(x,y).$$ ■

7.2 Open Sets, Closed Sets, and Bounded Sets

7.2.1 Definition
Let (X,d) be a metric space, $x \in X$ and $r > 0$. The set

$$B(x,r) := \{y \in X \mid d(x,y) < r\}$$

is called the **open ball** *with centre x and radius r.*
The set

$$\overline{B}(x,r) := \{y \in X \mid d(x,y) \leq r\}$$

is called the **closed ball** *with centre x and radius r.*

7.2.2 Lemma
Let x and y be distinct points in a metric space (X, d). Then there is an $\epsilon > 0$ such that $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$.

≺Since $x \neq y$, it follows that $d(x, y) > 0$. Choose ϵ such that $0 < \epsilon < \frac{d(x,y)}{2}$. Then $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$. Indeed, if $z \in B(x, \epsilon) \cap B(y, \epsilon)$, then

$$d(x, z) < \epsilon \text{ and } d(y, z) < \epsilon.$$

Therefore

$$0 < d(x, y) \leq d(x, z) + d(y, z) < \epsilon + \epsilon < \frac{d(x, y)}{2} + \frac{d(x, y)}{2} = d(x, y).$$

That is, $d(x, y) < d(x, y)$, which is absurd. ■

7.2.3 Definition
Let (X, d) be a metric space. A subset G of X is said to be **open** *if for each $x \in G$, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset G$.*

7.2.4 Definition
Let (X, d) be a metric space. A subset A of X is called a **neighbourhood** *of $x \in X$ if there is an open set $V \subset X$ such that $x \in V \subset A$.*

It is clear that a subset G of a metric space (X, d) is open if G is a neighbourhood of each of its points.

7.2.5 Examples

[1] An open ball in a metric space (X, d) is an open set. Indeed, let $B(x, r)$ be an open ball with centre x and radius r and let $y \in B(x, r)$. Then $d(x, y) < r$. Let $\epsilon = r - d(x, y)$. We now show that $B(y, \epsilon) \subset B(x, r)$. Let $z \in B(y, \epsilon)$. Then $d(y, z) < \epsilon$. Hence, by the triangle inequality,

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \epsilon = d(x, y) + r - d(x, y) = r.$$

That is, $z \in B(x, r)$, and so $B(y, \epsilon) \subset B(x, r)$.

[2] Let (X, d) be a discrete metric space. Then every subset of X is open. To see this, let G be a subset of X and $x \in G$. Then, with $0 < \epsilon < 1$, $B(x, \epsilon) = \{x\} \subset G$.

7.2.6 Theorem
Let (X, d) be a metric space.

(1) X and $\emptyset$ are open.

(2) A union of an arbitrary collection of open sets in X is open.

(3) An intersection of a finite collection of open sets in X is open.

≺Exercise.

7.2.7 Proposition
Let (X, d) be a metric space. Then a set A in X is open if and only if it is a union of open balls in X.

≺Assume that A is a union of open balls in X; i.e., $A = \bigcup_{x \in A} B(x, r_x)$. Since each open ball is an open set and a union of an arbitrary collection of open sets is open, it follows that A is an open set.

Conversely, assume that A is open in X. Then, for each $x \in A$, there is an $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset A$. Obviously $A = \bigcup_{x \in A} B(x, \epsilon_x)$. ■

7.2.8 Definition
A subset F of a metric space $(X, d)$is said to be **closed** *if its complement $X \setminus F$ is open.*

7.2.9 Example

[1] A closed ball in a metric space (X, d) is a closed set. Indeed, let $\overline{B}(x, r)$ be a closed ball with centre x and radius r and let $y \in X \setminus \overline{B}(x, r)$. Then $d(x, y) > r$. Let $\epsilon = d(x, y) - r$. We now show that $B(y, \epsilon) \subset X \setminus \overline{B}(x, r)$. Let $z \in B(y, \epsilon)$. Then $d(y, z) < \epsilon$. Hence, by the triangle inequality,

$$d(y, z) < \epsilon = d(x, y) - r \iff r < d(x, y) - d(y, z) \leq d(x, z).$$

Hence $z \notin \overline{B}(x, r)$ and so $z \in X \setminus \overline{B}(x, r)$.

[2] Let (X, d) be a discrete metric space. Then every subset of X is closed. To see this, let A be a subset of X. Since every subset of X is open, $X \setminus A$ is open. Hence $A = X \setminus (X \setminus A)$ is closed.

7.2.10 Theorem
Let (X, d) be a metric space.

(1) X and $\emptyset$ are closed.

(2) An intersection of an arbitrary collection of closed sets in X is closed.

(3) A union of a finite collection of closed sets in X is closed.

≺Exercise. ■

7.2.11 Proposition
Every singleton set in a metric space (X, d) is closed.

≺Let $x \in X$. We show that the set $\{x\}$ is closed. It suffices to show that the complement $X \setminus \{x\}$ is open. To that end, let $y \in X \setminus \{x\}$. Then $x \neq y$. By Lemma 7.2.2, there is an $\epsilon > 0$ such that $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$. Hence $B(y, \epsilon) \subseteq X \setminus \{x\}$, and so $X \setminus \{x\}$ is open. ■

7.2.12 Definition
Let S be a subset of a metric space (X, d), and $x \in X$. Then

(a) $x \in S$ is called an **interior point** *of S if there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset S$. The set of all interior points of a set S is denoted by S° or int(S).*

(b) $x \in X$ is called a **boundary point** *of S if for every $\epsilon > 0$ the open $B(x, \epsilon)$ contains points of S as well as points of $X \setminus S$. The set of boundary points of S is denoted by ∂S or bd(S).*

(c) $x \in S$ is called an **isolated point** *of S if there exists an $\epsilon > 0$ such that $B(x, \epsilon) \cap S = \{x\}$.*

(d) A point $x \in X$ is called an **accumulation point** *(or* **limit point***) of S if for every $\epsilon > 0$, the ϵ-ball, $B(x, \epsilon)$, contains a point of S distinct from x. The set of all accumulation points of S is called the* **derived set** *of S and is denoted by S'. That is, $S' = \{x \in X \mid (B(x, \epsilon) \setminus \{x\}) \cap S \neq \emptyset \text{ for all } \epsilon > 0\}$.*

(e) The **closure** *of the set S, denoted by $\overline{S}$, is the set $\overline{S} = S \cup S'$.*

7.2.13 Examples

[1] Let $X = \mathbb{R}^2$ and $S = \{(x_1, x_2) \in X \mid x_1^2 + x_2^2 < 1\}$.

7.2.14 Theorem

(Properties of Interior). *Let A and B be subsets of a metric space (X, d). Then*

(a) $A^\circ \subseteq A$;

(b) $A^{\circ\circ} = A^\circ$;

(c) If $A \subseteq B$, then $A^\circ \subseteq B^\circ$;

(d) $(A \cap B)^\circ = A^\circ \cap B^\circ$;

(e) $\displaystyle\bigcup_{i \in I} A_i^\circ \subseteq \left(\bigcup_{i \in I} A_i\right)^\circ$;

(f) $\displaystyle\left(\bigcap_{i \in I} A_i\right)^\circ \subseteq \bigcap_{i \in I} A_i^\circ$.

$\prec$

7.2.15 Theorem

(Properties of Closure). *Let A and B be subsets of a metric space (X, d). Then*

(a) $A \subseteq \overline{A}$;

(b) $\overline{\overline{A}} = \overline{A}$;

(c) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$;

(d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;

(e) $\displaystyle\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$;

(f) $\displaystyle\bigcup_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}$.

$\prec$

7.2.16 Theorem

A subset C of a metric space (X, d) is closed if and only if it contains all its accumulation points.

$\prec$Assume that C is closed and let $x \in C'$. We want to show that $x \in C$. If $x \notin C$, then $x \in X \setminus C$. Since C is closed, $X \setminus C$ is open. Therefore there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset X \setminus C$. This then implies that $B(x, \epsilon) \cap C = \emptyset$, which contradicts the fact that $x \in C'$. Thus $C' \subset C$.

To prove the converse, assume that $C' \subset C$. We want to show that C is closed, or equivalently, that $X \setminus C$ is open. To this end, let $x \in X \setminus C$. Then $x \notin C'$, and so there is an $\epsilon > 0$ such that

$$(B(x, \epsilon) \setminus \{x\}) \cap C = \emptyset.$$

Since $x \notin C$, we have that $B(x, \epsilon) \cap C = \emptyset$. Thus $B(x, \epsilon) \subset X \setminus C$, whence $X \setminus C$ is open. ■

7.2.17 Corollary
Let C be a subset of a metric (X, d). Then C is closed if and only if $\overline{C} = C$.

≺Assume that C is closed. Then, by Theorem 7.2.16, $C' \subset C$. Therefore $\overline{C} = C \cup C' \subset C \cup C = C$. But $C \subset C \cup C' = \overline{C}$. Thus $C = \overline{C}$.

Conversely, assume that $C = \overline{C}$. Then $C' \subset C \cup C' = \overline{C} = C$. Thus C contains all its accumulation points and, consequently, C is closed. ■

7.2.18 Definition
A subset A of a metric space (X, d) is said to be **bounded** *if $A \subseteq B(x, r)$ for some $x \in X$ and some $r > 0$.*

7.2.19 Proposition
A subset A of a metric space (X, d) is bounded if and only if there is a real number $M \geq 0$ such that $d(x, y) \leq M$ for all $x, y \in A$.

≺Exercise.

7.2.20 Definition
The **diameter** *of a subset A of a metric space (X, d) is defined as*

$$\text{diam}(A) := \sup\{d(x, y) \mid x, y \in A\}.$$

Note that a subset A of a metric space (X, d) is bounded if and only if $\text{diam}(A) < \infty$.

7.2.21 Proposition
Any subset of a discrete metric space (X, d) is bounded.

≺Let A be a subset of X. Clearly, by definition of the discrete metric, $d(x, y) \leq 1$ for all $x, y \in A$. Hence, A is bounded. ■

7.2.22 Proposition
A finite union of bounded subsets of a metric space (X, d) is bounded.

≺Let $U_1, U_2, \ldots, U_n$ be open subsets of X. Then, for each $i = 1, 2, \ldots, n$, there is an r_i such that $d(x, y) \leq r_i$ for all $x, y \in U_i$. Let $r = \max\{r_1, r_2, \ldots, r_n\}$ and $U = \bigcup_{i=1}^{n} U_i$. For each $i = 1, 2, \ldots, n$, choose $x_i \in U_i$. Let $s = \max\{d(x_i, x_j)$ for all $i, j = 1, 2, \ldots, n\}$. Let $x, y \in U$. Then $x \in U_i$ and $y \in U_j$ for some $i, j = 1, 2, \ldots, n$. Therefore

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \leq r + s + r = 2r + s.$$

That is, for all $x, y \in U$, $d(x, y) \leq M$, where $M = 2r + s$ and so U is bounded. ■

7.3 Convergence of Sequences in Metric Spaces

7.3.1 Definition
A sequence (x_n) in a metric space (X, d) is said to **converge** *to a point $x \in X$ if, given any $\epsilon > 0$, there is a natural number N (which depends on ϵ, in general) such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In this case the point x is called the* **limit** *of the sequence (x_n) and we write $\lim_{n\to\infty} x_n = x$.*
Equivalently, (x_n) converges to x if, given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $x_n \in B(x, \epsilon)$ for all $n \geq N$.

7.3.2 Theorem
(Limits of convergent sequences are unique). *Let (x_n) be a sequence in a metric space (X, d). If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, then $x = y$.*

$\prec$Assume that $x \neq y$ and let $0 < \epsilon < \dfrac{d(x, y)}{2}$. Then there are natural numbers N_1 and N_2 such that

$$d(x_n, x) < \epsilon \ \text{ for all } \ n \geq N_1 \ \text{ and}$$
$$d(x_n, y) < \epsilon \ \text{ for all } \ n \geq N_2.$$

Let $N = \max\{N_1, \ N_2\}$. Then, for all $n \geq N$,

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < 2\epsilon < d(x, y),$$

which is absurd. Hence $x = y$. ■

7.3.3 Proposition
A sequence (x_n) converges to x if and only if for each $\epsilon > 0$, the set $\{n \mid x_n \notin B(x, \epsilon)\}$ is finite.

7.3.4 Proposition
Every convergent sequence is bounded.

$\prec$

7.3.5 Proposition
If a sequence (x_n) converges to x, then every subsequence of (x_n) also converges to x.

$\prec$

7.3.1 Sequential Characterization of closed sets

7.3.6 Theorem
Let K be a nonempty subset of a metric space (X, d) and $x \in X$. Then

(a) $x \in \overline{K}$ if and only if there is a sequence $(x_n) \subset K$ such that $x_n \to x$ as $n \to \infty$.

(b) K is closed if and only if K contains the limit of every convergent sequence in K.

$\prec$

(a) Assume that $x \in \overline{K}$. Then either $x \in K$ or $x \in K'$. If $x \in K$, then the constant sequence $(x, \ x, \ x, \ \ldots)$ in K converges to x. If $x \in K'$, then, for each $n \in \mathbb{N}$, the open ball $B(x, \frac{1}{n})$ contains a point $x_n \in K$ distinct from x. It now follows that $d(x_n, x) < \frac{1}{n}$. Clearly, $(x_n) \subset K$ and $x_n \to x$ as $n \to \infty$.

Conversely, assume that there is a sequence $(x_n) \subset K$ such that $x_n \to x$ as $n \to \infty$. Then, either $x \in K$ or every ϵ-ball centred at x contains a point $x_n \neq x$, in which case $x \in K'$ Thus $x \in \overline{K}$.

(b) By Corollary 7.2.17, K is closed if and only if $K = \overline{K}$. Hence, (b) follows from (a). ■

7.3.2 Completeness in Metric Spaces

7.3.7 Definition
A sequence (x_n) in a metric space (X, d) is said to **Cauchy** *if, given any $\epsilon > 0$, there is a natural number N (which depends on ϵ, in general) such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.*

7.3.8 Proposition
A convergent sequence in a metric space (X, d) is Cauchy.

≺Let (x_n) be a sequence in X which converges to $x \in X$ and let $\epsilon > 0$. Then there is a natural number N such that $d(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq N$. For all $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, (x_n) is a Cauchy sequence in X. ■

7.3.9 Proposition
A Cauchy sequence in a metric space (X, d) is bounded.

≺Let (x_n) be a Cauchy sequence in X. Choose $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < 1 \quad \text{for all} \quad n, m \geq N.$$

Let $r = \max\{d(x_1, x_N),\ d(x_2, x_N),\ \ldots,\ d(x_{N-1}, x_N),\ 1\}$. Clearly $d(x_n, x_N) \leq r$ for all $n = 1,\ 2,\ \ldots,\ N-1$. If $n \geq N$, then $d(x_n, x_N) < 1 \leq r$. Thus, $d(x_n, X_N) \leq r$ for all $n \in \mathbb{N}$ and so (x_n) is bounded. ■

7.3.10 Proposition
Let (X, d) be a metric space. A Cauchy sequence in X which has a convergent subsequence is convergent.

≺Let (x_n) be a Cauchy sequence in X and (x_{n_k}) its subsequence which converges to $x \in X$. Then, for any $\epsilon > 0$, there are positive integers N_1 and N_2 such that

$$d(x_n, x_m) < \frac{\epsilon}{2} \text{ for all } n, m \geq N_1$$

and

$$d(x_{n_k}, x) < \frac{\epsilon}{2} \text{ for all } k \geq N_2.$$

Let $N = \max\{N_1,\ N_2\}$. If $k \geq N$, then since $n_k \geq k$,

$$d(x_k, x) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $x_n \to x$ as $n \to \infty$. ■

7.3.11 Definition
A metric space (X, d) is said to be **complete** *if every Cauchy sequence in X converges (to a point in X).*

7.3.12 Example
We have shown (see Theorem 4.2.19) that every Cauchy sequence of real numbers converges. It now follows that $\mathbb{R}$, with its usual metric, is a complete metric space.

7.3.13 Proposition
A closed subset F of a complete metric space(X, d) is complete.

≺Let (x_n) be a Cauchy sequence in F. Then (x_n) is a Cauchy sequence in X. Since X is complete, this sequence converges to some x in X. Since F is closed, $x \in F$. Hence F is complete. ■

7.4 Compactness in Metric Spaces

7.4.1 Definition
Let K be a subset of a metric space (X, d). An **open cover** *for K is a nonempty collection $\mathcal{U} = \{ U_i \mid i \in I \}$ of open subsets of X such that*

$$K \subseteq \bigcup_{i \in I} U_i.$$

Such an open cover is said to be **reducible to a finite subcover** *for K if there are $n \in \mathbb{N}$ indices $i_1, i_2, \ldots, i_n$ in I such that*

$$K \subseteq \bigcup_{k=1}^{n} U_{i_k}.$$

A set $K \subset X$ is said to be **compact** *if every open cover for K is reducible to a finite subcover.*

7.4.2 Theorem
A closed subset F of a compact metric space (X, d) is compact.

$\prec$Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover for F. Then $\mathcal{G} = \{U_i \cup (X \setminus F) \mid i \in I\}$ is an open cover for S. Since S is compact, the cover $\mathcal{G}$ is reducible to a finite subcover. That is, there are indices $i_1, i_2, \ldots, i_n$ such that

$$S \subset \bigcup_{k=1}^{n} U_{i_k} \cup (X \setminus F).$$

Since $F \subset S$ and $F \cap (X \setminus F) = \emptyset$, it follows that $F \subset \bigcup_{k=1}^{n} U_{i_k}$. Hence F is compact. ■

7.4.3 Proposition
A nonempty subset K of a discrete metric space (X, d) is compact if and only if K is finite.

$\prec$Assume that K is compact. Since each singleton set in a discrete metric space is open, the collection $\mathcal{C} = \{\{x\} \mid x \in K\}$ is an open cover for K. Since K is compact, there are elements $x_1, x_2, \ldots, x_n$ in K such that $K \subseteq \bigcup_{i=1}^{n} \{x_i\}$. Hence

$$K = K \cap \left(\bigcup_{i=1}^{n} \{x_i\} \right) = \{x_1, x_2, \ldots, x_n\},$$

a finite set.

Conversely, assume that K is finite. Then K is clearly compact as any finite set is compact. ■

7.4.4 Theorem
Every compact subset K of a metric space (X, d) is closed and bounded.

$\prec$<u>Closedness</u>: It suffices to show that $X \setminus K$ is open. To that end, let $x \in X \setminus K$. By Lemma 7.2.2, for each $y \in K$, there is an $\epsilon_y > 0$ such that $B(x, \epsilon_y) \cap B(y, \epsilon_y) = \emptyset$. The collection $\{B(y, \epsilon_y) \mid y \in K\}$ of open balls is an open cover for K. Since K is compact, there are elements $y_1, y_2, \ldots, y_n$ in K such that

$$K \subseteq \bigcup_{i=1}^{n} B\left(y_i, \epsilon_{y_i}\right).$$

Let $U = \bigcap_{i=1}^{n} B\left(x, \epsilon_{y_i}\right)$. Then $x \in U$ and, by Theorem 7.2.6(3), U is an open set. Hence there is a $\delta > 0$ such that $B(x, \delta) \subseteq U$. Since $B(x, \epsilon_{y_i}) \cap B(y_i, \epsilon_{y_i}) = \emptyset$ for each $i = 1, 2, \ldots, n$, it follows that $U \subset X \setminus K$ and so $x \in B(x, \delta) \subset X \setminus K$ and $X \setminus K$ is open.

Boundedness: The collection $\mathcal{G} = \{B(x, 1) \mid x \in K\}$ is an open cover for K. Since K is compact, there are elements $x_1, x_2, \ldots, x_n$ in K such that the sub-collection $\{B(x_j, 1) \mid j = 1, 2, \ldots, n\}$ of $\mathcal{G}$ covers K; i.e.,

$$K \subseteq \bigcup_{j=1}^{n} B(x_j, 1).$$

Since a finite union of bounded sets is bounded (Proposition 7.2.22), it follows that the set $\bigcup_{j=1}^{n} B(x_j, 1)$ is bounded. Hence K is bounded. ■

We saw earlier (Heine-Borel Theorem) that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. That is, the converse of Theorem 7.4.4 holds if $X = \mathbb{R}$. This converse does not hold in general. Here is a counterexample: Let K be an infinite subset of a discrete metric space (X, d). The K is closed and bounded but not compact (see Proposition 7.4.3).

7.4.5 Proposition
Let (X, d) be a compact metric space. Then any infinite subset of X has an accumulation point in X.

$\prec$Let K be an infinite subset of X with no accumulation point and let $x \in X$. Since x is not an accumulation point of K, there is an $\epsilon_x > 0$ such that $B(x, \epsilon_x) \cap K \subseteq \{x\}$. The collection $\mathcal{C} = \{B(x, \epsilon_x) \mid x \in X\}$ is an open cover for X. Since X is compact, there are elements $x_1, x_2, \ldots, x_n$ in X such that the finite sub-collection $\{B(x_j, \epsilon_{x_j}) \mid j = 1, 2, \ldots, n\}$ of $\mathcal{C}$ covers X; i.e.,

$$X \subseteq \bigcup_{j=1}^{n} B(x_j, \epsilon_{x_j}).$$

It follows that

$$K = K \cap X \subseteq K \cap \left(\bigcup_{j=1}^{n} B(x_j, \epsilon_{x_j}) \right) = \bigcup_{j=1}^{n} \left(K \cap B(x_j, \epsilon_{x_j})\right) \subseteq \bigcup_{j=1}^{n} \{x_j\} = \{x_1, x_2, \ldots, x_n\},$$

a finite set. This is a contradiction. Hence K has an accumulation point in K. ■

7.4.1 Sequential Compactness

7.4.6 Definition
A subset K of a metric space (X, d) is said to be **sequentially compact** *if every sequence in K has a subsequence that converges to a point in K.*

7.4.7 Exercise

[1] Let K be a bounded subset of a metric space (X, d). Show that

$$\text{diam}(K) = \inf\{M \mid d(x, y) \leq M \ \ \forall \ \ x, y \in K\}.$$

[2] Let K be a compact subset of a metric space (X, d). Show that the derived set K' of K is compact.